Multiscale analysis of tropical cyclones

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Asymptotic modelling framework

Structure of atmospheric vortices I: two scales

Structure of atmospheric vortices II: cascade of scales

Conclusions

Nondimensionalization

$$(\boldsymbol{x}, z) = \frac{1}{h_{\mathrm{sc}}} (\boldsymbol{x}', z') \,, \qquad t = \frac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}} t'$$

$$(\boldsymbol{u}, w) = \frac{1}{u_{\mathrm{ref}}} (\boldsymbol{u}', w') \,, \qquad (p, T, \rho) = \left(\frac{p'}{p_{\mathrm{ref}}}, \frac{T'}{T_{\mathrm{ref}}}, \frac{\rho' R T_{\mathrm{ref}}}{p_{\mathrm{ref}}} \right)$$

where

$$u_{\mathrm{ref}} = \frac{2}{\pi} \frac{g h_{\mathrm{sc}}}{\Omega a} \frac{\Delta\Theta}{T_{\mathrm{ref}}} \qquad \text{(thermal wind scaling)}$$

Dimensionless numbers, length scales, distinguished limit

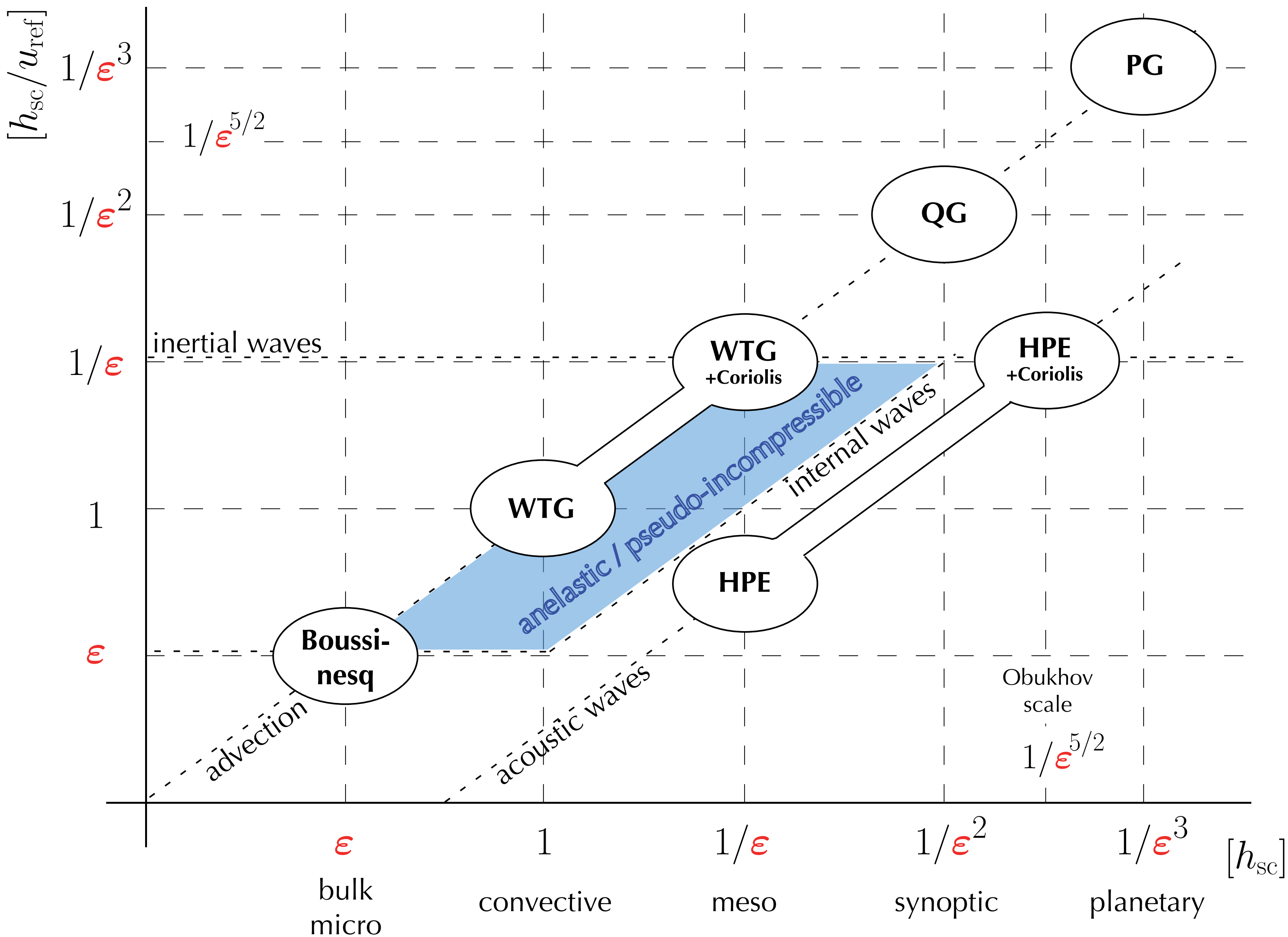
$$\begin{aligned}
\mathrm{Fr}_{\mathrm{int}} &\sim \varepsilon \\
\mathrm{Ro}_{h_{\mathrm{sc}}} &\sim \varepsilon^{-1} \\
\mathrm{Ro}_{L_{\mathrm{Ro}}} &\sim \varepsilon \\
\mathrm{Ma} &\sim \varepsilon^{3/2}
\end{aligned}$$

$$\begin{aligned}
L_{\mathrm{mes}} &= \varepsilon^{-1}\, h_{\mathrm{sc}} \\
L_{\mathrm{syn}} &= \varepsilon^{-2}\, h_{\mathrm{sc}} \\
L_{\mathrm{Ob}} &= \varepsilon^{-5/2} h_{\mathrm{sc}} \\
L_{\mathrm{p}} &= \varepsilon^{-3}\, h_{\mathrm{sc}}
\end{aligned}$$

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \boldsymbol{v}_{\shortparallel} \; + \; \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\shortparallel} + \frac{1}{\varepsilon^3 \rho} \; \nabla_{\shortparallel} p \; &= \; \boldsymbol{S}_{\boldsymbol{v}_{\shortparallel}} \,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) w \; + \; \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} \; &= \; S_w - \frac{1}{\varepsilon^3} \,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \rho \; + \; \rho \nabla \cdot \boldsymbol{v} \; &= \; 0 \,, \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \Theta \; &= \; S_\Theta \,.
\end{aligned}
$$

Scale-Dependent Models



R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

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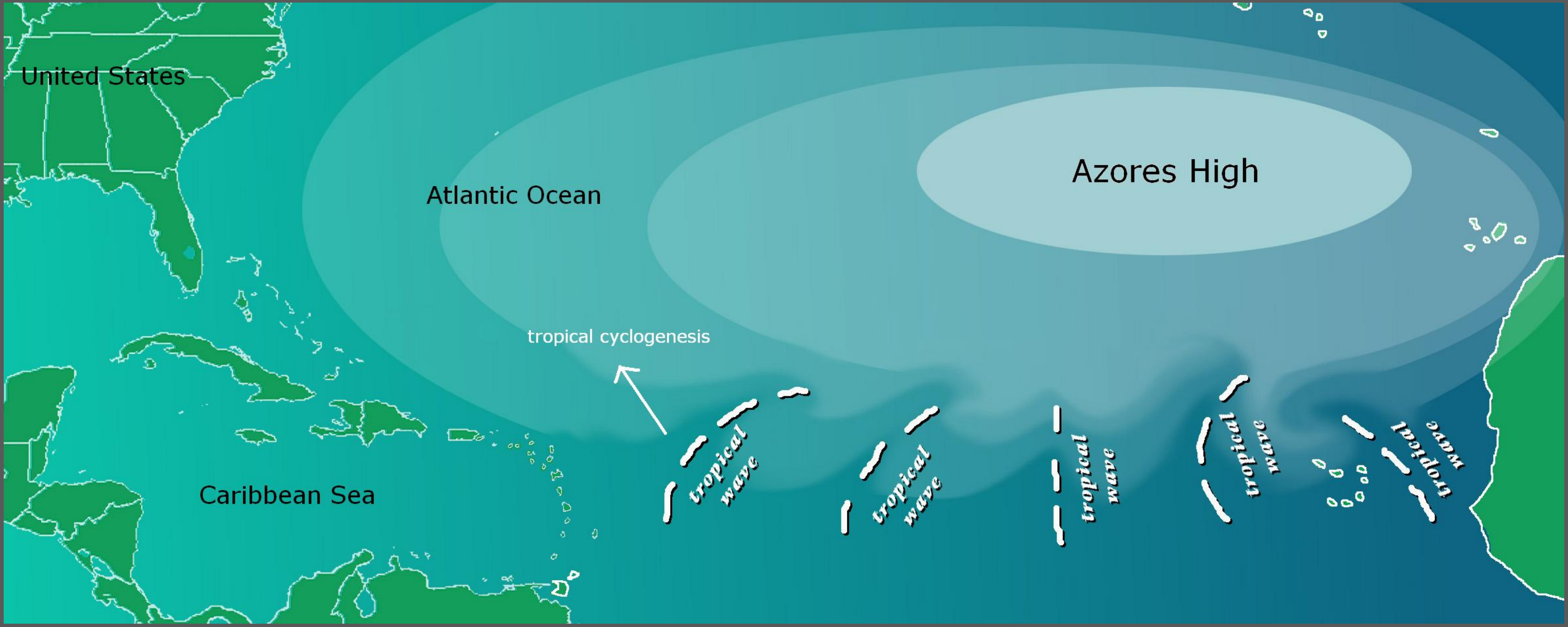
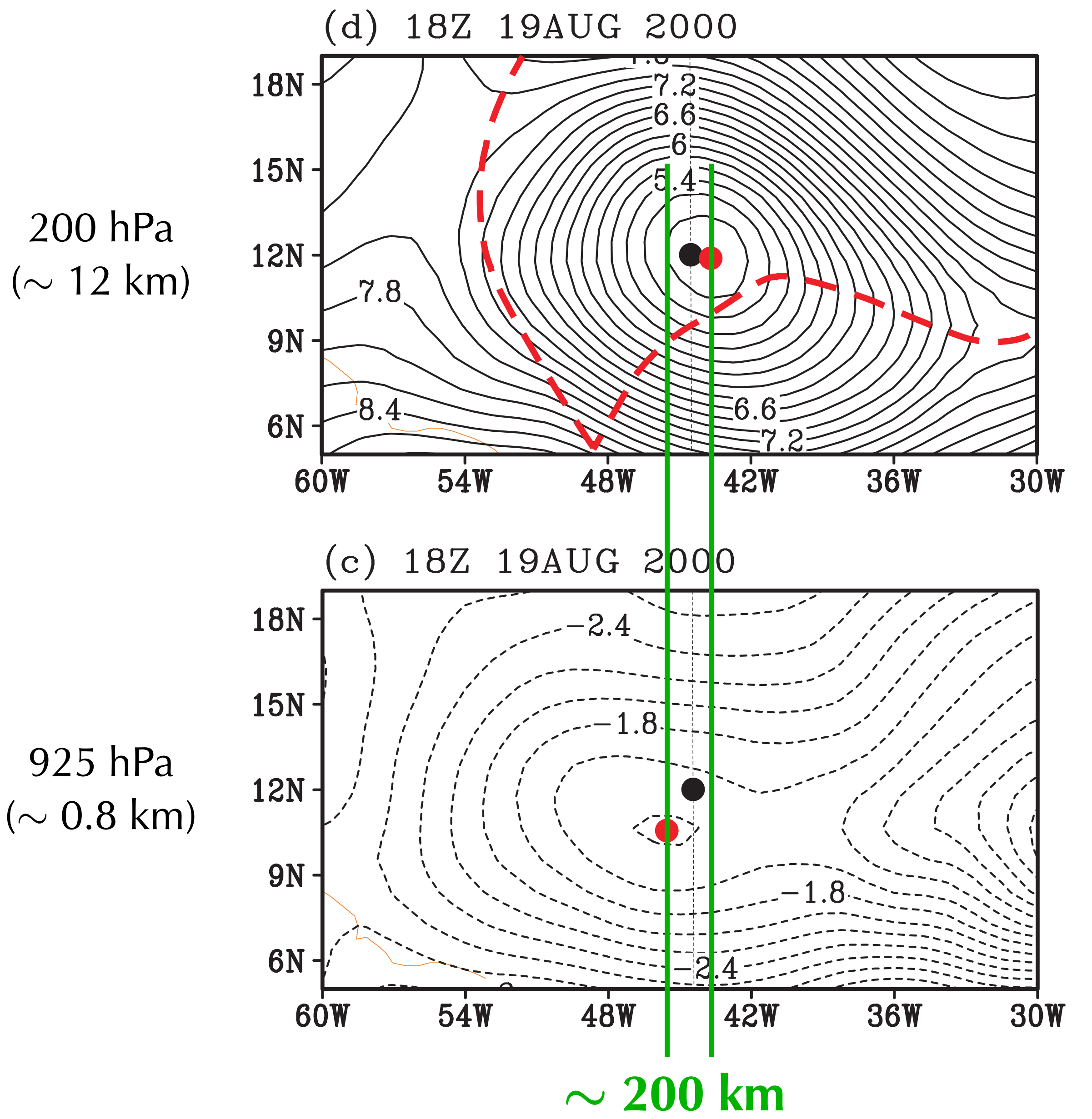
Tropical easterly african waves



http://www.aoml.noaa.gov/hrd/tcfaq/A4.html

Vortex tilt in the incipient hurricane stage

(Velocity potential)

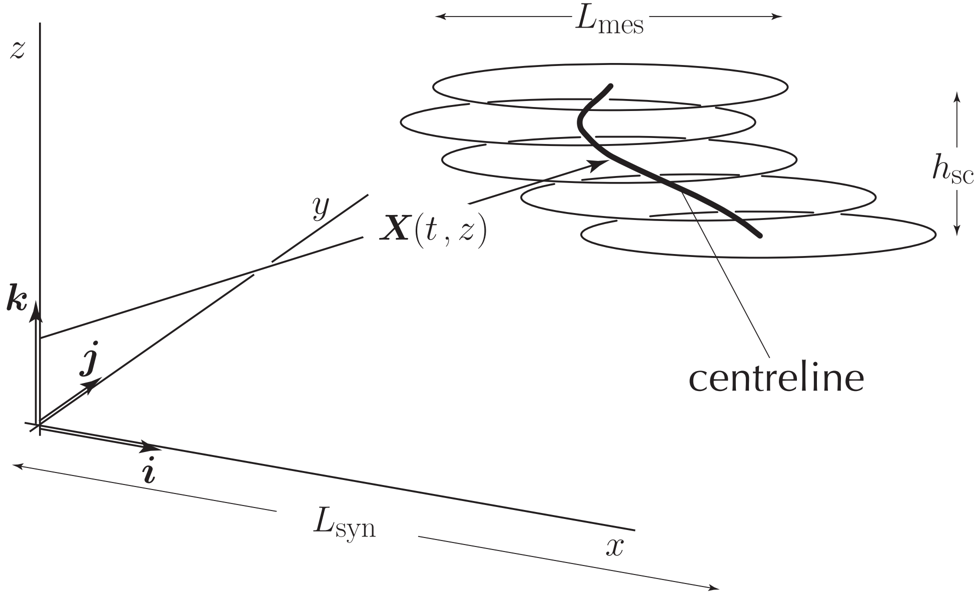


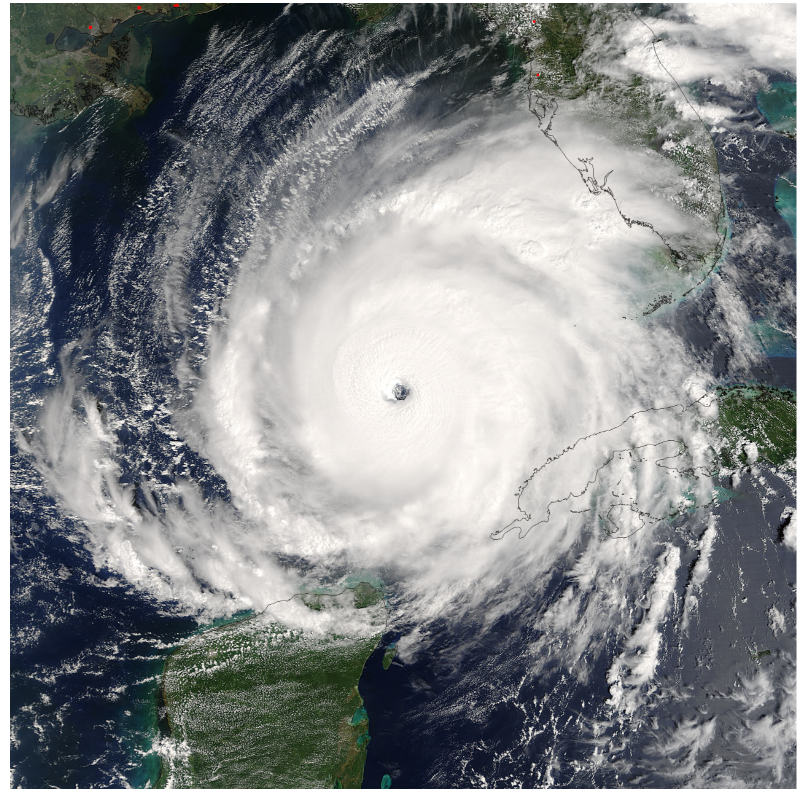
Dunkerton et al., Atmos. Chem. Phys., **9**, 5587–5646 (2009)

Radial momentum balance regimes

$- \frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**geostrophic**	$\mathrm{Ro} \ll 1$	**typical "weather"**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**gradient wind**	$\mathrm{Ro} = \mathcal{O}(1)$	**tropical storm** **incipient hurricane**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} = o(1)$	**cyclostrophic**	$\mathrm{Ro} \gg 1$	**hurricane**

Asymptotic scaling regime





$$t_{\mathrm{syn}} = \frac{h_{\mathrm{sc}}/u_{\mathrm{ref}}}{\varepsilon^2}; \qquad L_{\mathrm{syn}} = \frac{h_{\mathrm{sc}}}{\varepsilon^2}; \qquad |\boldsymbol{v}_{\shortparallel}| = \mathcal{O}(1);$$

farfield: classical QG theory

$$|\boldsymbol{v}_{\shortparallel}|\, L = \mathcal{O}\left(\varepsilon^{-2}\right); \qquad |\boldsymbol{v}_{\shortparallel}|/fL = \mathcal{O}(\varepsilon)$$

$$L_{\mathrm{mes}} = \frac{h_{\mathrm{sc}}}{\varepsilon^{3/2}}; \qquad |\boldsymbol{v}_{\shortparallel}| = \mathcal{O}\left(\frac{1}{\varepsilon^{1/2}}\right)$$

core: gradient wind scaling

$$|\boldsymbol{v}_{\shortparallel}|\, L = \mathcal{O}\left(\varepsilon^{-2}\right); \qquad |\boldsymbol{v}_{\shortparallel}|/fL = \mathcal{O}(1)$$

Result of matched asymptotic expansion analysis:

3D Theory for

vortex motion, vortex core dynamics*,

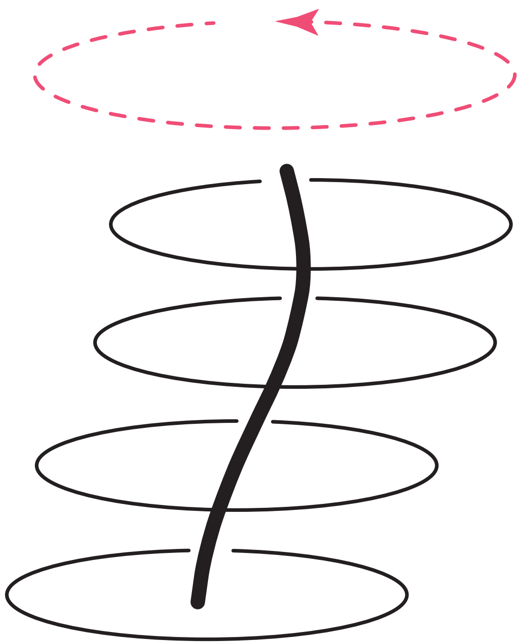
and the role of subscale moist processes*

* **Includes strong vortex tilt**

* **Modelled by prescribed heating patterns here**

Vortex motion

Adiabatic case ($Q_\Theta \equiv 0$)



- Linear **small displacement*** theory extended to **large displacements**
- Precession and stationary tilt in background shear* explained analytically

* Jones (1995, 2004); Montgomery & Co-workers (2001, 2004–2007)

Vortex motion

$$\boldsymbol{X}(\tau, z)/L_{\mathrm{mes}} = \overline{\boldsymbol{X}}(\tau)/\sqrt{\varepsilon} + \widehat{\boldsymbol{X}}(\tau, z)\,.$$

where

$$d\overline{\boldsymbol{X}}/d\tau = \overline{\boldsymbol{v}}_{\mathrm{qg}}$$

$$\partial\widehat{\boldsymbol{X}}/\partial\tau = \underbrace{\widehat{\boldsymbol{X}} \cdot (\boldsymbol{\nabla}_{\shortparallel}\overline{\boldsymbol{v}}_{\mathrm{qg}}) + \widehat{\boldsymbol{v}}^{*}_{\mathrm{qg}}}_{\text{background advection}} \underbrace{- \left(\ln\frac{1}{\sqrt{\varepsilon}} + \frac{1}{2}\right)(\boldsymbol{k} \times \chi)^{*} + (\boldsymbol{k} \times \boldsymbol{\Psi})}_{\text{self-induced motion}}$$

$$\chi = \frac{f^2}{4\pi\overline{\rho}\Gamma}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}\left(\frac{\overline{\rho}\Gamma^2}{d\overline{\Theta}/dz}\frac{\boldsymbol{\partial}\widehat{\boldsymbol{X}}}{\boldsymbol{\partial z}}\right) \qquad \text{(LIA-like expression)}$$

$\boldsymbol{\Psi}$:
- integral expression representing net vertical transport of horizontal momentum
- depend on core structure, tilt, diabatic source terms

* includes effect of vortex on background flow (β-gyres)

* Analogous to local-induction-approximation LIA

Adiabatic lifting and WTG

(0th & 1st circumferential Fourier modes: $w = w_0 + w_{11}\cos\theta + w_{12}\sin\theta + \dots$)

gradient wind balance (0th) and hydrostatics (1st) in the tilted vortex

$$\frac{1}{\overline{\rho}}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} + f\,u_\theta\,, \qquad \Theta_{1\boldsymbol{k}} = -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial r}\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}\right)_{1\boldsymbol{k}}$$

potential temperature transport (1st)

$$-(-1)^k\frac{u_\theta}{r}\Theta_{1\boldsymbol{k}^*} + w_{1k}\frac{d\overline{\Theta}}{dz} = Q_{\Theta,1\boldsymbol{k}} \qquad (\boldsymbol{k}^* = 3-k)$$

1st-mode phase relation: vertical velocity – diabatic sources & vortex tilt

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \left(\boldsymbol{e}_r\cdot\frac{\partial\widehat{\boldsymbol{X}}^{\perp}}{\partial z}\right)_{\boldsymbol{k}}\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

Spin-up by asynchronous heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,1\mathbf{1}}\frac{\partial \widehat{X}}{\partial z} + Q_{\Theta,1\mathbf{2}}\frac{\partial \widehat{Y}}{\partial z}\right)}$$

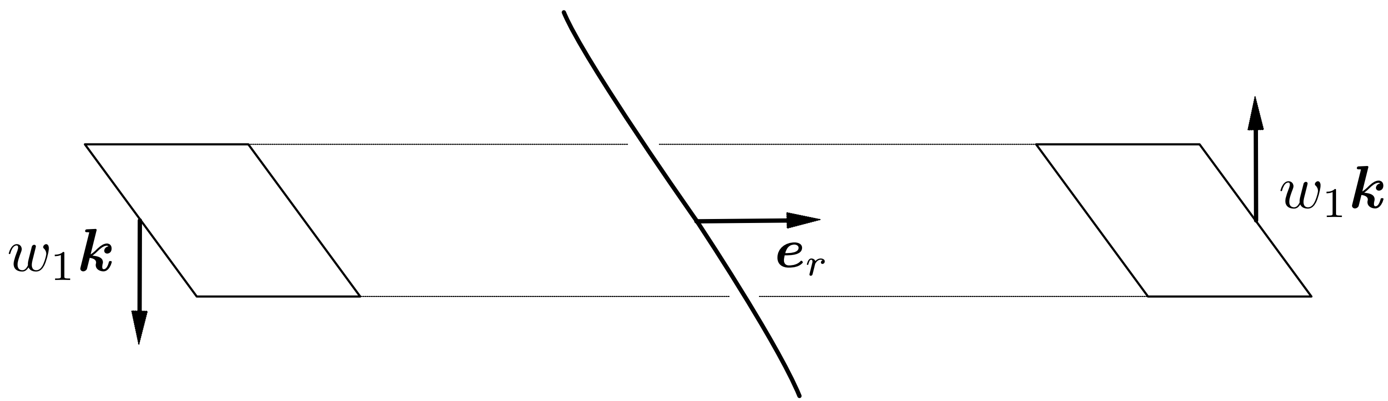
$$\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} = \widehat{X}\cos\theta + \widehat{Y}\sin\theta$$

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

Spin-up by asynchronous heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,11}\frac{\partial \widehat{X}}{\partial z} + Q_{\Theta,12}\frac{\partial \widehat{Y}}{\partial z}\right)}$$



Apparent radial transport in a tilted vortex

Recent results

Theory is valid uniformly for

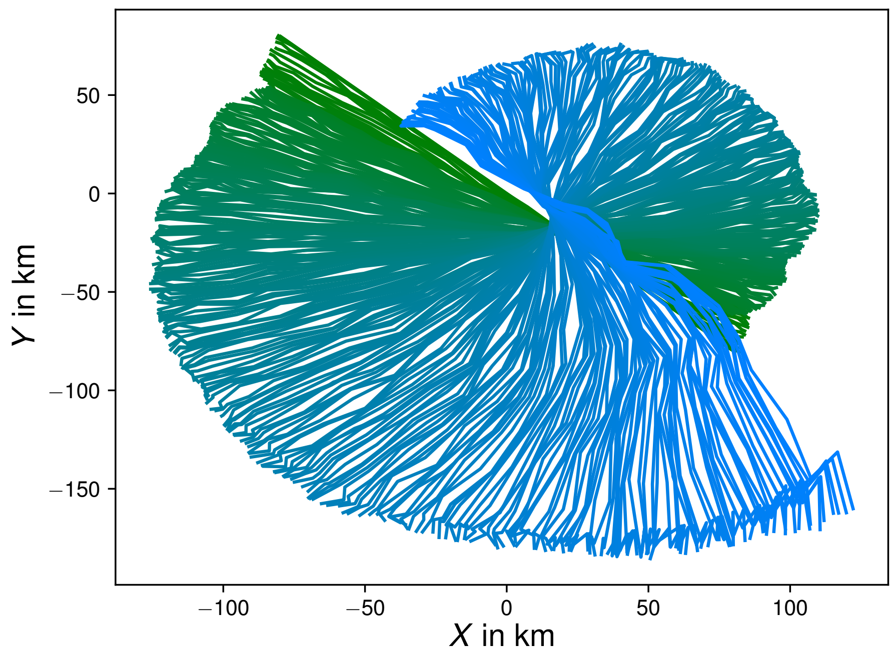
large vortex Rossby numbers $(f \to 0)$

as long as the internal wave Froude number is small

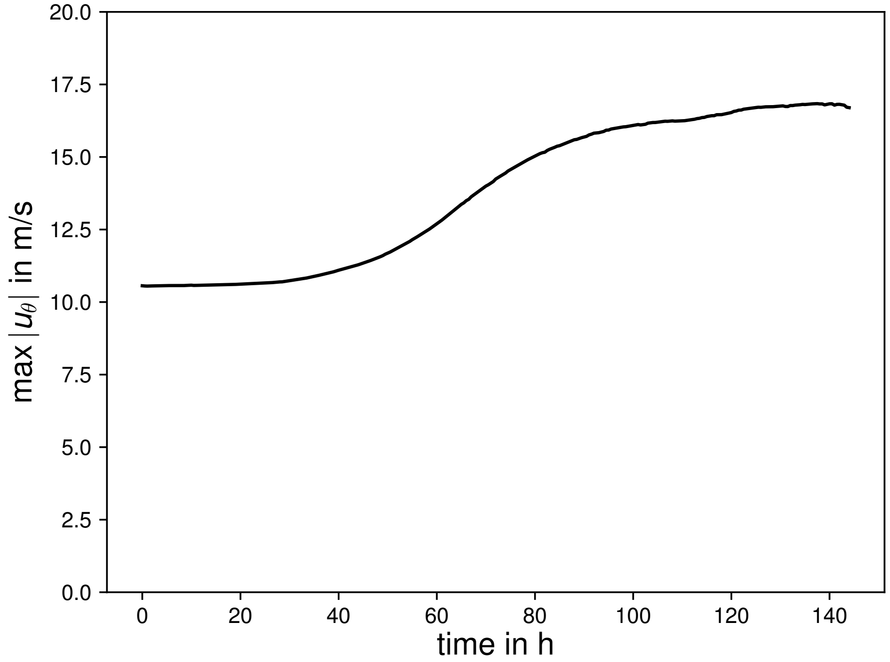
Dörffel et al., preprint, arXiv:1708.07674 (2017)

Recent results

Qualitative corroboration through 3D-numerics (benign case)



Centerline evolution

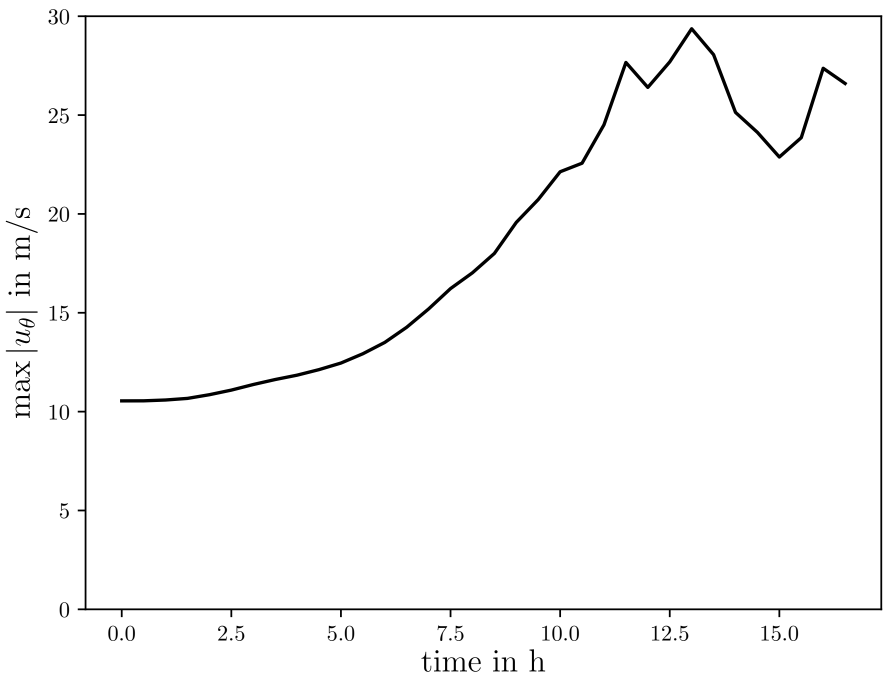


Intensification

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[-\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}\right)_{\boldsymbol{k}}\frac{u_\theta^0}{r}\left(\frac{{u_\theta^0}^2}{r}+f\,u_\theta^0\right)+\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}}\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r}+f\,u_\theta\right)\right]$$

Recent results

Qualitative corroboration through 3D-numerics (violent case)*



$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[-\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}\right)_{\boldsymbol{k}}\frac{u_\theta}{r}\left(\frac{{u_\theta}^2}{r}+f\,u_\theta\right)+\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}}\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r}+f\,u_\theta\right)\right]$$

* Ultimately leaves asymptotic regime!

Dörffel et al., preprint, arXiv:1708.07674 (2017)

Recent results

Compatibility with Lorenz′ APE theory

$$\Big(re_{\mathrm{k}}\Big)_t + \Big(ru_{r,0}[e_{\mathrm{k}} + p']\Big)_r + \Big(rw_0[e_{\mathrm{k}} + p']\Big)_z = \frac{r\overline{\rho}}{N^2\overline{\Theta}^2}\Big(\Theta_0' Q_{\Theta,0} + \boldsymbol{\Theta}_1' \cdot \boldsymbol{Q}_{\Theta,1}\Big)$$

Asymptotic modelling framework

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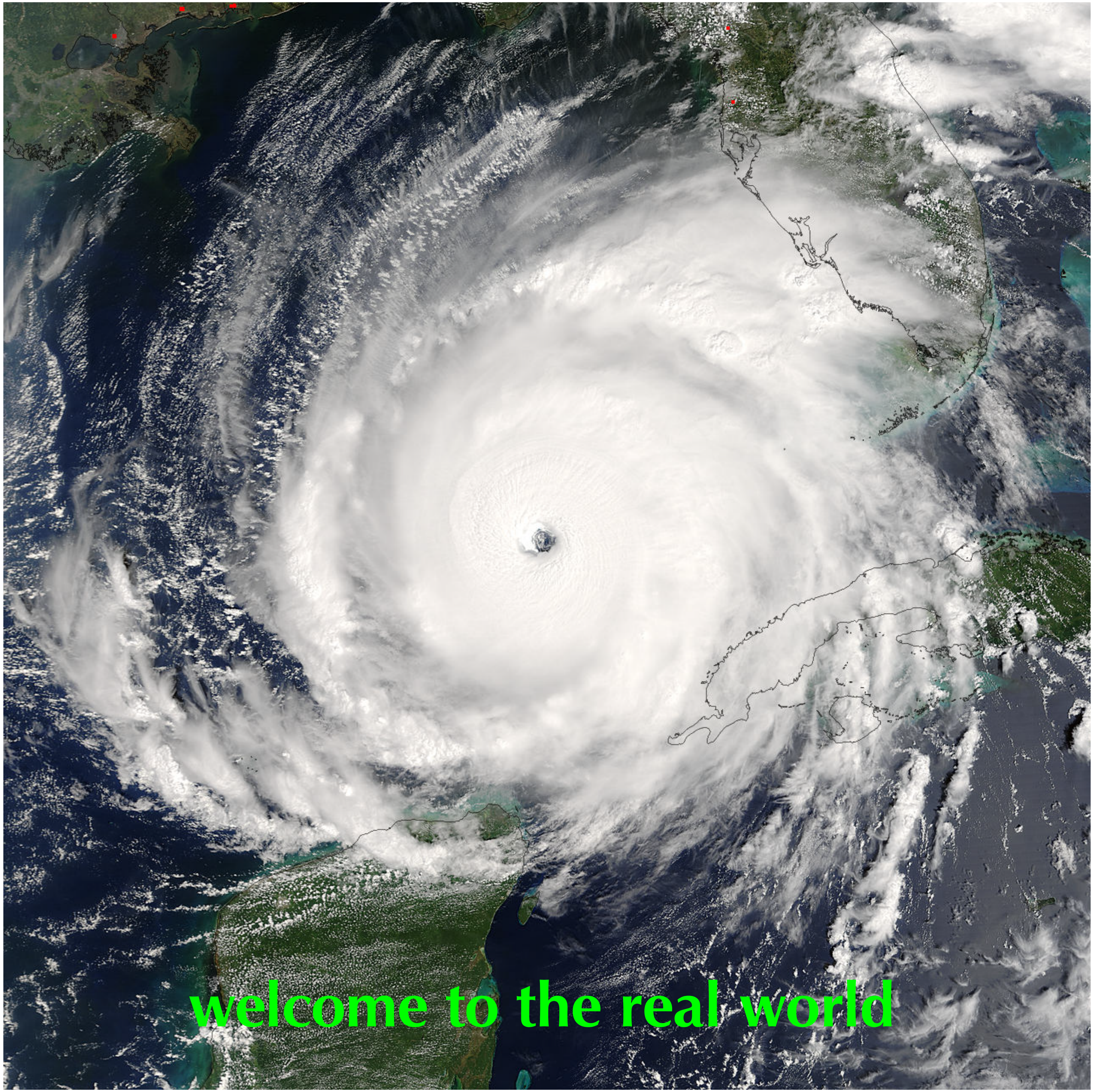
Conclusions

Cascade of scales:

radial structure

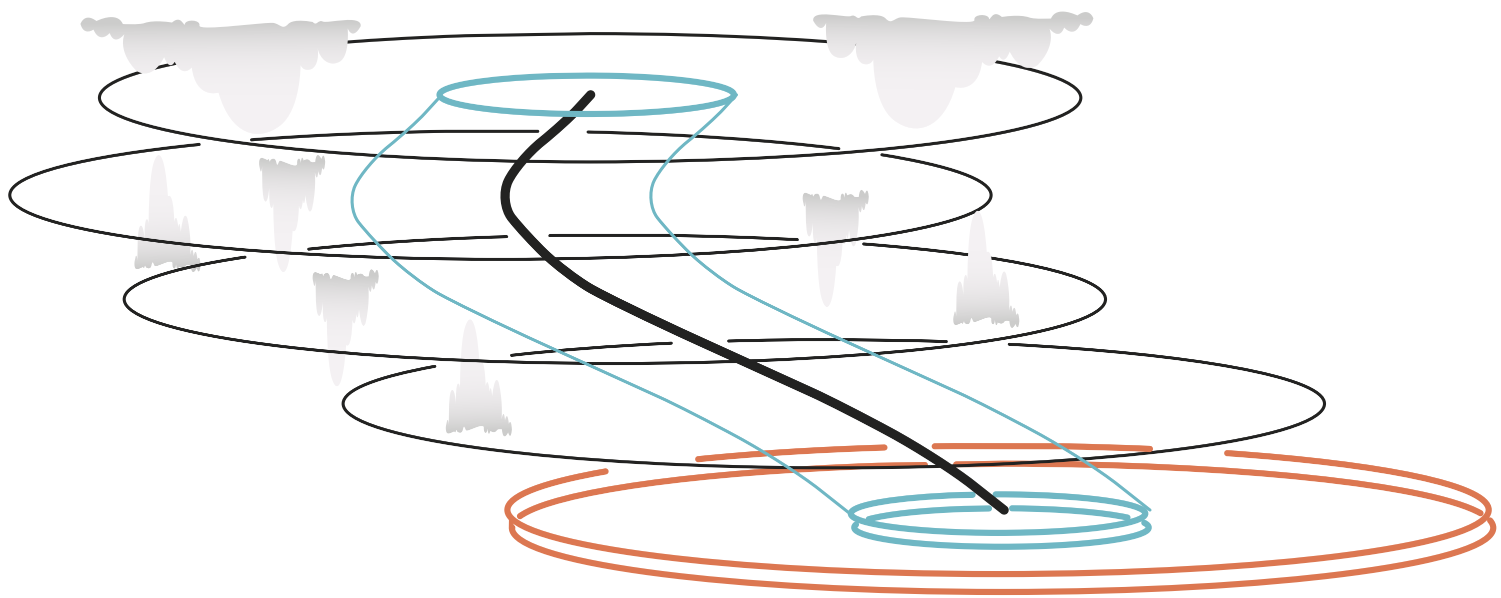
- > 2 radial layers (eye)
- vortex Rossby waves
- spiral rainbands
- “spotty” cloud patterns
- ...



Cascade of scales:

vertical structure

- **boundary layer**
- **convective updrafts**
- secondary circulation
- tropopause cap
- ...



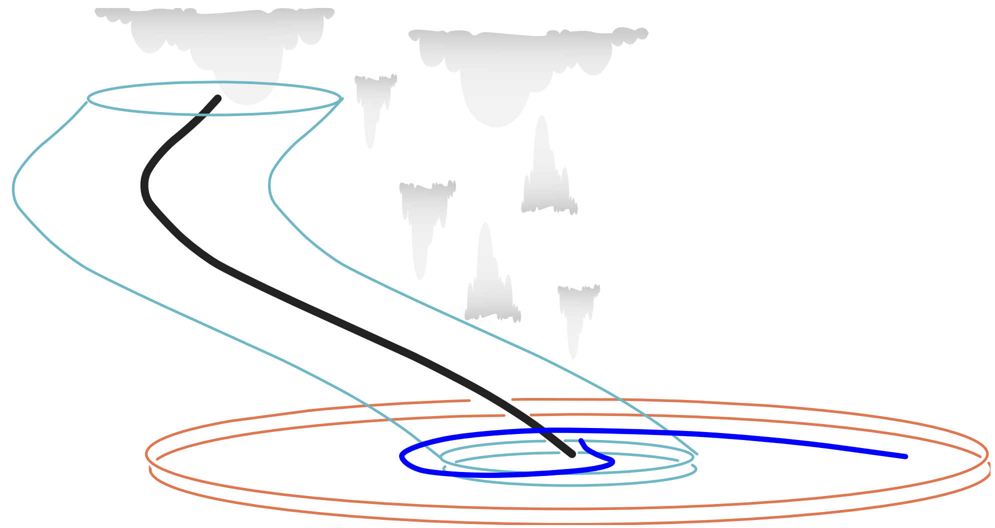
Boundary layer

- turbulent friction disturbs momentum balance

$$-\frac{u_\theta^2}{r} - f\,u_\theta + \frac{1}{\rho}\frac{\partial p}{\partial r} = \underline{\frac{\partial}{\partial z}\left(K\frac{\partial u_r}{\partial z}\right)}$$

$$-2\frac{u_\theta u_r}{r} + f\,u_r = \underline{\frac{\partial}{\partial z}\left(K\frac{\partial u_\theta}{\partial z}\right)}$$

- implies Ekman-type radial inflow
- boundary layer expells mass vertically



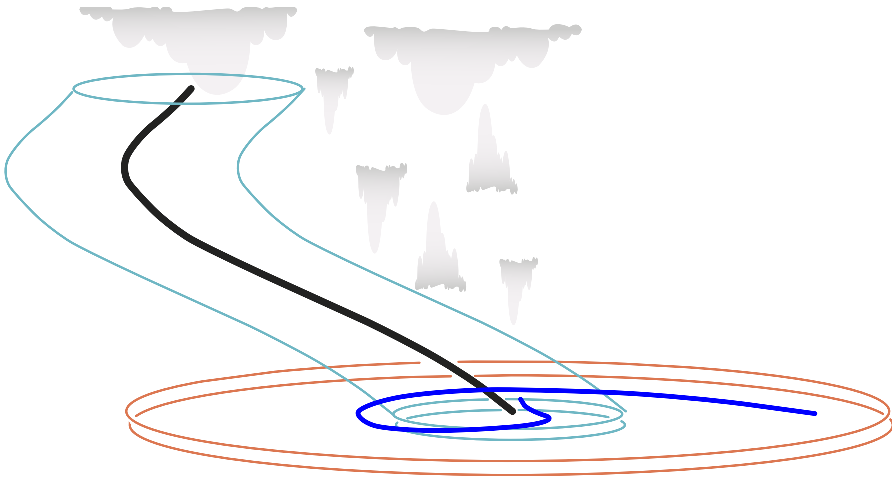
Orders of magnitude:

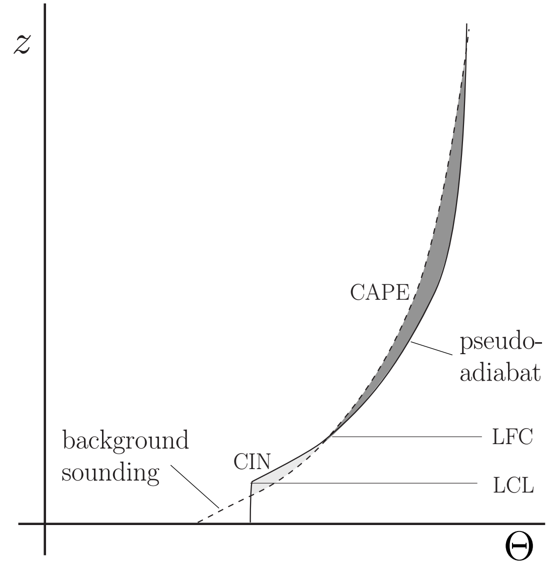
$$u_r \sim u_\theta \sim \frac{u_{\mathrm{ref}}}{\boldsymbol{\delta}}; \qquad L \sim \frac{h_{\mathrm{sc}}}{\boldsymbol{\delta}^2}; \qquad h_{\mathrm{bl}} \sim \boldsymbol{\delta}^2 h_{\mathrm{sc}}; \qquad w \sim \boldsymbol{\delta}^{\boldsymbol{\alpha}} u_{\mathrm{ref}}$$

$$2\pi\left(\frac{h_{\mathrm{sc}}}{\boldsymbol{\delta}^2}\right)\left(\frac{u_{\mathrm{ref}}}{\boldsymbol{\delta}}\right) \sim \pi\left(\frac{h_{\mathrm{sc}}}{\delta^2}\right)^2 \boldsymbol{\delta}^{\boldsymbol{\alpha}} u_{\mathrm{ref}} \qquad \Rightarrow \qquad w_{\mathrm{bl-top}} \sim \boldsymbol{\delta}^3 u_{\mathrm{ref}}$$

Same order of magnitude as observed in the tilted bulk vortex

Convective updrafts





Orders of magnitude

$$w_{\text{upd}} \leq \sqrt{2\,\text{CAPE}}$$

$$\text{CAPE} \sim 100 \ldots 3\,600\ \text{m}^2/\text{s}^2; \qquad w_{\text{upd}} \sim 10 \ldots 60\ \text{m/s} \sim \frac{u_{\text{ref}}}{\delta^{\beta}}; \qquad \beta \geq 0$$

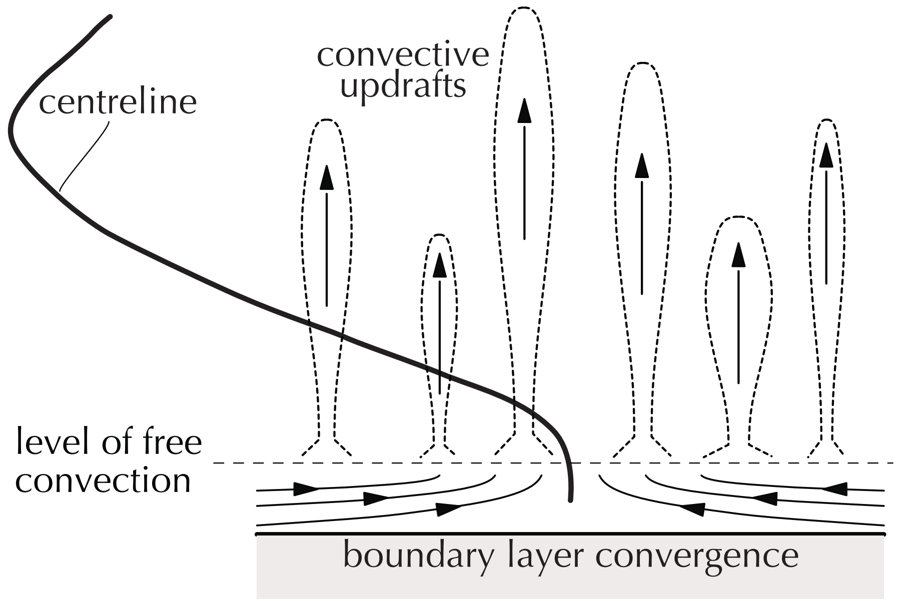
vertical velocities

$$w_{\text{bulk}} \sim w_{\text{bl-top}} \ll w_{\text{upd}}$$

but vertical mass fluxes

$$\dot{m}_{\text{bulk}} \sim \dot{m}_{\text{bl-top}} \sim \dot{m}_{\text{upd}} \qquad \Rightarrow \qquad \boldsymbol{w_{\text{bulk}} \sim w_{\text{bl-top}} \sim \overline{w}_{\text{upd}}}$$

Convective updrafts

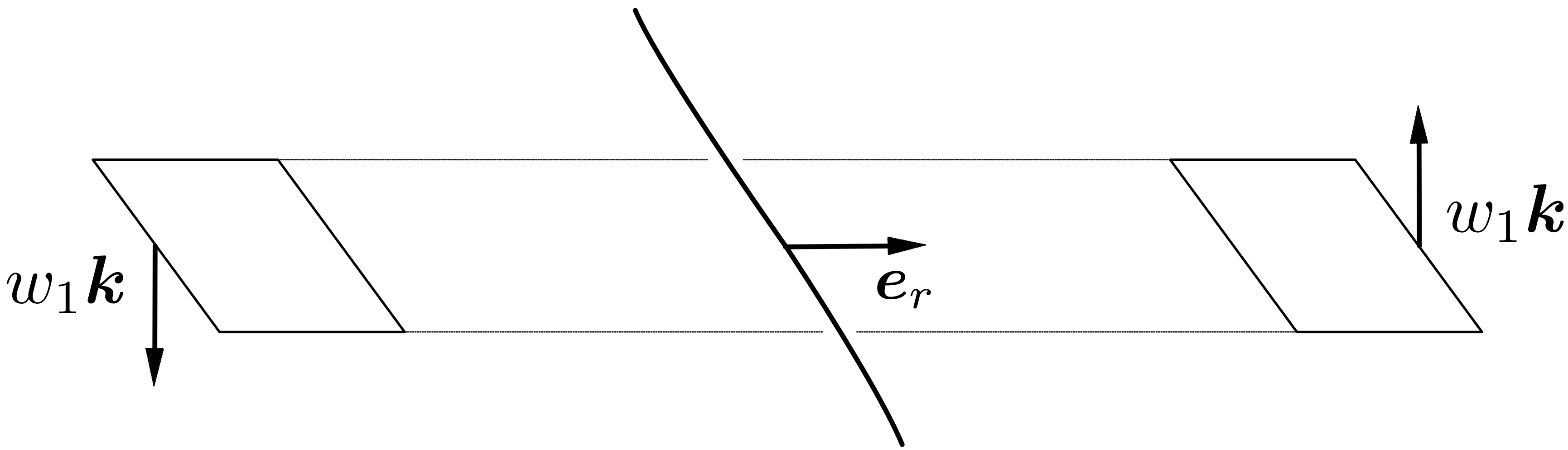


Convection concentrates in narrow towers (area fraction $\sigma \ll 1$)

Dry dynamics between towers

Comparable average vertical mass fluxes

From angular momentum conservation for a centered torus ...



From angular momentum conservation for a centered torus ...

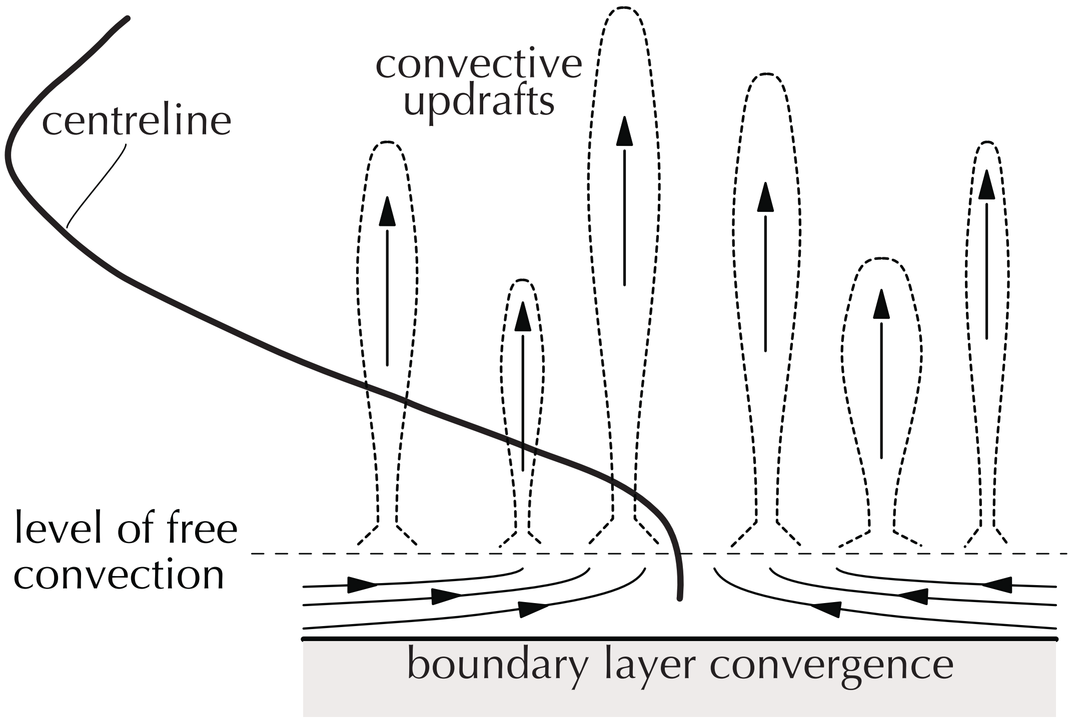
Spin-up by asynchronous convection

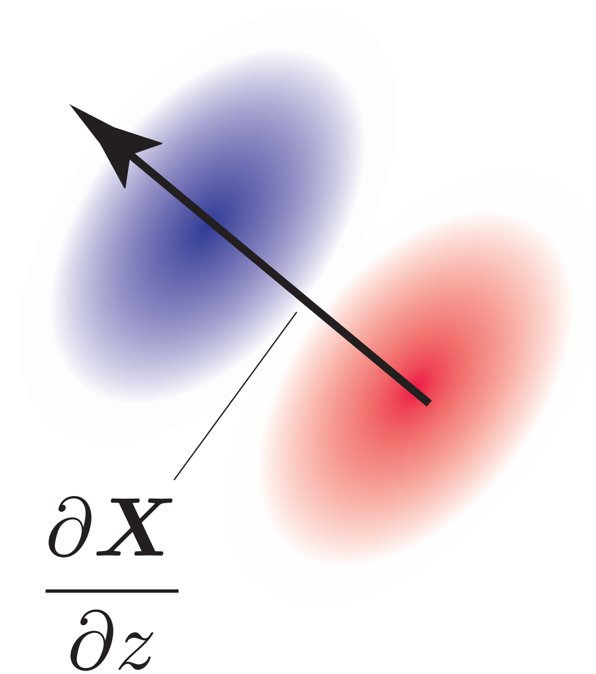
$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\, \boldsymbol{u_{r,*}} \left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle \boldsymbol{w} \, \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right) \right\rangle_\theta = \underline{\overline{\boldsymbol{w}}_{\mathrm{upd},1\mathbf{1}} \frac{\partial \widehat{X}}{\partial z} + \overline{\boldsymbol{w}}_{\mathrm{upd},1\mathbf{2}} \frac{\partial \widehat{Y}}{\partial z}} \qquad \textbf{!!}$$

Area averaged updraft fluxes take role of WTG-induced vertical velocities in the dry vortex theory

Convection/heating arrangement for most rapid intensification





- max efficiency for $w_{\mathrm{upd}} > \boldsymbol{\delta} u_{\mathrm{ref}}$
- "decorrelation" by circumferential advection for $w_{\mathrm{upd}} \leq \boldsymbol{\delta} u_{\mathrm{ref}}$

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Addressed to both graduate students and researchers this monograph provides in-depth analyses of vortex-dominated flows via matched and multiscale asymptotics, and it demonstrates how insight gained through these analyses can be exploited in the construction of robust, efficient, and accurate numerical techniques. The dynamics of slender vortex filaments is discussed in detail, including fundamental derivations, compressible core structure, weakly nonlinear limit regimes, and associated numerical methods. Similarly, the volume covers asymptotic analysis and computational techniques for weakly compressible flows involving vortex-generated sound and thermoacoustics.

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