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Navier–Stokes Equations and Nonlinear Functional Analysis

Second Edition

ROGER TEMAM
Université de Paris-Sud, Orsay, France
Indiana University, Bloomington, Indiana

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Preface to the Second Edition

Since publication of the first edition of this book in 1983, a very active area in the theory of Navier–Stokes equations has been the study of these equations as a dynamical system in relation to the dynamical system approach to turbulence. A large number of results have been derived concerning the long-time behavior of the solutions, the attractors for the Navier–Stokes equations and their approximation, the problem of the existence of exact inertial manifolds and approximate inertial manifolds, and new numerical algorithms stemming from dynamical systems theory, such as the nonlinear Galerkin method. Numerical simulations of turbulence and other numerical methods based on different approaches have also been studied intensively during this decade.

Most of the results presented in the first edition of this book are still relevant; they are not altered here. Recent results on the numerical approximation of the Navier–Stokes equation or the study of the dynamical system that they generate are addressed thoroughly in more specialized publications.

In addition to some minor alterations, the second edition of *Navier–Stokes Equations and Nonlinear Functional Analysis* has been updated by the addition of a new appendix devoted to inertial manifolds for Navier–Stokes equations. In keeping with the spirit of these notes, which was to arrive as rapidly and as simply as possible at some central problems in the Navier–Stokes equations, we choose to add this section addressing one of the topics of extensive research in recent years.

Although some related concepts and results had existed earlier, inertial manifolds were first introduced under this name in 1985 and systematically studied for partial differential equations of the Navier–Stokes type since that date. At this time the theory of inertial manifolds for Navier–Stokes equations is not complete, but there is already available a set of results which deserves to be known, in the hope that this will stimulate further research in this area.

Inertial manifolds are a global version of central manifolds. When they exist they encompass the complete dynamics of a system, reducing the dynamics of an infinite system to that of a smooth, finite-dimensional one called the inertial system. In the Appendix we describe the concepts and recall the definitions and some typical results; we show the existence of inertial manifolds for the Navier–Stokes equations with an enhanced viscosity. We also describe a tentative route for proving the existence of inertial systems for the actual two-dimensional Navier–Stokes equations and for the two-dimensional version of the Navier–Stokes equations corresponding to the flow around a sphere (flow of a thin layer of fluid around a sphere), a subject of obvious interest for geophysical flows and climate problems.

As indicated earlier, another aspect of inertial manifolds not presented here is the use of approximation of inertial manifolds for the development of new multilevel algorithms adapted to the resolution of the many scales present in turbulent flows. These aspects are addressed (and will be addressed further) in publications more numerically or computationally oriented; some bibliographic references are given in the Appendix.

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Introduction

The Navier–Stokes equations are the equations governing the motion of usual fluids like water, air, oil, . . . , under quite general conditions, and they appear in the study of many important phenomena, either alone or coupled with other equations. For instance, they are used in theoretical studies in aeronautical sciences, in meteorology, in thermo-hydraulics, in the petroleum industry, in plasma physics, etc. From the point of view of continuum mechanics the Navier–Stokes equations (N.S.E.) are essentially the simplest equations describing the motion of a fluid, and they are derived under a quite simple physical assumption, namely, the existence of a linear local relation between stresses and strain rates. These equations, which are recalled in § 1, are nonlinear. The nonlinear term $(u \cdot \nabla)u$ contained in the equations comes from kinematical considerations (i.e., it is the result of an elementary mathematical operation) and does not result from assumptions about the nature of the physical model; consequently this term cannot be avoided by changing the physical model.

While the physical model leading to the Navier–Stokes equations is simple, the situation is quite different from the mathematical point of view. In particular, because of their nonlinearity, the mathematical study of these equations is difficult and requires the full power of modern functional analysis. Even now, despite all the important work done on these equations, our understanding of them remains fundamentally incomplete.

Three types of problems appear in the mathematical treatment of these equations. Although they are well known, we recall them briefly for the nonspecialist.

1) *Existence, uniqueness and regularity.* It has been known since the work of J. Leray [1] that, provided the data are sufficiently smooth (see § 3), the initial value problem for the time-dependent Navier–Stokes equations possesses a unique smooth solution on some interval of time $(0, T_*)$; according to J. Leray [3] and E. Hopf [1], this solution can be extended for subsequent time as a (possibly) less regular function. A major question as yet unanswered is whether the solution remains smooth all the time. In the case of an affirmative answer the question of existence and uniqueness would be considerably clarified. In the case of a negative answer, then it would be important to have information on the nature of the singularities, and to know whether the weak solutions are unique and, should they be not unique, how to characterize the "physical" ones.

All these and other related questions are interesting not only for mathematical understanding of the equations but also for understanding the phenomenon of turbulence. Recall here that J. Leray's conjecture [1]–[3] on turbulence, and his motivation for the introduction of the concept of weak solutions, was that the solutions in three space dimensions are not smooth, the velocity or the vorticity (the curl of the velocity) becoming infinite at some points or on some "small" sets where the turbulence would be located.

All the mathematical problems that we have mentioned are as yet open. Let us mention, however, some recent studies on the Hausdorff dimension of the set of singularities of solutions (the set where the velocity is infinite), studies initiated by B. Mandelbrot [1] and V. Scheffer [1]–[4] and developed by L. Caffarelli–R. Kohn–L. Nirenberg [1] (see also C. Foias–R. Temam [5]). The studies are meant to be some hopeful steps towards the proof of regularity if the solutions are smooth, or else some steps in the study of the singular set if singularities do develop *spontaneously*.

2) *Long time behavior.* If the volume forces and the given boundary values of the velocity are independent of time, then time does not appear explicitly in the Navier–Stokes equations and the equations become an autonomous infinite dimensional dynamical system. A question of interest is then the behavior for $t \to \infty$ of the solution of the time-dependent N.S.E. A more detailed description of this problem is contained in § 9, but, essentially, the situation is as follows. If the given forces and boundary values of the velocity are small then there exists a unique stable stationary solution and the time-dependent solution converges to it as $t \to \infty$. On the other hand, if the forces are large, then it is very likely from physical evidence and from our present understanding of bifurcation phenomena that, as $t \to \infty$, the solution tends to a time periodic one or to a more complicated attracting set. In the latter case, the long time behavior of the solution representing the "permanent" regime, could well appear chaotic. This is known to happen even for very simple dynamical systems in finite dimensional spaces, such as the Lorenz model (cf. also O. Lanford [1]) or the examples of mappings of the unit interval of $\mathbb{R}$ into itself, discussed by M. Feigenbaum [1] and O. Lanford [2]. Such chaotic behavior is another way to explain turbulence; it is based on the ideas of dynamical systems and strange attractors, following D. Ruelle [1], [2], D. Ruelle–F. Takens [1] and S. Smale [2]. Actually, for the moment, these two ways for the description of turbulence are not mutually exclusive, as singularities *and* long time chaotic behavior could perhaps be both present in the Navier–Stokes equations. Let us mention also that, as observed by D. Ruelle [3], the strange attractor point of view is not sufficient to explain the chaotic structure in space of the physical flow[1].

A great number of mathematical problems relating to the behavior for $t \to \infty$ of the solutions of the N.S.E. are open. They include convergence to a stable stationary or time periodic solution in connection with bifurcation theory[2] or convergence to a more complicated attractor (cf. C. Foias–R. Temam [5], [8], [9] and also J. Mallet-Paret [1]). Of course in three space dimensions the

[1] We specify "physical flow" since the word "flow" is also used in the dynamical system context to describe the trajectory of $u(t)$ in the function space.

[2] From the abundant literature we refer, for instance, to C. Bardos–C. Bessis [1], P. Rabinowitz [1] and the references therein.

difficulties are considerable, and the problem is perhaps out of reach for the moment since we do not even know whether the initial value problem for the N.S.E. is well posed.

3) *Numerical solution.* We mentioned at the beginning that the N.S.E. play an important role in several scientific and engineering fields. The needs there are usually not for a qualitative description of the solution but rather for a quantitative one, i.e., for the values of some quantities related to the solution. Since the exact resolution of the N.S.E. is totally out of reach (we actually know only a very small number of exact solutions of these equations), the data necessary for engineers can be provided only through numerical computations. Also, for practical reasons there is often a need for accurate solutions of the N.S.E. and reliance on simplified models is inadequate.

Here again the problem is difficult and the numerical resolution of the N.S.E. will require (as in the past) the simultaneous efforts of mathematicians, numerical analysts and specialists in computer science. Several significant problems can already be solved numerically, but much time and effort will be necessary until we master the numerical solution of these equations for realistic values of the physical parameters. Besides the need for the development of appropriate algorithms and codes and the improvement of computers in memory size and computation speed, there is another difficulty of a more mathematical (as well as practical) nature. The solutions of the N.S.E. under realistic conditions are so highly oscillatory (chaotic behavior) that even if we were able to solve them with a great accuracy we would be faced with too much useless information. One has to find a way, with some kind of averaging, to compute mean values of the solutions and the corresponding desired parameters.

As mentioned before, some analytical work is necessary for the numerical resolution of the N.S.E. But, conversely, one may hope that if improved numerical methods are available, they may help the mathematician in the formulation of realistic conjectures about the N.S.E., as happened for the Korteweg–de Vries equations.

After these general remarks on fluid dynamics and the Navier–Stokes equations, we describe the content of this monograph, which constitutes a very modest step towards the understanding of the outstanding problems mentioned above. The monograph contains 14 sections grouped into three parts (§§ 1–8, 9–12, 13–14), corresponding respectively to the three types of problems which we have just presented.

Part I (§§ 1–8) contains a set of results related to the existence, uniqueness and regularity of the weak and strong solutions. The material in Part I may appear technical, but most of it is essential for a proper understanding of more qualitative or more concrete questions. In § 1 the N.S.E. are recalled and we give a brief description of the boundary value problems usually associated with them. Sections 2 and 3 contain a description of the classical existence and uniqueness results for weak and strong solutions. We have tried to simplify the presentation of these technical results. In particular, we have chosen to emphasize the case of the flow in a cube in $\mathbb{R}^n$, $n=2$ or 3, with *space periodic* boundary conditions. This case, which is not treated in the available books or survey articles, leads to many technical simplifications while retaining the main difficulties of the problem (except for the boundary layer question, which is not considered here). Of course the necessary modifications for the case of the flow in a bounded region of $\mathbb{R}^n$ are given. The results in §§ 2 and 3 are essentially self-contained, except for some very technical points that can be found elsewhere in the literature, (cf. in particular O. A. Ladyzhenskaya [1], J. L. Lions [1] and R. Temam [6] to which we will refer more briefly as [RT]).

New (or recent) developments, related either to weak solutions or to strong solutions, are then presented in §§ 4 to 8. New a priori estimates for weak solutions are proved in § 4 following C. Foias–C. Guillopé–R. Temam [1]. They imply in particular, as noticed by L. Tartar, that the L^∞-norm of a weak solution is L^1 in time, and this allows us in § 8 to define the *Lagrangian representation* of the flow associated with a weak solution. In § 5 we present the result established in C. Foias–R. Temam [5] concerning the *fractional dimension of the singular set* of a weak solution.

In § 6 we derive, after R. Temam [8], the necessary and sufficient conditions for regularity, at time $t=0$ of the (strong) solutions to the N.S.E. The result relates to the question of *compatibility conditions* between the given initial and boundary values of the problem. This question, which is a well-known one for other initial and boundary value problems including linear ones, apparently has not been solved for the N.S.E.; of course the question of regularity at $t=0$ has nothing to do with the singularities which may develop at positive time.

Finally in § 7 we prove a result of *analyticity* in time of the solutions, following with several simplifications C. Foias–R. Temam [5]. The proof, which could be used for other nonlinear evolution problems, is simple and is closely related to the methods used elsewhere in these notes.

Part II (§§ 9–12) deals with questions related to the behavior for $t\to\infty$ of the solutions of the N.S.E. Section 9 explains the physical meaning of these problems through the example of the Couette–Taylor experiment. Section 10 is devoted to stationary solutions of the N.S.E.; contains a brief proof of existence and uniqueness (for small Reynolds number), and the fact that the solution to the N.S.E. tends, as $t\to\infty$, to the unique stationary solution when the Reynolds number is small. This section contains also a proof of a *finiteness property of the set of stationary solutions* based on topological methods.

The results in §§ 11 and 12 (following C. Foias–R. Temam [5]) are related to the behavior for $t\to\infty$ of the solutions to the N.S.E. at arbitrary (or large) Reynolds numbers. They indicate that *a turbulent flow is somehow structured* and depends (for the cases considered, see below), on a finite number of parameters. These properties include a squeezing property of the trajectories in the function space (§ 11) and the seemingly important fact that, as $t\to\infty$, a solution to the Navier–Stokes equations converges to a functional invariant set (an ω-limit set, or an attractor), which has a *finite Hausdorff dimension* (§ 12).

Intuitively, this means that under these circumstances all but a finite set of modes of the flow are damped.

Both results are proved for a space dimension $n=2$ or 3, without any restriction for $n=2$, but for $n=3$ with the restriction that the solution considered has a bounded H^1-norm *for all time.* It is shown (cf. § 12.3 and C. Foias–R. Temam [13]) that if this boundedness assumption is not satisfied then there exists a weak solution to the N.S.E. which displays singularities. Alternatively, this means that the *relevant results in* §§ 11 *and* 12 (and some of the results in Part III) *fail to be true in dimension* 3 *only if Leray's conjecture on turbulence is verified* (existence of singularities).

Although the problems studied in Part II are totally different from those studied in Part I, the techniques used are essentially the same as in the first sections, and particularly in §§ 2 and 3.

Part III (§§ 13 and 14) presents some results related to the numerical approximation of the Navier–Stokes equations. At moderate Reynolds numbers, the major difficulties for the numerical solution of the equations are the nonlinearity and the constraint $div\, u=0$. In § 13 we present one of the algorithms which have been derived in the past to overcome these difficulties, and which has been recently applied to large scale engineering computations. Section 14 contains some remarks related to the solution of the N.S.E. for large time: it is shown (and this is not totally independent of the result in Part II) that the behavior of the solution for large time depends on a finite number of parameters and an estimate of the number of parameters is given. A more precise estimate of the number of parameters in terms of the Reynolds number and further developments will appear elsewhere (C. Foias–R. Temam [9], C. Foias–O. Manley–R. Temam–Y. Trève [1], C. Foias–R. Temam [11], [12]).

At the level of methods and results, there is again a close relation between Part III and Part I (§§ 3 and 13 in particular), and there is a connection as already mentioned with Part II (§§ 12 and 14).

It is not the purpose of these notes to make an exhaustive presentation of recent results on the Navier–Stokes equations. We have only tried to present some typical results, and the reader is referred to the bibliographical comments in the text and at the end for further developments. In particular we have refrained from developing the stochastic aspect of the Navier–Stokes equations, which would have necessitated the introduction of too many different tools. The interested reader can consult A. Bensoussan–R. Temam [1], C. Foias [1], C. Foias–R. Temam [6] [7], M. Viot [1], M. I. Vishik [1], M. I. Vishik–A. V. Fursikov [1], [2], [3].

Our aim while writing these notes was to try to arrive as rapidly and as simply as possible at some central problems in the Navier–Stokes equations. We hope that they can stimulate some interest in these equations. One can hope also that the demand of new technologies and the accelerated improvement of the opportunities offered by new (existing or projected) computers will help stimulate further interest in these problems in the future.

In conclusion I would like to thank all those who helped in the preparation

of these notes: C. Foias for his collaboration which led to several articles on which these notes are partly based, C. Guillopé, Ch. Gupta, J. C. Saut, D. Serre and the referee for reading the manuscript and for the comments they made. Finally I would like to thank the mathematical secretary at Dekalb University and Mrs. Le Meur at Orsay for kindly typing the manuscript.

PART I

Questions Related to the Existence, Uniqueness and Regularity of Solutions

Orientation. In this first Part, which contains §§ 1 to 8, we present the Navier–Stokes equations of viscous incompressible fluids, and the main boundary value problems which are usually associated with these equations. Then we study the case of the flow in a bounded domain with periodic or zero boundary conditions, and we give in this case the functional setting of the equation, and various results on existence, uniqueness and regularity of time-dependent solutions. We emphasize the case of the flow with space periodic boundary conditions, treating more briefly the zero boundary conditions which are much more often considered in the literature; see for instance O. A. Ladyzhenskaya [1], J. L. Lions [1], and R. Temam [6] to which we will refer more briefly as [RT].

In § 1 we recall the Navier–Stokes equations and the corresponding boundary value problems. In § 2 we present the appropriate functional setting. In § 3 we recall the main existence and uniqueness results (which are essentially classical), with the details of various a priori estimates used frequently in the sequel and we sketch the proof of existence and uniqueness. Section 4 contains some new a priori estimates, used in particular in § 8. Section 5 includes some results on the Hausdorff dimension of the singular set of a weak solution. Section 6 presents the necessary and sufficient conditions of regularity of the solution at time $t = 0$ (the compatibility conditions). Section 7 shows under appropriate assumptions that the solution is analytic in time with values in $D(A)$. Finally, in § 8 we exhibit the Lagrangian representation of the flow associated with a weak solution of the Navier–Stokes equations.

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1 Representation of a Flow. The Navier–Stokes Equations

Let us assume that a fluid fills a region Ω of space. For the Eulerian representation of the flow of this fluid, we consider three functions $\rho = \rho(x, t)$, $p = p(x, t)$, $u = u(x, t)$, $x = (x_1, x_2, x_3) \in \Omega$, $t \in \mathbb{R}$, where $\rho(x, t)$ (or $p(x, t)$) is the density (or the pressure) of the fluid at point x at time t and $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity of the particle of fluid which is at point x at time t. One may also consider Lagrangian representation of the flow, in which case we introduce functions $\tilde{\rho} = \tilde{\rho}(a, t)$, $\tilde{p} = \tilde{p}(a, t)$, $\tilde{u} = \tilde{u}(a, t)$; here $\tilde{u}(a, t)$ is the velocity of the particle of fluid which was at point $a \in \Omega$ at some reference time t_0, and the meanings of $\tilde{\rho}(a, t)$, $\tilde{p}(a, t)$ are similar. The Lagrangian representation of a flow is less often used, but we will make some comments on it in § 8.

If the fluid is Newtonian, then the functions ρ, p, u are governed by the momentum conservation equation (1.1) (Navier–Stokes equation), by the continuity equation (1.2) (mass conservation equation) and by some constitutive law connecting ρ and p:

$$\rho\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i}\right) - \mu \Delta u - (3\lambda + \mu) \operatorname{grad} \operatorname{div} u + \operatorname{grad} p = f, \tag{1.1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho u) = 0, \tag{1.2}$$

where $\mu > 0$ is the kinematic viscosity, λ another physical parameter and $f = f(x, t)$ represents a density of force per unit volume. If the fluid is homogeneous and incompressible, then ρ is a constant independent of x and t, and the equations reduce to

$$\rho\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i}\right) - \mu \Delta u + \operatorname{grad} p = f, \tag{1.3}$$

$$\operatorname{div} u = 0. \tag{1.4}$$

Usually we take $\rho = 1$, set $\nu = \mu$ and, using the differential operator $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ arrive at

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f. \tag{1.5}$$

We can also consider (1.5) as the nondimensional form of the Navier–Stokes equation (1.3), which is obtained as follows. We set $\rho = \rho_* \rho'$, $p = p_* p'$, $u = u_* u'$, $x = L_* x'$, $t = T_* t'$, $f = (p_* U_*/T_*) f'$, where ρ_*, L_*, T_* are respectively a reference density, a reference length and a reference time for the flow, and $U_* = L_*/T_*$, $p_* = U_*^2 \rho_*$. By substitution into (1.3) we get (1.5) for the reduced quantities $u'(x',t')$, $p'(x',t')$, $f'(x',t')$, but in this case the inverse of ν represents the *Reynolds number* of the flow:

$$\mathrm{Re} = \frac{\rho_*}{\mu} L_* U_*. \tag{1.6}$$

The equations (1.4), (1.5) are our basic equations. We note that they make sense mathematically (and in some way physically) if Ω is an open set in $\mathbb{R}^2$, $u=(u_1,u_2)$, $f=(f_1,f_2)$. Since it is useful to consider this situation too, and in order to cover both cases simultaneously, we assume from now on that

(1.7) Ω is an open set of $\mathbb{R}^n$, $n=2$ or 3, with boundary Γ. We assume furthermore that Ω is located locally on one side of Γ and that Γ is Lipschitzian or sometimes of class C^r for a specified r.

One of the first mathematical questions concerning the equations (1.4), (1.5) is the determination of a well-posed boundary value problem associated with these equations. This is still an open problem, but it is believed (and has been proved for $n=2$) that (1.4), (1.5) must be completed by the following initial and boundary conditions (for flow for $t>0$, $x\in\Omega$).

- *Initial condition:*

$$u(x,0)=u_0(x), \qquad x\in\Omega \quad (u_0 \text{ given}). \tag{1.8}$$

- *Boundary condition:*

$$u(x,t)=\phi(x,t), \qquad x\in\Gamma, \quad t>0 \quad (\Omega \text{ bounded}, \phi \text{ given}). \tag{1.9}$$

If Ω is unbounded (and in particular for $\Omega=\mathbb{R}^n$), we add to (1.9) a condition at infinity:

$$u(x,t)\to\psi(x,t) \quad \text{as } |x|\to+\infty, \tag{1.10}$$

ψ given[1].

Instead of (1.9) (and (1.10)) it is interesting to consider another boundary condition which has no physical meaning:

$$u(x+Le_i,t)=u(x,t) \quad \forall x\in\mathbb{R}^n, \quad \forall t>0, \tag{1.11}$$

where $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$, and L is the period in the ith direction; $Q=(]0,L[)^n$ is the cube of the period[2]. The advantage of the boundary condition (1.11) is that it leads to a simpler functional setting, while many of the mathematical difficulties remain unchanged (except of course those related to the boundary layer difficulty, which vanish). In fact, in § 2 we will describe in detail the corresponding functional setting of the problem (i.e. for (1.4), (1.5), (1.8), (1.11)), and we will mention only briefly the case with a

[1] For special unbounded sets Ω further conditions must be added to (1.9), (1.10); cf. J. G. Heywood [1], O. A. Ladyzhenskaya–V. A. Solonnikov [1], [2].

[2] Of course one may consider different periods $L_i,\dots,L_n$ in the different directions, and in this case $Q=\prod_{i=1}^n(]0,L_i[)$.

boundary, which is presented in detail in the references already quoted (Ladyzhenskaya [1], Lions [1], [RT]).

Remark 1.1. In the periodic case (i.e. (1.11)), it is convenient to introduce the average of u on the cube of the period,

$$m_u(t) = \frac{1}{\text{meas } Q} \int_Q u(x, t)\, dx, \tag{1.12}$$

and to set

$$u = m_u + \bar{u}. \tag{1.13}$$

The average m is explicitly determined in terms of the data. By integration of (1.5) on Q, using (1.4), the Stokes formula and the fact that the integrals on the boundary ∂Q of Q vanish because of (1.11), we get

$$\frac{d}{dt} m_u(t) = m_f(t) \qquad \left(= \frac{1}{\text{meas } Q} \int_Q f(x, t)\, dx \right), \tag{1.14}$$

so that

$$m_u(t) = m_{u_0} + \int_0^t m_f(s)\, ds. \tag{1.15}$$

By substitution of (1.13) into (1.5) we find

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} - \nu\, \Delta \bar{u} + (m_u \cdot \nabla)\bar{u} = \bar{f} (= f - m_f). \tag{1.16}$$

The quantity m_u being known, the study of (1.16) is very similar to that of (1.5). Therefore in the periodic case we will assume for simplicity that the average flow vanishes, $m_u = 0$.

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2 Functional Setting of the Equations

In this section we describe the functional setting of the equations, insisting more particularly on the space periodic case (boundary condition (1.11)).

2.1. Function spaces. We denote by $L^2(\Omega)$ the space of real valued functions on Ω which are L^2 for the Lebesgue measure $dx = dx_1 \cdots dx_n$; this space is endowed with the usual scalar product and norm

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \qquad |u| = \{(u, u)\}^{1/2}.$$

We denote by $H^m(\Omega)$ the Sobolev space of functions which are in $L^2(\Omega)$, together with all their derivatives of order $\leqq m$. This is as usual a Hilbert space for the scalar product and the norm

$$(u, v)_m = \sum_{[\alpha]\leqq m} (D^\alpha u, D^\alpha v), \qquad |u|_m = \{(u, u)_m\}^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, $[\alpha] = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}};$$

$H_0^1(\Omega)$ is the Hilbert subspace of $H^1(\Omega)$, made of functions vanishing on Γ. The reader is referred to R. Adams [1] and J. L. Lions–E. Magenes [1] for the theory of Sobolev spaces.

We denote by $H_p^m(Q)$, $m \in \mathbb{N}$, the space of functions which are in $H_{\mathrm{loc}}^m(\mathbb{R}^n)$ (i.e., $u|_{\mathcal{O}} \in H^m(\mathcal{O})$ for every open bounded set $\mathcal{O}$) and which are periodic with period Q:

$$u(x + Le_i) = u(x), \qquad i = 1, \ldots, n. \tag{2.1}$$

For $m = 0$, $H_p^0(Q)$ coincides simply with $L^2(Q)$ (the restrictions of the functions in $H_p^0(Q)$ to Q are the whole space $L^2(Q)$). For an arbitrary $m \in \mathbb{N}$, $H_p^m(Q)$ is a Hilbert space for the scalar product and the norm

$$(u, v)_m = \sum_{[\alpha]\leqq m} \int_Q D^\alpha u(x) D^\alpha v(x)\,dx.$$

The functions in $H_p^m(Q)$ are easily characterized by their Fourier series expansion

$$H_p^m(Q) = \left\{u,\ u = \sum_{k\in\mathbb{Z}^n} c_k e^{2i\pi k\cdot x/L},\ \bar{c}_k = c_{-k},\ \sum_{k\in\mathbb{Z}^n} |k|^{2m}\,|c_k|^2 < \infty\right\}, \tag{2.2}$$

and the norm $|u|_m$ is equivalent to the norm $\{\sum_{k\in\mathbb{Z}^n} (1 + |k|^{2m})\,|c_k|^2\}^{1/2}$. We also set

$$\dot{H}_p^m(Q) = \{u \in H_p^m(Q) \text{ of type (2.2)},\ c_0 = 0\}. \tag{2.3}$$

We observe that the right-hand side of (2.2) makes sense more generally for $m \in \mathbb{R}$, and we actually define $H_p^m(Q)$, $m \in \mathbb{R}$, $m \geqq 0$, by (2.2); it is a Hilbert space for the norm indicated above. For $m \in \mathbb{R}$, we define $\dot{H}_p^m(Q)$ with (2.3); this is a Hilbert space for the norm $\{\sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k|^2\}^{1/2}$, and $\dot{H}_p^m(Q)$ and $\dot{H}_p^{-m}(Q)$ are in duality for all $m \in \mathbb{R}$.

Two spaces frequently used in the theory of Navier–Stokes equations are

$$V = \{u \in \mathbb{H}_p^1(Q), \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\},$$

$$H = \{u \in \mathbb{H}_p^0(Q), \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\},$$

where $\mathbb{H}_p^m(Q) = \{H_p^m(Q)\}^n$; more generally, for any function space X we denote by $\mathbb{X}$ the space X^n endowed with the product structure. We equip V with the scalar product and the Hilbert norm

$$((u, v)) = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right), \qquad \|u\| = \{((u, u))\}^{1/2}.$$

This norm is equivalent to that induced by $(H_p^1(Q))^n$, and V is a Hilbert space for this norm. It is easy to see that the dual V' of V is

$$V' = \{u \in \mathbb{H}_p^{-1}(Q) = (\mathbb{H}_p^1(Q))', \operatorname{div} u = 0 \text{ in } \mathbb{R}^n\};$$

$\|\cdot\|_{V'}$ will denote the dual norm of $\|\cdot\|$ on V'. We have

$$V \subset H \subset V',$$

where the injections are continuous and each space is dense in the following one. One can also show by mollification that the space of smooth functions,

$$\mathcal{V} = V \cap \mathscr{C}^\infty(\mathbb{R}^n)^n, \tag{2.4}$$

is dense in V, H and V'.

Remark 2.1. i) Using the trace theorem, one can show that $u \in V$ if and only if its restriction $u|_Q$ to Q belongs to

$$\{v \in H^1(Q), \operatorname{div} u = 0 \text{ in } Q, v|_{\Gamma_{j+n}} = v|_{\Gamma_j}, j = 1, \ldots, n\} \tag{2.5}$$

where we have numbered the faces $\Gamma_1, \ldots, \Gamma_{2n}$ of Q as follows:

$$\Gamma_j = \partial Q \cap \{x_j = 0\}, \qquad \Gamma_{j+n} = \partial Q \cap \{x_j = L\}, \tag{2.6}$$

and $v|_{\Gamma_j}$ is an improper notation for the trace of v on Γ_j. The characterization of $u|_Q$ for $u \in H$ is more delicate and relies on a trace theorem given in [RT, Chap. I., Thm. 1.2][1]: $u \in H$ if and only if u belongs to

$$\{v \in \mathbb{L}^2(Q), \operatorname{div} v = 0 \text{ in } Q, v \cdot \boldsymbol{\nu}|_{\Gamma_{j+n}} = -v \cdot \boldsymbol{\nu}|_{\Gamma_j}, j = 1, \ldots, n\}. \tag{2.7}$$

[1] Let $\mathcal{O}$ be an open bounded set of $\mathbb{R}^n$ of class $\mathscr{C}^2$. Then if $v \in \mathbb{L}^2(\mathcal{O})$ and $\operatorname{div} v \in L^2(\mathcal{O})$, we can define $\gamma_\nu v \in H^{-1/2}(\partial\mathcal{O})$, which coincides with $v \cdot \boldsymbol{\nu}|_{\partial\mathcal{O}}$ if v is smooth ($\boldsymbol{\nu}$ denotes the unit outward normal on $\partial\mathcal{O}$). Furthermore, for every u in $H^1(\mathcal{O})$, we have the generalized Stokes formula $(v, \nabla u) + (\operatorname{div} v, u) = \langle \gamma_\nu v, \gamma_0 u \rangle$, where $\gamma_0 u = u|_{\partial\mathcal{O}}$ denotes the trace of u on $\partial\mathcal{O}$, and $\langle \cdot, \cdot \rangle$ is the pairing between $H^{1/2}(\partial\mathcal{O}) = \gamma_0 H^1(\mathcal{O})$ and its dual $H^{-1/2}(\partial\mathcal{O})$.

ii) Let G be the orthogonal complement of H in $\dot{\mathbb{H}}_p^0 = (L_p^2(Q)/\mathbb{R})^n$. We have

$$G = \{u \in H_p^0(Q),\ u = \nabla q,\ q \in H_p^1(Q)\}. \tag{2.8}$$

2.2. The Stokes problem and the operator A. The Stokes problem associated with the space periodicity condition (1.11) is the following one.

(2.9) Given $f \in \dot{\mathbb{H}}_p^0(Q)$ or $\dot{\mathbb{H}}_p^{-1}(Q)$, find $u \in \dot{\mathbb{H}}_p^1(Q)$ and $p \in L^2(Q)$ such that

$$-\Delta u + \nabla p = f \quad \text{in } Q, \qquad \nabla \cdot u = 0 \quad \text{in } Q.$$

It is easy to solve this problem explicitly using Fourier series. Let us introduce the Fourier expansions of u, p and f;

$$u = \sum_{k \in \mathbb{Z}^n} u_k e^{2i\pi k \cdot x/L}, \quad p = \sum_{k \in \mathbb{Z}^n} p_k e^{2i\pi k \cdot x/L}, \quad f = \sum_{k \in \mathbb{Z}^n} f_k e^{2i\pi k \cdot x/L}.$$

Equations (2.9) reduce for every $k \neq 0$ to

$$-\frac{4\pi^2 |k|^2}{L^2} u_k + \frac{2i\pi k}{L} p_k = f_k, \tag{2.10}$$

$$k \cdot u_k = 0. \tag{2.11}$$

Taking the scalar product of (2.10) with k and using (2.11) we find the p_k's:

$$p_k = \frac{Lk \cdot f_k}{2i\pi |k|^2}, \qquad k \in \mathbb{Z}^n, \quad k \neq 0; \tag{2.12}$$

then (2.10) provides the u_k's:

$$u = -\frac{L^2}{4\pi^2 |k|^2}\left(f_k - \frac{(k \cdot f_k)k}{|k|^2}\right), \qquad k \in \mathbb{Z}^n, \quad k \neq 0. \tag{2.13}$$

By the definition (2.2) of $H_p^m(Q)$, if $f \in \dot{\mathbb{H}}_p^0(Q)$ then $u \in \dot{\mathbb{H}}_p^2(Q)$ and $p \in \dot{H}_p^1(Q)$; if $f \in \dot{\mathbb{H}}_p^{-1}(Q)$ then $u \in \dot{\mathbb{H}}_p^1(Q)$ and $p \in \dot{H}_p^0(Q)$. Now if f belongs to H, then $k \cdot f_k = 0$ for every k so that $p = 0$ and

$$u_k = -\frac{f_k L^2}{4\pi^2 |k|^2}.$$

We define in this way a one-to-one mapping $f \to u$ from H onto

$$D(A) = \{u \in H, \Delta u \in H\} = \dot{\mathbb{H}}_p^2(Q) \cap H.$$

Its inverse from $D(A)$ onto H is denoted by A, and in fact

$$Au = -\Delta u \quad \forall\, u \in D(A). \tag{2.14}$$

If $D(A)$ is endowed with the norm induced by $\dot{\mathbb{H}}_p^0(Q)$, then A becomes an isomorphism from $D(A)$ onto H. If follows that the norm $|Au|$ on $D(A)$ is equivalent to the norm induced by $\dot{\mathbb{H}}_p^2(Q)$.

The operator A can be seen as an unbounded positive linear selfadjoint operator on H, and we can define the powers A^α, $\alpha \in \mathbb{R}$, with domain $D(A^\alpha)$ in

H. We set

$$V_\alpha = D(A^{\alpha/2});$$

V_α is a closed subspace of $\dot{\mathbb{H}}^\alpha_p(Q)$, and in fact

$$V_\alpha = \{v \in \dot{\mathbb{H}}^\alpha_p(Q), \operatorname{div} v = 0\}. \tag{2.15}$$

In particular, $V_2 = D(A)$, $V_1 = V$, $V_0 = H$, $V_{-1} = V'$; A is an isomorphism from $V_{\alpha+2}$ onto V_α, $D(A)$ onto H, V onto V', and so forth. The norm $|A^{\alpha/2}u|$ on V_α is equivalent to the norm induced by $\dot{\mathbb{H}}^\alpha_p(Q)$,

$$c\,|u|_{2\alpha} \leqq |A^\alpha u| \leqq c'\,|u|_{2\alpha} \quad \forall\, u \in D(A^\alpha), \tag{2.16}$$

with[2] c, c' depending on L and α.

The operator A is an isomorphism from $V_{\alpha+2}$ onto V_α for all $\alpha \in \mathbb{R}$. We recall also that the injection of V_α into $V_{\alpha-\varepsilon}$ is compact for every $\alpha \in \mathbb{R}$, $\varepsilon > 0$. Indeed if u_m is a sequence converging weakly to 0 in V_α, then

$$\sum_{k \in \mathbb{Z}^n} |u_{mk}|^2\,|k|^{2\alpha} \leqq c, \quad \text{a constant independent of } m,$$

$$u_{mk} \to 0 \quad \text{as } m \to \infty \quad \forall\, k \in \mathbb{Z}^n.$$

For every $K \in \mathbb{R}$, $K > 0$,

$$|u_m|^2_{\alpha-\varepsilon} = \sum_{k \in \mathbb{Z}^n} |u_{mk}|^2\,|k|^{2\alpha'} \leqq \sum_{|k| \leqq K} |u_{mk}|^2\,|k|^{2\alpha'} + \frac{1}{K^{2\varepsilon}} \sum_{|k| > K} |u_{mk}|^2\,|k|^{2\alpha},$$

$$\limsup_{m\to\infty} |u_m|^2_{\alpha-\varepsilon} \leqq \frac{c}{K^{2\varepsilon}},$$

and since this lim sup is arbitrarily small, $u_m \to 0$ in $V_{\alpha-\varepsilon}$.

Eigenfunctions of A. The operator A^{-1} is linear continuous from H into $D(A)$, and since the injection of $D(A)$ in H is compact, A^{-1} can be considered as a compact operator in H. As an operator in H it is also selfadjoint. Hence it possesses a sequence of eigenfunctions w_j, $j \in \mathbb{N}$, which form an orthonormal basis of H,

$$\begin{aligned} &Aw_j = \lambda_j w_j, \qquad w_j \in D(A), \\ &0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \cdots, \qquad \lambda_j \to \infty \quad \text{for } j \to \infty. \end{aligned} \tag{2.17}$$

In fact the sequence of w_j's and λ_j's is the sequence of functions $w_{k,\alpha}$ and numbers $\lambda_{k,\alpha}$,

$$w_{k,\alpha} = \left(e_\alpha - \frac{k_\alpha k}{|k|^2}\right) e^{2i\pi k\cdot x/L}, \qquad \lambda_{k,\alpha} = 4\pi^2\,|k|^2 L^2,$$

---

[2] The letters c, c', c_i, c_i' indicate various positive constants. The letters c_i represent well-defined constants while the constants represented by the letters c, c', c_i' may be different in different places in the text.

where $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$, $k\neq 0$, $\alpha=1,\dots,n$, and $e_1,\dots,e_n$ represents the canonical basis of $\mathbb{R}^n$.

2.3. Sobolev inequalities. The trilinear form *b*. We recall some Sobolev inequalities and some basic properties of Sobolev spaces.

If $\mathcal{O}$ is an open set of $\mathbb{R}^n$ and its boundary $\partial\mathcal{O}$ is sufficiently regular (say Lipschitzian), and if $\frac{1}{2}-m/n>0$, then

$$H^m(\mathcal{O})\subset L^q(\mathcal{O}),\qquad \frac{1}{q}=\frac{1}{2}-\frac{m}{n},\tag{2.18}$$

the injection being continuous. In particular, there exists a positive constant c depending on m, n and L such that

$$|u|_{L^q(Q)}\leqq c(m,n,L)\,|u|_m\quad \forall\, u\in H^m_p(Q),\quad m<\frac{n}{2}.\tag{2.19}$$

For $m>n/2$, $H^m_p(Q)\subset\mathscr{C}_p(Q)$ (the space of real continuous functions with period Q) with a continuous injection.

If $m_1, m_2\in\mathbb{R}$, $m_1\leqq m_2$ and $\theta\in\,]0,1[$, the discrete Hölder inequality[3] gives

$$\sum_{k\in\mathbb{Z}^n}|k|^{2((1-\theta)m_1+\theta m_2)}\,|u_k|^2\leqq\Big(\sum_{k\in\mathbb{Z}^n}|k|^{2m_1}\,|u_k|^2\Big)^{(1-\theta)}\cdot\Big(\sum_{k\in\mathbb{Z}^n}|k|^{2m_2}\,|u_k|^2\Big)^{\theta},$$

so that

$$|u|_{(1-\theta)m_1+\theta m_2}\leqq|u|_{m_1}^{(1-\theta)}\,|u|_{m_2}^{\theta}\quad\forall\, u\in H^{m_2}_p(Q),\quad m_1\leqq m_2.\tag{2.20}$$

If $(1-\theta)m_1+m_2>n/2$, the continuous imbedding of $H^m_p(Q)$ into $\mathscr{C}_p(Q)$ shows that there exists a constant c depending only on θ, m_1, m_2, n, L such that

$$\begin{aligned}&|u|_{L^\infty(Q)}\leqq c(\theta,m_1,m_2,n,L)(|u|_{m_1}^{(1-\theta)}\,|u|_{m_2}^{\theta})\\ &\forall\, u\in H^{m_2}_p(Q),\quad m_1\leqq m_2,\quad 0<\theta<1,\quad (1-\theta)m_1+\theta m_2>\frac{n}{2}.\end{aligned}\tag{2.21}$$

Actually (cf. S. Agmon [1]), the inequality (2.21) is also valid if

$$0\leqq m_1<\frac{n}{2}<m_2,\qquad (1-\theta)m_1+\theta m_2=\frac{n}{2},\quad \text{i.e., }\theta=\frac{n/2-m_1}{m_2-m_1}.\tag{2.22}$$

In particular, if $n=2$,

$$|u|_{L^\infty(Q)}\leqq\begin{cases}c\;|u|^{1/2}\,|u|_2^{1/2}\\ c\;|u|_1^{3/4}\,|u|_2^{1/4}\end{cases}\quad\forall\, u\in\mathbb{H}^2_p(Q),\tag{2.23}$$

and because of (2.16)

$$|u|_{L^\infty(Q)}\leqq\begin{cases}c|u|^{1/2}\,|Au|^{1/2}\\ c\|u\|^{3/4}\,|Au|^{1/4}\end{cases}\quad\forall\, u\in D(A);\tag{2.24}$$

[3] $|\sum_i a_ib_i|\leqq(\sum_i|a_i|^p)^{1/p}(\sum_i|b_i|^{p'})^{1/p'}$, $p=1/\theta$, $p'=1/(1-\theta)$.

if $n=3$

$$(2.25)\qquad |u|_{L^\infty(Q)} \leqq \begin{cases} c\,|u|^{1/4}\,|u|_2^{3/4} \\ c\,|u|_1^{1/2}\,|u|_2^{1/2} \end{cases} \quad \forall\, u \in \mathbb{H}^2_{\mathrm{p}}(Q),$$

$$(2.26)\qquad |u|_{L^\infty(Q)} \leqq \begin{cases} c\,|u|^{1/4}\,|Au|^{3/4} \\ c\,\|u\|^{1/2}\,|Au|^{1/2} \end{cases} \quad \forall\, u \in D(A).$$

The form b. We now show how to apply these properties of Sobolev spaces to the study of the form b.

Let $\mathcal{O}$ be an open bounded set of $\mathbb{R}^n$ which will be either Ω or Q. For $u, v, w \in \mathbb{L}^1(\mathcal{O})$, we set

$$(2.27)\qquad b(u, v, w) = \sum_{i,j=1}^{n} \int_{0} u_i D_i v_j w_j\, dx,$$

whenever the integrals in (2.22) make sense. In particular, we have:

LEMMA 2.1. *Let $\mathcal{O}=\Omega$ or Q. The form b is defined and is trilinear continuous on $\mathbb{H}^{m_1}(\mathcal{O}) \times \mathbb{H}^{m_2+1}(\mathcal{O}) \times \mathbb{H}^{m_3}(\mathcal{O})$ where $m_i \geqq 0$, and*

$$(2.28)\qquad \begin{aligned} &m_1+m_2+m_3 \geqq \frac{n}{2} \quad \text{if } m_i \neq \frac{n}{2}, \quad i=1,2,3,\\ &m_1+m_2+m_3 > \frac{n}{2} \quad \text{if } m_i = \frac{n}{2} \quad \text{for some } i. \end{aligned}$$

Proof. If $m_i < n/2$ for $i=1,2,3$, then by (2.18) $H^{m_i}(\mathcal{O}) \subset L^{q_i}(\mathcal{O})$ where $1/q_i = \frac{1}{2} - m_i/n$. Due to (2.28), $(1/q_1 + 1/q_2 + 1/q_3) \leqq 1$, the product $u_i(D_i v_j) w_j$ is integrable and $b(u, v, w)$ makes sense. By application of Hölder's inequality we get

$$(2.29)\qquad \begin{aligned} &|b(u, v, w)| \leqq \sum_{i,j=1}^{n} |u_i|_{L^{q_1}(\mathcal{O})}\, |D_i v_j|_{L^{q_2}(\mathcal{O})}\, |w_j|_{L^{q_3}(\mathcal{O})},\\ &|b(u, v, w)| \leqq c_1\, |u|_{m_1}\, |v|_{m_2+1}\, |w|_{m_3}. \end{aligned}$$

If one or more of the m_i's is larger than $n/2$ we proceed as before, with the corresponding q_i replaced by $+\infty$ and the other q_j's equal to 2. If some of the m_i's are equal to $n/2$, we replace them by $m_i' < m_i$, $m_i - m_i'$ sufficiently small so that the corresponding inequality (2.29) still holds. □

Remark 2.2. i) As a particular case of Lemma 2.1 and (2.29), b is a trilinear continuous form on $V_{m_1} \times V_{m_2+1} \times V_{m_3}$, m_i as in (2.28) and

$$(2.30)\qquad |b(u, v, w)| \leqq c_1\, |u|_{m_1}\, |v|_{m_2+1}\, |v|_{m_3} \quad \forall\, u \in V_{m_1},\ v \in V_{m_2+1},\ w \in V_{m_3}.$$

In particular, b is a trilinear continuous form on $V \times V \times V$ and even on $V \times V \times V_{1/2}$.

ii) We can supplement (2.29)–(2.30) by other inequalities which follow from (2.29)–(2.30) and (2.20)–(2.26). For instance the following inequalities combining (2.30) and (2.20) will be useful.

$$(2.31)\qquad \begin{aligned} &|b(u, v, w)| \leqq c_2\, |u|^{1/2}\, \|u\|^{1/2}\, \|v\|^{1/2}\, |Av|^{1/2}\, |w| \\ &\qquad\qquad \forall\, u \in V,\ v \in D(A),\ w \in H, \quad \text{if } n=2, \end{aligned}$$

$$|b(u, v, w)| \leqq c_3 \|u\| \|v\|^{1/2} |Av|^{1/2} |w| \qquad \forall\, u \in V,\ v \in D(A),\ w \in H, \quad \text{if } n = 3. \tag{2.32}$$

iii) Less frequently, we will use the following inequalities. We observe that $u_i(D_i v_j) w_j$ is summable if (for instance) $u_i \in L^\infty(\mathcal{O})$, $D_i v_j$, $w_j \in L^2(\mathcal{O})$, and

$$\left| \int_{\mathcal{O}} u_i D_i v_j w_j \, dx \right| \leqq |u_i|_{L^\infty(\mathcal{O})} |D_i v_j| \, |w_j|.$$

In this manner, and using also (2.24), we get in the case $n = 2$

$$|b(u, v, w)| \leqq c \times \begin{cases} |u|^{1/2} |Au|^{1/2} \|v\| \, |w| & \forall\, u \in D(A),\ v \in V,\ w \in H, \\ |u| \, \|v\| \, |w|^{1/2} |Aw|^{1/2} & \forall\, u \in H,\ v \in V,\ w \in D(A), \end{cases}$$

and in the case $n = 3$, using (2.26)

$$|b(u, v, w)| \leqq c \times \begin{cases} |u|^{1/4} |Au|^{3/4} \|v\| \, |w| & \forall\, u \in D(A),\ v \in V,\ w \in H, \\ |u| \, \|v\| \, \|w\|^{1/2} |Aw|^{1/2} & \forall\, u \in H,\ v \in V,\ w \in D(A), \end{cases}$$

Finally we recall a fundamental property of the form b:

$$b(u, v, w) = -b(u, w, v) \quad \forall\, u, v, w \in V. \tag{2.33}$$

This property is easily established for $u, v, w \in \mathcal{V}$ (cf. [RT, p. 163]) and follows by continuity for $u, v, w \in V$. With $v = w$, (2.33) implies

$$b(u, v, v) = 0 \quad \forall\, u, v \in V. \tag{2.34}$$

The operator B. For $u, v, w \in V$ we define $B(u, v) \in V'$ and $Bu \in V'$ by setting

$$\langle B(u, v), w \rangle = b(u, v, w), \qquad Bu = B(u, u). \tag{2.35}$$

Since b is a trilinear continuous form on V, B is a bilinear continuous operator from $V \times V$ into V'. More generally, by application of Lemma 2.1 we see that

(2.36) B is a bilinear continuous operator from $V_{m_1} \times V_{m_2+1}$ (or from $\mathbb{H}^{m_1}(\mathcal{O}) \times \mathbb{H}^{m_2+1}(\mathcal{O})$) into V_{-m_3}, where m_1, m_2, m_3 satisfy the assumptions in Lemma 2.1.

It is clear that various estimates for the norm of the bilinear operator $B(\cdot, \cdot)$ can be derived from the above estimates for b.

2.4. Variational formulation of the equations. As indicated before, we are interested in the boundary value problem (1.4), (1.5), (1.8), (1.11), when the average of u on Q is 0; u_0 and f are given, and we are looking for u and p.

Let $T > 0$ be given, and let us assume that u and p are sufficiently smooth, say $u \in \mathscr{C}^2(\mathbb{R}^n \times [0, T])$, $p \in \mathscr{C}^1(\mathbb{R}^n \times [0, T])$, and are classical solutions of this problem. Let $u(t)$ and $p(t)$ be respectively the functions $\{x \in \mathbb{R}^n \mapsto u(x, t)\}$, $\{x \in \mathbb{R}^n \mapsto p(x, t)\}$. Obviously $u \in L^2(0, T; V)$, and if v is an element of $\mathcal{V}$ then by multiplying (1.5) by v, integrating over Q and using (1.4), (1.11) and the

Stokes formula we find (cf. [RT, Chap. III] for the details) that

$$\frac{d}{dt}(u, v)+\nu((u, v))+b(u, u, v)=(f, v). \tag{2.37}$$

By continuity, (2.37) holds also for each $v \in V$.

This suggests the following weak formulation of the problem, due to J. Leray [1], [2], [3].

Problem 2.1 (weak solutions). *For f and u_0 given,*

$$f \in L^2(0, T; V'), \tag{2.38}$$

$$u_0 \in H, \tag{2.39}$$

find u satisfying

$$u \in L^2(0, T; V) \tag{2.40}$$

and

$$\frac{d}{dt}(u, v)+\nu((u, v))+b(u, u, v)=\langle f, v\rangle \quad \forall\, v \in V, \tag{2.41}$$

$$u(0)=u_0. \tag{2.42}$$

If u merely belongs to $L^2(0, T; V)$, the condition (2.42) need not make sense. But if u belongs to $L^2(0, T; V)$ and satisfies (2.41), then (cf. below) u is almost everywhere on $[0, T]$ equal to a continuous function, so that (2.42) is meaningful.

We can write (2.41) as a differential equation in V' by using the operators A and B. We recall that A is an isomorphism from V onto V' and B is a bilinear continuous operator from $V \times V$ into V'. If $u \in L^2(0, T; V)$, the function $Bu: \{t \mapsto Bu(t)\}$ belongs (at least) to $L^1(0, T; V')$. Consequently (2.41) is equivalent to

$$\frac{d}{dt}\langle u, v\rangle=\langle f-\nu Au-Bu, v\rangle,$$

and since $f-\nu Au-Bu \in L^1(0, T; V')$, $u'=du/dt$ belongs to $L^1(0, T; V')$, and

$$\frac{du}{dt}+\nu Au+Bu=f \text{ in } V'. \tag{2.43}$$

Furthermore, (cf. [RT, Chap. III, § 1]) u is almost everywhere equal to a continuous function from $[0, T]$ into V', and (2.42) makes sense.

We refer to [RT] (and the sequel) for more details and in particular for the relation of Problem 2.1 to the initial problem (1.4), (1.5), (1.8), (1.11). One can show that if u is a solution of Problem 2.1, then there exists p such that (1.4), (1.5), (1.8), (1.11) are satisfied in a weak sense (cf. [RT, p. 307]).

Of course a weak solution of the Navier–Stokes equations (Problem 2.1) may or may not possess further regularity properties. For convenience we will introduce a class of more regular solutions which we call strong solutions.

Problem 2.2 (strong solutions). *For f and u_0 given,*

$$f \in L^2(0, T; H), \tag{2.44}$$

$$u_0 \in V, \tag{2.45}$$

find u satisfying

$$u \in L^2(0, T; D(A)) \cap L^\infty(0, T; V) \tag{2.46}$$

and (2.41)–(2.43).

Further regularity properties of strong solutions are investigated in § 4 (for the space periodic case) and § 6 (for the bounded case). Let us observe here that, by application of (2.32) and (2.41)–(2.42),

$$|B(\psi, \psi)| \leqq c_3 \|\psi\|^{3/2} |A\psi|^{1/2} \quad \forall \psi \in D(A). \tag{2.47}$$

Hence if u is a strong solution, then for almost every t,

$$|Bu(t)| \leqq c_3 \|u(t)\|^{3/2} |Au(t)|^{1/2}, \tag{2.48}$$

and the function $Bu:\{t \mapsto Bu(t)\}$ belongs to $L^4(0, T; H)$. Since f and Au belong to $L^2(0, T; H)$, $u' = f - \nu Au - Bu$ belongs to this space too, and

$$u' \in L^2(0, T; H). \tag{2.49}$$

The two conditions $u \in L^2(0, T; D(A))$, $u' \in L^2(0, T; H)$ imply by interpolation (cf. J. L. Lions–E. Magenes [1] or [RT, Chap. III § 1.4), that u is almost everywhere equal to a continuous function from $[0, T]$ into V:

$$u \in \mathscr{C}([0, T]; V). \tag{2.50}$$

2.5. Flow in a bounded domain. Although up to now we have concentrated on the space periodic case, all of what follows applies as well to the case of flow in a bounded domain with a fixed boundary (1.4), (1.5), (1.8), (1.9) with $\phi = 0$, Ω bounded, provided we properly define the different spaces and operators.

In this case (cf. [RT],

$$\begin{aligned}
&\mathcal{V} = \{v \in \mathscr{C}_c^\infty(\Omega)^n, \text{ div } v = 0\},\\
&V = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(\Omega)\\
&\quad = \{v \in \mathbb{H}_0^1(\Omega), \text{ div } v = 0\},\\
&H = \text{the closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\Omega)\\
&\quad = \{v \in \mathbb{L}^2(\Omega), \text{ div } v = 0, \ \gamma_\nu v = v \cdot \nu|_\Gamma = 0\},\\
&H^\perp(\text{in } \mathbb{L}^2(\Omega)) = G = \{v, v = \nabla p, p \in H^1(\Omega)\},\\
&D(A) = V \cap \mathbb{H}^2(\Omega) = \{v \in \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega), \text{ div } v = 0\},\\
&Au = -P\,\Delta u \quad \forall u \in D(A),
\end{aligned}$$

where P is the orthogonal projector in $\mathbb{L}^2(\Omega)$ onto H.

Most of the abstract results in § 2.2 remain valid in this case. We cannot use

Fourier series any more, so that neither can $A^{-1}f$ be explicitly written, nor can the eigenfunctions of A be calculated. Still, A is an isomorphism from V onto V' and from $D(A)$ onto H, but this last result is a nontrivial one relying on the theory of regularity of solutions of elliptic systems (cf. S. Agmon–A. Douglis–L. Nirenberg [1]), L. Cattabriga [1], V. A. Solonnikov [1], I. I. Vorovitch–V. I. Yodovich [1]). The spaces $V_\alpha = D(A^{\alpha/2})$, $\alpha > 0$, (which will not be used too often) are still closed subspaces of $\mathbb{H}^\alpha(\Omega)$, but their characterization is more involved than (2.15) and contains boundary conditions on Γ. All the inequalities in § 2.3 apply to the bounded case, replacing just $H_p^m(Q)$ by $H^m(\Omega)$ (assuming that (1.7) is satisfied with an appropriate r). The proof of (2.20)[4], which was elementary, relies in the bounded case on the theory of interpolation, as well as on the definition of $H^m(\Omega)$, $m \in \mathbb{R} \backslash \mathbb{N}$ (cf. J. L. Lions–E. Magenes [1]). Finally, with these definitions of $A, V, H, \dots$, § 2.4 applies to the bounded case without any modification.

[4] In the bounded case, (2.20) becomes

$$|u|_{(1-\theta)m_1+\theta m_2} \leqq c\, |u|_{m_1}^{(1-\theta)} |u|_{m_2}^{\theta} \quad \forall\, u \in H^{m_2}(\Omega).$$

3 Existence and Uniqueness Theorems (Mostly Classical Results)

In this section we derive basic a priori estimates for the solutions of Navier–Stokes equations, and we recall the classical existence or uniqueness theorems of weak or strong solutions. The only recent result is the theorem of generic solvability of Navier–Stokes equations, given in § 3.4 and due to A. V. Fursikov [1].

3.1. A priori estimates. We assume that u is a *sufficiently regular solution* of Problems 2.1–2.2, and we establish a priori estimates on u, i.e., majorizations of some norms of u in terms of the data $u_0, f, \ldots$.

i) By (2.41), for every $t \in (0, T)$ and $v \in V$,

$$(u'(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = \langle f(t), v \rangle. \tag{3.1}$$

Replacing v by $u(t)$ we get, with (2.34),

$$(u'(t), u(t)) + \nu \|u(t)\|^2 = \langle f(t), u(t) \rangle.$$

Hence

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu \|u(t)\|^2 = \langle f(t), u(t) \rangle \leqq \|f(t)\|_{V'} \|u(t)\| \tag{3.2}$$

$$\leqq \frac{\nu}{2}\|u(t)\|^2 + \frac{1}{2\nu}\|f(t)\|_{V'}^2,$$

$$\frac{d}{dt}|u(t)|^2 + \nu \|u(t)\|^2 \leqq \frac{1}{\nu}\|f(t)\|_{V'}^2. \tag{3.3}$$

By integration in t from 0 to T, we obtain, after dropping unnecessary terms,

$$\int_0^T \|u(t)\|^2 \, dt \leqq K_1, \tag{3.4}$$

$$K_1 = K_1(u_0, f, \nu, T) = \frac{1}{\nu}\left(|u_0|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2 \, dt\right). \tag{3.5}$$

Then by integration in t of (3.3) from 0 to $s, 0 < s < T$, we obtain

$$|u(s)|^2 \leqq |u_0|^2 + \frac{1}{\nu}\int_0^s \|f(t)\|_{V'}^2 \, dt,$$

$$\sup_{s \in [0,T]} |u(s)|^2 \leqq K_2, \tag{3.6}$$

$$K_2 = K_2(u_0, f, \nu, T) = \nu K_1. \tag{3.7}$$

ii) Assuming again that u is smooth, in (3.1) we replace v by $Au(t)$:

$$(u'(t), Au(t)) + \nu((u(t), Au(t))) + b(u(t), u(t), Au(t)) = (f(t), Au(t)).$$

Since

$$\langle A\phi, \psi\rangle = ((\phi, \psi)) \quad \forall\, \phi, \psi \in V, \tag{3.8}$$

this relation can be written

$$\begin{aligned}(3.9)\quad &\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\,|Au(t)|^2 + b(u(t), u(t), Au(t))\\ &\qquad = (f(t), Au(t)) \leqq \frac{\nu}{4}|Au(t)|^2 + \frac{1}{\nu}|f(t)|^2.\end{aligned}$$

Now the computations are different, depending on the dimension.

iii) *Dimension* $n = 2$. We use the relation (2.31); (3.9) implies

$$\frac{d}{dt}\|u(t)\|^2 + \frac{3}{2}\nu\,|Au(t)|^2 \leqq \frac{2}{\nu}|f(t)|^2 + 2c_2\,|u(t)|^{1/2}\,|Au(t)|^{3/2}\,\|u(t)\|.$$

The right-hand side can be majorized by

$$\frac{2}{\nu}|f(t)|^2 + \frac{\nu}{2}|Au(t)|^2 + c_1'\,|u(t)|^2\,\|u(t)\|^4,$$

using Young's inequality in the form

$$ab \leqq \varepsilon a^p + c_\varepsilon b^{p'}, \qquad 1 < p < \infty, \quad \varepsilon > 0, \quad p' = \frac{p}{p-1}, \quad c_\varepsilon = \frac{(p-1)}{p^{p'}\varepsilon^{1/p-1}}, \tag{3.10}$$

with $p = \frac{4}{3}$ and $\varepsilon = \nu/2$. We obtain

$$\frac{d}{dt}\|u(t)\|^2 + \nu\,|Au(t)|^2 \leqq \frac{2}{\nu}|f(t)|^2 + c_1'\,|u(t)|^2\,\|u(t)\|^4. \tag{3.11}$$

Momentarily dropping the term $\nu\,|Au(t)|^2$, we have a differential inequality,

$$\begin{aligned}(3.12)\quad &y' \leqq a + \theta y,\\ &y(t) = \|u(t)\|^2, \quad a(t) = \frac{2}{\nu}|f(t)|^2, \quad \theta(t) = c_1'\,|u(t)|^2\,\|u(t)\|^2,\end{aligned}$$

from which we obtain by the technique of Gronwall's lemma:

$$\frac{d}{dt}\left(y(t)\exp\left(-\int_0^t \theta(\tau)\,d\tau\right)\right) \leqq a(t)\exp\left(-\int_0^t \theta(\tau)\,d\tau\right),$$

$$y(t) \leqq y(0)\exp\left(\int_0^t \theta(\tau)\,d\tau\right) + \int_0^t a(s)\exp\left(\int_s^t \theta(\tau)\,d\tau\right)ds,$$

or

$$\begin{aligned}(3.13)\quad &\|u(t)\|^2 \leqq \|u_0\|^2 \exp\left(\int_0^t c_1'\,|u(\tau)|^2\,\|u(\tau)\|^2\,d\tau\right)\\ &\qquad + \frac{2}{\nu}\int_0^t |f(s)|^2 \exp\left(\int_s^t c_1'\,|u(\tau)|^2\,\|u(\tau)\|^2\,d\tau\right)ds.\end{aligned}$$

With (3.4)–(3.7),

$$\sup_{t\in[0,T]} \|u(t)\|^2 \leqq K_3, \tag{3.14}$$

(3.15)[1]
$$K_3 = K_3(u_0, f, \nu, L) = \left(\|u_0\|^2 + \frac{2}{\nu}\int_0^T |f(s)|^2\, ds\right) \exp\,(c_1' K_1 K_2).$$

We come back to (3.11), which we integrate from 0 to T:

$$\nu\int_0^T |Au(t)|^2\, dt \leqq \|u_0\|^2 + \frac{2}{\nu}\int_0^T |f(t)|^2\, dt + c_1' \sup_{t\in[0,T]} |u(t)|^2\, \|u(t)\|^4,$$

$$\int_0^T |Au(t)|^2\, dt \leqq K_4, \tag{3.16}$$

$$K_4 = K_4(u_0, f, \nu, L) = \frac{1}{\nu}\left(\|u_0\|^2 + \frac{2}{\nu}\int_0^T |f(t)|^2\, dt + c_1' K_2 K_3^2\right). \tag{3.17}$$

iv) Before treating the case $n=3$, let us mention an improvement of the preceding estimates in the periodic case, for $n=2$. This improvement, *which does not extend to the case of the flow in a bounded domain or if $n=3$*, is based on:

LEMMA 3.1. *In the periodic case and if $n=2$,*

$$b(\phi, \phi, A\phi) = 0 \quad \forall\, \phi \in D(A) = \mathbb{H}_p^2(Q) \cap V. \tag{3.18}$$

Proof. In the periodic case $A\phi = -\Delta\phi$, and

$$\begin{aligned} b(\phi, \phi, A\phi) &= -\sum_{i,j=1}^{2} \int_Q \phi_i (D_i\phi_j)\, \Delta\phi_j\, dx \\ &= \sum_{i,j,k=1}^{2} \int_Q \phi_i (D_i\phi_j) D_k^2 \phi_j\, dx, \end{aligned}$$

and, by integration by parts, using the Stokes formula,

$$b(\phi, \phi, A\phi) = \sum_{i,j,k} \int_Q \phi_i D_{ik}\phi_j D_k\phi_j\, dx + \sum_{i,j,k} \int_Q D_k\phi_i D_i\phi_j D_k\phi_j\, dx. \tag{3.19}$$

Now both integrals vanish, the first one because

$$\begin{aligned} \sum_i \int_Q \phi_i D_{ik}\phi_j D_k\phi_j\, dx &= \sum_i \int_Q \phi_i D_i \frac{(D_k\phi_j)^2}{2}\, dx \\ &= -\int_Q \operatorname{div} \phi \frac{(D_k\phi_j)^2}{2}\, dx, \end{aligned}$$

and the second one because the sum $\sum_{i,j,k=1}^2 D_k\phi_i D_i\phi_j D_k\phi_j$ vanishes identically (straightforward calculation). □

[1] c_2 (and therefore c_1') depends on the domain $\mathcal{O}$, i.e., on L if $\mathcal{O} = Q$.

This lemma allows us to transform (3.9) into the simpler *energy equality*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+\nu\,|Au(t)|^2=(f(t),Au(t)), \tag{3.20}$$

from which we derive, as for (3.4)–(3.7),

$$\frac{d}{dt}\|u(t)\|^2+\nu\,|Au(t)|^2\leqq\frac{1}{\nu}|f(t)|^2, \tag{3.21}$$

$$\sup_{t\in[0,T]}\|u(t)\|^2\leqq K_3'=\|u_0\|^2+\frac{1}{\nu}\int_0^T|f(t)|^2\,dt, \tag{3.22}$$

$$\int_0^T|Au(t)|^2\,dt\leqq K_4'=\frac{1}{\nu}K_3'. \tag{3.23}$$

v) *Dimension* $n=3$. In the case $n=3$, we derive results which are similar to that in part iii) (but weaker).

After (3.9) we use (2.32) instead of (2.31), and we obtain

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2+\frac{3}{2}\nu\,|Au(t)|^2&\leqq\frac{2}{\nu}|f(t)|^2+2c_3\|u(t)\|^{3/2}|Au(t)|^{3/2}\\&\leqq\frac{2}{\nu}|f(t)|^2+\frac{\nu}{2}|Au(t)|^2+c_3'\|u(t)\|^6\end{aligned}$$

(by Young's inequality),

$$\frac{d}{dt}\|u(t)\|^2+\nu\,|Au(t)|^2\leqq\frac{2}{\nu}|f(t)|^2+c_3'\|u(t)\|^6. \tag{3.24}$$

But

$$\|v\|\leqq\frac{1}{\sqrt{\lambda_1}}|Av|\quad\forall\,v\in D(A), \tag{3.25}$$

and

$$\frac{d}{dt}\|u(t)\|^2+\nu\lambda_1\|u(t)\|^2\leqq\frac{2}{\nu}|f(t)|^2+c_3'\|u(t)\|^6. \tag{3.26}$$

This is similar to (3.11), but instead of (3.12), the comparison differential inequality is now

$$\begin{aligned}&y'\leqq c_4'y^3,\\&y(t)=1+\|u(t)\|^2,\qquad c_4'=\max\left(c_3',\frac{2}{\nu}\sup_{t\in[0,T]}|f(t)|^2\right).\end{aligned} \tag{3.27}$$

We conclude that

$$y(t)\leqq\frac{y(0)}{\sqrt{1-2y(0)^2c_4't}}$$

as long as $t < 1/(2y(0)^2 c_4')$, and thus

$$1+\|u(t)\|^2 \leqq 2(1+\|u_0\|^2)$$

for

$$0 \leqq t \leqq \frac{3}{8c_4'} \frac{1}{(1+\|u_0\|^2)^2}.$$

LEMMA 3.2. *There exists a constant K_6 $(=3/8c_4')$ depending only on f, ν, Q, T such that*

$$\|u(t)\|^2 \leqq 2(1+\|u_0\|^2) \tag{3.28}$$

for

$$t \leqq T_1(\|u_0\|) = \frac{K_6}{(1+\|u_0\|^2)^2}. \tag*{(3.29)[2]}$$

It follows that if $n=3$ and u is a sufficiently regular solution of Problems 2.1–2.2 then (assuming $T_1 \leqq T$)

$$\sup_{t\in[0,T_1]} \|u(t)\|^2 \leqq K_7 = 2(1+\|u_0\|^2), \tag{3.30}$$

T_1 given by Lemma 3.2, and from (3.24),

$$\begin{gathered}\int_0^{T_1} |Au(t)|^2\,dt \leqq K_8,\\ K_8 = \frac{1}{\nu}\left(\|u_0\|^2 + \frac{2}{\nu}\int_0^{T_1} |f(t)|^2\,dt + c_3' K_7^3\right).\end{gathered} \tag{3.31}$$

3.2. Existence and uniqueness results. There are many different existence and uniqueness results for Navier–Stokes equations. The next two theorems collect the most typical results, obtained in particular by J. Leray [1], [2], [3], E. Hopf [1], O. A. Ladyzhenskaya [1], J. L. Lions [1], J. L. Lions–G. Prodi [1], and J. Serrin [1].

THEOREM 3.1 (weak solutions). *For f and u_0 given,*

$$f \in L^2(0,T;V'), \qquad u_0 \in H, \tag{3.32}$$

there exists a weak solution u to the Navier–Stokes equations (Problem 2.1) satisfying

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \tag{3.33}$$

as well as (2.39) (or (2.43)) and (2.40).

Furthermore, if $n=2$, u is unique and

$$u \in \mathscr{C}([0,T];H), \tag{3.34}$$

$$u' \in L^2(0,T;V'). \tag{3.35}$$

[2] $t \leqq T_1(\|u_0\|)$ and obviously $t \leqq T$.

If $n=3$, u is weakly continuous from $[0,T]$ *into* H:

$$u\in\mathscr{C}([0,T];H_w), \tag{3.36}$$

and

$$u'\in L^{4/3}(0,T;V'). \tag{3.37}$$

THEOREM 3.2 (strong solutions, $n=2,3$). i) *For $n=2$, f and u_0 given,*

$$f\in L^\infty(0,T;H),\qquad u_0\in V, \tag{3.38}$$

there exists a unique strong solution to the Navier–Stokes equations (*Problem* 2.2), *satisfying*[3]:

$$u\in L^2(0,T;D(A)),\qquad u'\in L^2(0,T;H), \tag{3.39}$$

$$u\in\mathscr{C}([0,T];V). \tag{3.40}$$

ii) *For $n=3$, f and u_0 given, satisfying* (3.38), *there exists* $T_*=T_*(u_0)=\min(T,T_1(\|u_0\|))$, $T_1(\|u_0\|)$ *given by* (3.29), *and, on* $[0,T_*]$, *there exists a unique strong solution u to the Navier–Stokes equations, satisfying* (3.39), (3.40) *with T replaced by T_*.*

Remark 3.1. i) The theory of existence and uniqueness of solutions is not complete for $n=3$: we do not know whether the weak solution is unique (or what further condition could perhaps make it unique); we do not know whether a strong solution exists for an arbitrary time T. See, however, § 3.4.

ii) We recall that as long as a strong solution exists ($n=3$), it is *unique* in the class of weak solutions (cf. J. Sather and J. Serrin in J. Serrin [1], or [RT, Thm. III.3.9]).

iii) Due to a regularizing effect of the Navier–Stokes equations for strong solutions, those solutions can have further regularity properties than (3.39)–(3.40) if the data are sufficiently smooth: regularity properties will be investigated in § 4 for the space periodic case and in § 6 for the bounded case.

Let us also mention that if $n=2$ and in (3.38) we assume only that $u_0\in H$, then the solution u is in $L^2_{\mathrm{loc}}((0,T];D(A))$ and $\mathscr{C}((0,T];V)$.

Remark 3.2. The strong solutions (and the weak solution if $n=2$) satisfy the energy equality (3.2). For $n=3$ we know only that there exists a weak solution which verifies the energy inequality

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2\leqq\langle f(t),u(t)\rangle\quad\text{on }(0,T).$$

It is not known whether all the weak solutions satisfy this inequality or whether this inequality is actually an equality.

Remark 3.3. Let us assume that the conditions (3.38) are verified. Then for $n=2$, if u is a weak solution to the Navier–Stokes equations (Problem 2.1), u is automatically a strong solution by uniqueness and Theorem 3.2. For $n=3$,

[3] $f\in L^2(0,T;H)$ is sufficient for $n=2$.

the same is true on $(0, T_*(u_0))$, but u will be a strong solution on the whole interval $[0, T]$ if u satisfies some further regularity property.

For instance, *if a weak solution u belongs to $L^4(0, T; V)$, then u is a strong solution:* indeed u is a strong solution on some interval $(0, T_*)$, $T_* \leqq T$, Let us assume that the largest possible value of T_* is $T' < T$. Then we must have

$$\limsup_{t \to T'-0} \|u(t)\| = +\infty.$$

On the other hand, the energy inequality (3.24), which is valid on $(0, T'-\varepsilon)$ for all $\varepsilon > 0$, implies

$$\frac{d}{dt}(1+\|u(t)\|^2) \leqq c(1+\|u(t)\|^2)^3,$$

and by the technique of Gronwall's lemma,

$$1+\|u(t)\|^2 \leqq (1+\|u_0\|^2) \exp\left(c\int_0^T (1+\|u(s)\|^2)^2\, ds\right), \qquad 0<t<T'.$$

Thus $\|u(t)\|$ is bounded for $t \to T'-0$, contradicting the assumption that $T' < T$.

3.3. Outlines of the proofs. The proofs of Theorem 3.1 and 3.2 can be found in the original papers or in the books of O. A. Ladyzhenskaya [1], J. L. Lions [1], [RT]. We finish this section with some outlines of the proofs which we need for the sequel.

i) We implement a Galerkin method, using as a basis of H the eigenfunctions w_j, $j \in \mathbb{N}$, of the operator A (cf. (2.17)). For every integer m, we are looking for an approximate solution u_m of Problems 2.1–2.2,

(3.41) $$u_m = \sum_{j=1}^{m} g_{jm}(t) w_j,$$

$$u_m : [0, T] \mapsto W_m = \text{the space spanned by } w_1, \ldots, w_m.$$

The function u_m satisfies, instead of (2.39)–(2.40),

(3.42) $$\frac{d}{dt}(u_m, v) + \nu((u_m, v)) + b(u_m, u_m, v) = (f, v) \quad \forall\, v \in W_m,$$

(3.43) $$u_m(0) = P_m u_0,$$

where

(3.44)[4] P_m is the orthogonal projector in H onto W_m.

The semiscalar equation (3.42) is equivalent to the ordinary differential system

(3.45) $$\frac{du_m}{dt} + \nu A u_m + P_m B u_m = P_m f.$$

[4] Due to the definition of the w_j's, P_m is also the orthogonal projector on W_m, in $V, V', D(A), \ldots$.

The existence and uniqueness of a solution u_m to (3.42)–(3.45) defined on some interval $(0, T_m)$, $T_m > 0$, is clear; in fact the following a priori estimate shows that $T_m = T$.

ii) The passage to the limit $m \to \infty$ (and the fact that $T_m = T$) is based on obtaining a priori estimates on u_m. The first estimate, needed for Theorem 3.1, is obtained by replacing v by u_m $(= u_m(t))$ in (3.42). Using (2.34) we get

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \nu\,\|u_m(t)\|^2 = (f, u_m(t)). \tag{3.46}$$

This relation is the same as (3.2), and we deduce from it the bounds on u_m analogous to (3.4)–(3.7):

$$\int_0^T \|u_m(t)\|^2\,dt \leqq \frac{1}{\nu}\left(|u_{0m}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,dt\right),$$

$$\sup_{t\in[0,T]} |u_m(t)|^2 \leqq \left(|u_{0m}|^2 + \frac{1}{\nu}\int_0^T \|f(t)\|_{V'}^2\,dt\right).$$

Since $|u_{0m}| = |P_m u_0| \leqq |u_0|$, we get for u_m exactly the same bounds as (3.4)–(3.7), from which we conclude that

(3.47) *u_m remains in a bounded set of $L^2(0, T;\, V) \cap L^\infty(0, T;\, H)$.*

Then, as usual in the Galerkin method, we extract a subsequence $u_{m'}$ weakly convergent in $L^2(0, T;\, V)$ and $L^\infty(0, T;\, H)$,

$$u_{m'} \to u \begin{cases} \text{in } L^2(0, T;\, V) & \text{weakly and} \\ \text{in } L^\infty(0, T;\, H) & \text{weak-star.} \end{cases} \tag{3.48}$$

The passage to the limit in (3.42), (3.43) allows us to conclude that u is a solution of Problem 2.1 and proves the existence in Theorem 3.1. However, this step necessitates a further a priori estimate and the utilization of a compactness theorem; we will come back to this point in a more general situation in § 13. Finally, we refer to [RT] for the proof of the other results in Theorem 3.1.

iii) For $n = 2$, the only new element in Theorem 3.2 is (3.39)–(3.40). We obtain that $u \in L^2(0, T;\, D(A)) \cap L^\infty(0, T;\, V)$ by deriving further a priori estimates on u_m, in fact, a priori estimates similar to (3.14)–(3.17). We obtain them by taking the scalar product of (3.45) with Au_m. Since P_m is selfadjoint in H and $P_m A u_m = A P_m u_m = A u_m$, we obtain, using (3.8),

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu\,|Au_m|^2 + b(u_m, u_m, Au_m) = (f, Au_m). \tag{3.49}$$

This relation is similar to (3.9). Exactly as in § 3.1, we obtain the bounds (3.14)–(3.17) for u_m, with u_0 replaced by u_{0m}. Since $u_{0m} = P_m u_0$ and P_m is an orthogonal projector in V,

$$\|u_{0m}\| = \|P_m u_0\| \leqq \|u_0\|, \tag{3.50}$$

and we then find for u_m exactly the same bounds as for u in (3.14)–(3.17):

(3.51) *u_m remains in a bounded set of $L^2(0, T; D(A)) \cap L^\infty(0, T; V)$* $\quad (n=2)$.

By extraction of a subsequence as in (3.48), we find that u is in $L^2(0, T; D(A)) \cap L^\infty(0, T; V)$. The continuity property (3.40) results from (3.39) by interpolation (cf. J. L. Lions–E. Magenes [1] or [RT]). The fact that $u' \in L^2(0, T; H)$ follows from (2.43),

$$u' = f - \nu Au - Bu, \tag{3.52}$$

and the properties of B; f and Au are obviously in $L^2(0, T; H)$ and for B we notice in the relation (2.31) that

$$|Bu(t)| \leqq c_2 |u(t)|^{1/2} \|u(t)\| \, |Au(t)|^{1/2}, \tag{3.53}$$

so that Bu belongs to $L^4(0, T; H)$ and u' belongs to $L^2(0, T; H)$.

The proof of Theorem 3.2 in the case $n = 3$ is exactly the same, except that we use the estimates (3.30), (3.31) valid on $[0, T_1(\|u_0\|)]$ instead of the estimates (3.14)–(3.17) valid on the whole interval $[0, T]$. Also, instead of using (2.31) and obtaining (3.53), we use the relation (2.32) which gives us

$$|Bu(t)| \leqq c_3 \|u(t)\|^{3/2} |Au(t)|^{1/2}; \tag{3.54}$$

Bu is still in $L^2(0, T; H)$, and u' is too.

Remark 3.4. Except for Lemma 3.1 and (3.20)–(3.23), the a priori estimates and the existence and uniqueness results are absolutely the same for both the space periodic case and the flow in a bounded domain with $u = 0$ on the boundary (cf. § 2.5). If $u = \phi \neq 0$ on the boundary and/or Ω is unbounded, similar results are valid. We refer to the literature for the necessary modifications.

3.4. Generic solvability of the Navier–Stokes equations. We do not know whether Problem 2.2 is solvable for an arbitrary pair u_0, f, but this is generically true in the following sense (A. V. Fursikov [1]):

THEOREM 3.3. *For $n = 3$, given $\nu, \mathcal{O}$ $(= Q$ or $\Omega)$ and u_0 belonging to V, there exists a set F, included in $L^2(0, T; H)$ and dense in*

$$L^q(0, T; V') \quad \forall q, \quad 1 \leqq q < \tfrac{4}{3}, \tag{3.55}$$

such that for every $f \in F$, Problem 2.2 corresponding to u_0, f, possesses a unique solution (strong solution).

Proof. i) Since $L^2(0, T; H)$ is dense in $L^q(0, T; V')$, $1 \leqq q < \frac{4}{3}$, it suffices to show that every $f \in L^2(0, T; H)$ can be approximated in the norm of $L^q(0, T; V')$ by a sequence of f_m's, $f_m \in L^2(0, T; H)$, such that Problem 2.2 for u_0, f_m possesses a unique solution.

For that purpose, given f, we consider the Galerkin approximation u_m described in § 3.3 above (cf. (3.41)–(3.45)). It is clear that u_m' is in $L^2(0, T; W_m)$ and hence in $L^2(0, T; D(A))$ and that u_m is continuous from $[0, T]$ into W_m and hence into $D(A)$. Now for every m we consider also the solution v_m of the linearized problem

$$\frac{dv_m}{dt} + \nu A v_m = 0, \tag{3.56}$$

$$v_m(0) = (I - P_m)u_0. \tag{3.57}$$

It is standard that the linear problem (3.56)–(3.57) possess a unique solution satisfying in particular

$$v_m \in L^2(0, T; D(A)) \cap L^\infty(0, T; V). \tag{3.58}$$

We then set $w_m = u_m + v_m$ and observe that w_m satisfies

$$w_m \in L^2(0, T; D(A)) \cap L^\infty(0, T; V), \tag{3.59}$$

and by adding (3.56) to (3.45) and (3.57) to (3.43),

$$w_m(0) = u_0, \tag{3.60}$$

$$\frac{dw_m}{dt} + \nu A w_m + B w_m = f + g_m, \tag{3.61}$$

$$g_m = -(I - P_m)f + B(v_m) + B(v_m, u_m) + B(u_m, v_m) + (I - P_m)B(w_m). \tag{3.62}$$

The proof will be complete if we show that, for $m \to \infty$,

$$g_m \to 0 \quad \text{in } L^q(0, T; V'), \quad 1 \leqq q < \tfrac{4}{3}. \tag{3.63}$$

ii) Since $|(I - P_m)f(t)| \to 0$ for $m \to \infty$ for almost every t, and $|(I - P_m)f(t)| \leqq |f(t)|$, it is clear by the Lebesgue dominated convergence theorem that $(I - P_m)f \to 0$ in $L^2(0, T; H)$ for $m \to \infty$.

Multiplying (3.56) by Av_m we get

$$\frac{d}{dt}\|v_m(t)\|^2 + 2\nu\,|Av_m(t)|^2 = 0,$$

$$\|v_m(t)\|^2 + 2\nu\int_0^t |Av_m(s)|^2\,ds \leqq \|(I - P_m)u_0\|^2,$$

from which it follows that, for $m \to \infty$,

$$v_m \to 0 \quad \text{in } L^2(0, T; D(A)) \quad \text{and} \quad L^\infty(0, T; V). \tag{3.64}$$

Using (3.64), the estimate (3.47) for u_m and Lemma 2.1 (with $m_1 = 0$, $m_2 = 1$, $m_3 = 1$), we find

$$\int_0^T \|B(u_m, v_m)\|_{V'}^2\,dt \leqq c\int_0^T |u_m(t)|^2\,|Av_m(t)|^2\,dt \to 0.$$

In a similar manner we prove that

$$B(v_m, u_m) \quad \text{and} \quad B(v_m, v_m) \to 0 \quad \text{in } L^2(0, T; V')$$

for $m \to \infty$, and the proof of (3.63) is reduced to that of

$$(I - P_m)B(u_m) \to 0 \quad \text{in } L^q(0, T; V'), \quad \text{for } m \to \infty. \tag{3.65}$$

iii) The proof of (3.65) is technical and we give only a sketch of it.

The sequence u_m converges to some limit u (cf. (3.48))[5]. Since $Bu \in L^{4/3}(0, T; V')$[6], and since, by Lebesgue's theorem, $(I-P_m)\, Bu \to 0$ in $L^{4/3}(0, T; V')$, it suffices to show that

$$Bu_m \to Bu \quad \text{strongly in } L^q(0, T; V'). \tag{3.66}$$

Due to (2.33) and Lemma 2.1,

$$|b(\phi, \phi, \psi)| = |b(\phi, \psi, \phi)| \leqq c\, |\phi|_{3/4}^2 \|\psi\| \quad \forall \phi, \psi;$$

therefore B is a bilinear continuous operator from $V_{3/4} \times V_{3/4}$ into V' (see also (2.36)) and

$$\|B\phi\|_{V'} \leqq c\, |\phi|_{3/4}^2 \quad \forall \phi. \tag{3.67}$$

It follows from the proof of Theorem 3.1 (cf. the cited references) that u_m converges to u strongly in $L^2(0, T; V_{1-\varepsilon})$ and $L^{1/\varepsilon}(0, T; H)$ for all $\varepsilon > 0$. By (2.20) with $m_1 = 0, m_2 = 1-\varepsilon, \theta m_2 + (1-\theta) m_1 = \theta(1-\varepsilon) = \frac{3}{4}$, and the Hölder inequality,

$$\int_0^T |u_m - u|_{3/4}^\rho \, dt \leqq \left(\int_0^T |u_m - u|^{1/\varepsilon} \, dt \right)^{\varepsilon(1-\theta)} \left(\int_0^T |u_m - u|_{1-\varepsilon}^2 \, dt \right)^{2\theta},$$

$1/\rho = \varepsilon(1-\theta) + \theta/2$. We conclude that $u_m \to u$ in $L^\rho(0, T; V_{3/4})$, and with (3.67), that $Bu_m \to Bu$ in $L^{\rho/2}(0, T; V')$. Finally, since $\rho < 8/3$, $\rho \to 8/3$, as $\varepsilon \to 0$, we choose ε sufficiently small so that $\rho/2 = q$. □

Remark 3.5. As indicated in Remark 3.1, the theory of existence and uniqueness of solutions is not complete when $n = 3$, while it is totally satisfactory for $n = 2$.

It was Leray's conjecture on turbulence, which is not yet proved nor disproved, that the solutions of Navier–Stokes equations do develop singularities (cf. also B. Mandelbrot [1], [2]). It seems useful to study the properties of weak solutions of Navier–Stokes equations with the hope of either proving that they are regular, or studying the nature of their singularities if they are not.

The results in §§ 4, 5 and 8 tend in this direction. Of course they (as would Theorem 3.3) would lose all of their interest if the existence of strong solutions were demonstrated.

[5] In (3.48) a subsequence $u_{m'}$ of u_m converges to u, but this is sufficient.

[6] Cf. [RT]; this is how (3.37) is proved.

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4 New A Priori Estimates and Applications

In this section we establish some properties of weak solutions of Navier–Stokes equations. The results are proved for the space periodic case, but they do not all extend to the bounded case (cf. the comments, following § 14). *We assume throughout this section that $n = 3$ and the boundary condition is space periodicity.*

4.1. Energy inequalities and consequences. We derive formal energy inequalities assuming that u_0, f, u are sufficiently regular.

LEMMA 4.1. *If u is a smooth solution of Problems* 2.1–2.2 (*space periodic case,* $n = 3$), *then for each $t > 0$, for any $r \geqq 1$,*

$$\frac{d}{dt}|u(t)|_r^2 + \nu\,|u(t)|_{r+1}^2 \leqq L_r(1 + |u(t)|_1^2\,|u(t)|_r^{4r/(2r-1)}) \tag{4.1}$$

where the constant L_r depends on the data, ν, Q and $N_{r-1}(f) = |f|_{L^\infty(0,T;V_{r-1})}$.

Moreover, for any $r \geqq 3$, we have

$$\frac{d}{dt}|u(t)|_r^2 + \nu\,|u(t)|_{r+1}^2 \leqq L_r'(1 + |u(t)|_1^2)^{2r+1}, \tag{4.2}$$

where L_r' depends also on ν, Q and $N_{r-1}(f)$.

Proof. i) We take the scalar product in H of (2.43) with $A^r u$, and we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_r^2 + \nu\,|u|_{r+1}^2 = (f, u)_r - (Bu, A^r u).$$

By the definition (2.42) of B and as $Au = -\Delta u$ in the space periodic case (cf. (2.14)), we can write

$$\frac{1}{2}\frac{d}{dt}|u|_r^2 + \nu\,|u|_{r+1}^2 = (f, u)_r - (-1)^r b(u, u, \Delta^r u). \tag{4.3}$$

The first term in the right-hand side of (4.3) is majorized by

$$|f(t)_{r-1}\,|u(t)|_{r+1} \leqq \frac{\nu}{4}|u(t)|_{r+1}^2 + \frac{1}{\nu}N_{r-1}(f)^2. \tag{4.4}$$

The second term is a sum of integrals of the type

$$\int_Q u_i D_i u_j\,\Delta u_j\,dx \quad \text{or} \quad \int_Q u_i D_i u_j D_i^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u_j\,dx, \qquad \alpha_i \in \mathbb{N}, \quad \alpha_1 + \alpha_2 + \alpha_3 = r.$$

We can integrate by parts using Stokes' formula; the boundary terms on ∂Q cancel each other due to the periodicity of u, and the integrals take the form

$$\int_Q D^\alpha(u_i D_i u_j) D^\alpha u_j\,dx, \qquad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}. \tag{4.5}$$

With Leibniz' formula, we see that these integrals are sums of integrals of the form

$$\int_Q u_i D_i D^\alpha u_j D^\alpha u_j \, dx, \tag{4.6}$$

and of integrals of the form

$$\int_Q \delta^k u_i \delta^{r-k} D_i u_j D^\alpha u_j, \qquad k = 1, \dots, r, \tag{4.7}$$

where δ^k is some differential operator D^α with $[\alpha] = \alpha_1 + \alpha_2 + \alpha_3 = k$.

The sum for $i = 1, 2, 3$ of the integrals (4.6) vanishes because of the condition $\operatorname{div} u = 0$. Then it remains to estimate the integrals (4.7).

ii) *Proof of* (4.1). By Hölder's inequality, the modulus of (4.7) is less than or equal to

$$|\delta^k u_i(t)|_{L^3(Q)} \, |\delta^{r+1-k} u_j(t)|_{L^6(Q)} \, |D^\alpha u(t)|.$$

From the Sobolev injection theorems ($H^1(Q) \subset L^6(Q)$, $H^{1/2}(Q) \subset L^3(Q)$, cf. (2.18)), this term is less than or equal to

$$c_1' \, |u(t)|_{k+1/2} \, |u(t)|_{r+2-k} \, |u(t)|_r,$$

c_1' depending on k, r, Q.

We then apply the interpolation inequality (2.20) with $m_1 = 1$, $m_2 = r+1$, $\theta = k/r - 1/2r$, $(1-\theta)m_1 + \theta m_2 = k + \frac{1}{2}$ and $\theta = 1 - (k-1)/r$, $(1-\theta)m_1 + \theta m_2 = r + 2 - k$, to get

$$|u(t)|_{k+1/2} \leqq c_2' \, |u(t)|_1^{1-k/r+1/2r} \, |u(t)|_{r+1}^{k/r-1/2r},$$
$$|u(t)|_{r+2-k} \leqq c_2' \, |u(t)|_1^{k-1/r} \, |u(t)|_{r+1}^{1-k-1/r}.$$

Therefore the modulus of the integrals of type (4.7) is bounded by an expression

$$c_3' \, |u(t)|_1^{1-1/2r} \, |u(t)|_{r+1}^{1+1/2r} \, |u(t)|_r.$$

Hence

$$|b(u, u, \Delta^r u)| \leqq c_4' \, |u(t)|_1^{1-1/2r} \, |u(t)|_{r+1}^{1+1/2r} \, |u(t)|_r$$
$$\leqq \frac{\nu}{4} |u(t)|_{r+1}^2 + c_5' \, |u(t)|_1^2 \, |u(t)|^{4r/2r-1} \tag{4.8}$$

(by Young's inequality).

This relation with (4.3) and (4.4) gives (4.1).

iii) *Proof of* (4.2). We majorize $b(u, u, \Delta^r u)$ in a slightly different way. We write

$$|u(t)|_r \leqq c_5' \, |u(t)|_1^{1/r} \, |u(t)|_{r+1}^{1-1/r} \tag{4.9}$$

by application of (2.20) with $m_1=1$, $m_2=m+1$, $\theta=1-1/r$. Then (4.8) gives

$$
\begin{aligned}
|b(u,u,\Delta^r u)| &\leqq c_6' |u(t)|_1^{1+1/2r} |u(t)|_{r+1}^{2-1/2r} \\
&\leqq \frac{\nu}{4} |u(t)|_{r+1}^2 + c_7' |u(t)|_1^{4r+2} \quad \text{(by Young's inequality).}
\end{aligned}
\tag{4.10}
$$

This relation combined with (4.3)–(4.4) gives (4.2). □

An immediate consequence of (4.2) is that u remains in V_r as long as $\|u(t)\|$ remains bounded. This is expressed in

LEMMA 4.2. *If* $u_0 \in V_r$ *and* $f \in L^\infty(0,T;V_{r-1})$, $r \geqq 1$, *then the solution* u *to Problem* 2.2 *given by Theorem* 3.2 ii) *belongs to* $\mathscr{C}([0,T_*];V_r)$.

If $u_0 \in V$ *and* $f \in L^\infty(0,T;V_{r-1})$, $r \geqq 1$, *then* $u \in \mathscr{C}((0,T_*];V_r)$.

Proof. i) We consider the case $u_0 \in V_r$, and we first show that u belongs to $L^\infty(0,T_*;V_r)$. For that it suffices to prove that the Galerkin approximation u_m of u constructed in § 3 remains bounded in $L^\infty(0,T_*;V_r)$ as $m \to \infty$. We take the scalar product in H of (3.45) with $A^r u_m = (-1)^r \Delta^r u_m$, and since P_m is selfadjoint in H and $P_m A^r u_m = A^r u_m$, we get

$$
\frac{1}{2}\frac{d}{dt}|u_m|_r^2 + \nu |u_m|_{r+1}^2 = (f,u_m)_r - (-1)^r b(u_m,u_m,\Delta^r u_m).
$$

This is similar to (4.3) and thus, exactly as in Lemma 4.1, we get the analogue of (4.2):

$$
\frac{d}{dt}|u_m(t)|_r^2 + \nu |u_m(t)|_{r+1}^2 \leqq L_r'(1+|u_m(t)|_1^2)^{2r+1};
$$

due to (3.51) we conclude that

$$
\begin{gathered}
|u_m(t)|_r^2 \leqq c_1' + |u_m(0)|_r^2 \quad \text{for } 0 \leqq t \leqq T_*, \\
\int_0^{T_*} |u_m(t)|_{r+1}^2\,dt \leqq c_2' + |u_m(0)|_r^2.
\end{gathered}
$$

Now P_m is a projector in V_r too, and $|P_m u_0|_r \leqq |u_0|_r$, so that u_m remains in a bounded set of $L^\infty(0,T_*;V_r)$ and $L^2(0,T_*;V_{r+1})$, and

$$
u \in L^\infty(0,T_*;V_r) \cap L^2(0,T_*;V_{r+1}). \tag{4.11}
$$

We then check that Bu, and therefore $u' = f - \nu Au - Bu$, belongs to $L^2(0,T_*;V_{r-1})$; the continuity of u from $[0,T_*]$ into V_r follows by interpolation (cf. J. L. Lions–E. Magenes [1] or [RT, § III.1]).

ii) For $u_0 \in V$, we observe that the solution u of Problem 2.2 belongs to $L^2(0,T_*;D(A))$. Thus $u(t) \in D(A) = V_r$ almost everywhere on $(0,T_*)$, and we can find t_1 arbitrarily small such that $u(t_1) \in V_2$. The first part of the proof shows that $u \in \mathscr{C}([t_1,T_*];V_2] \cap L^2(t_1;T_*;V_3)$. Hence $u(t_2) \in V_3$ for some $t_2 \in [t_1,T_*]$, t_2 arbitrarily close to t_1, and $u \in \mathscr{C}([t_2,T_*];V_3) \cap L^2(t_2,T_*;V_4)$. By induction we arrive at $u \in \mathscr{C}([t_{r-1},T_*];V_r) \cap L^2(t_{r-1},T_*;V_{r-1})$ and, since t_{r-1} is arbitrarily close to 0, the result is proved. □

4.2. Structure of the singularity set of a weak solution. Let $m \geqq 1$. We say that a solution u of Problem 2.1–2.2 is $\mathbb{H}^m$-*regular* on (t_1, t_2) $(0 \leqq t_1 \leqq t_2)$ if $u \in \mathscr{C}((t_1, t_2); \mathbb{H}^m(Q))$. We say that an $\mathbb{H}^m$-regularity interval (t_1, t_2) is *maximal* if there does not exist an interval of $\mathbb{H}^m$-regularity greater than (t_1, t_2).

The local existence of an $\mathbb{H}^m$-regular solution is given by Lemma 4.2: if $u_0 \in V_m$ and $f \in L^\infty(0, T; V_{m-1})$, then there exists an $\mathbb{H}^m$-regular solution of the Navier–Stokes equations defined on some interval $(0, t_0)$. Also, it follows easily from Lemma 4.2 that if (t_1, t_2) is a maximal interval of $\mathbb{H}^m$-regularity of a solution u, then

$$\limsup_{t \to t_2 - 0} |u(t)|_m = +\infty. \tag{4.12}$$

We are now able to give some indications on the set of $\mathbb{H}^m$-regularity of a weak solution.

THEOREM 4.1 ($n = 3$, space periodic case). *We assume that* $u_0 \in H$, $f \in L^\infty(0, T; V_{m-1})$, $m \geqq 1$, *and that* u *is a weak solution of the Navier–Stokes equations* (*Problem* 2.1). *Then* u *is* $\mathbb{H}^m$-*regular on an open set of* $(0, T)$ *whose complement has Lebesgue measure* 0.

Moreover, the set of $\mathbb{H}^m$-*regularity of* u *is independent of* m, *i.e. is the same for* $r = 1, \ldots, m$.

Proof. Since u is weakly continuous from $[0, T]$ into H, $u(t)$ is well defined for every t and we can define

$$\Sigma_r = \{t \in [0, T], u(t) \notin \mathbb{H}^r(Q)\},$$

$$\Omega_r = \{t \in [0, T], u(t) \in \mathbb{H}^r(Q)\},$$

$$\mathcal{O}_r = \{t \in (0, T), \exists\, \varepsilon > 0, u \in \mathscr{C}((t - \varepsilon, t + \varepsilon); \mathbb{H}^r(Q))\}.$$

It is clear that $\mathcal{O}_r$ is open for every r.

For $r = 1$, since $u \in L^2(0, T; V)$, Σ_1 has Lebesgue measure 0. If t_0 belongs to Ω_1 and not to $\mathcal{O}_1$ then, according to Theorem 3.2 ii), t_0 is the left end of an interval of $\mathbb{H}^1$-regularity, i.e. one of the connected components of $\mathcal{O}_1$. Thus $\Omega_1 \backslash \mathcal{O}_1$ is countable and $[0, T] \backslash \mathcal{O}_1$ has Lebesgue measure 0.

The theorem is proved for $m = 1$. We will now complete the proof by showing that $\mathcal{O}_m = \mathcal{O}_1$.

If (t_1, t_2) is a connected component of $\mathcal{O}_1$ (a maximal interval of $\mathbb{H}^1$-regularity), then for every t_1' in this interval, $u(t_1') \in V$ and, according to Lemma 4.2, there exists a unique $\mathbb{H}^m$-regular solution defined on some interval (t_1', t_2'), $t_1' < t_2' \leqq t_2$. Since uniqueness holds also in the class of weak solutions (cf. Remark 3.1 ii)) this solution coincides with u, i.e., (t_1', t_2') is an interval of $\mathbb{H}^m$-regularity of u. According to (4.2), u remains bounded in $\mathbb{H}^m$ as long as u is bounded in V. Therefore, using Lemma 4.2 also, we see that $t_2' = t_2$, and since t_1' is arbitrarily close to t_1, (t_1, t_2) is an interval of $\mathbb{H}^m$-regularity. This proves that $\mathcal{O}_m = \mathcal{O}_1$, and $\mathcal{O}_r = \mathcal{O}_1$, $r = 1, \ldots, m-1$. $\square$

4.3. New a priori estimates.

THEOREM 4.2 ($n = 3$, space periodic case). *We assume that* $u_0 \in H$, $f \in L^2(0, T; V_{m-1})$ *and that* u *is a weak solution of the Navier–Stokes equations*

(*Problem* 2.1). *Then u satisfies*

$$u \in L^{\alpha_r}(0, T; \mathbb{H}_p^r(Q)), \qquad \int_0^T |u(t)|_r^{\alpha_r}\, dt \leqq c_r, \tag{4.13}$$

$r = 1, \ldots, m+1$, *where the constants* c_r *depend on the data* ν, Q, u_0, f, *and the* α_r*'s are given by*

$$\alpha_r = \frac{1}{2r-1}. \tag{4.14}$$

Proof. i) Let (α_i, β_i), $i \in \mathbb{N}$, be the connected components of $\mathcal{O}_1$, which are also maximal intervals of $\mathbb{H}^m$-regularity of u.

On each interval (α_i, β_i), the inequalities (4.1) are satisfied, $r = 1, \ldots, m$, and we write them in the slightly stronger form

$$\frac{d}{dt}|u(t)|_r^2 + \nu\, |u(t)|_{r+1}^2 \leqq L_r(1+|u(t)|_1^2)(1+|u(t)|_r^2)^{2r/(2r-1)}. \tag{4.15}$$

Then we deduce

$$\frac{(d/dt)\,|u(t)|_r^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}} + \frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}} \leqq L_r(1+|u(t)|_1^2).$$

By integration in t from α_i to β_i, we get

$$-(2r-1)\frac{1}{(1+|u(\beta_i-0)|_r^2)^{1/(2r-1)}} + (2r-1)\frac{1}{(1+|u(\alpha_i+0)|_r^2)^{1/(2r-1)}}$$
$$+\nu\int_{\alpha_i}^{\beta_i} \frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}}\, dt \leqq L_r \int_{\alpha_i}^{\beta_i} (1+|u(t)|_1^2)\, dt.$$

From (4.12), the first term in the left-hand side of this inequality vanishes, since (α_i, β_i) is a maximal interval of $\mathbb{H}^m$-regularity. Thus

$$\int_{\alpha_i}^{\beta_i} \frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}}\, dt \leqq \frac{L_r}{\nu}\int_{\alpha_i}^{\beta_i} (1+|u(t)|_1^2)\, dt.$$

By summation of these relations for $i \in \mathbb{N}$, we find, since $u \in L^2(0, T; V)$,

$$\int_0^T \frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}}\, dt \leqq c_r, \qquad r = 1, \ldots, m. \tag{4.16}$$

ii) The proof of (4.13) is now made by induction. The result is true for $r = 1$. We assume that it is true for $1, \ldots, r$, and prove it for $r+1$ ($r \leqq m$).

We have

$$\int_0^T |u(t)|_{r+1}^{\alpha_{r+1}}\, dt = \int_0^T \left[\frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r+1)}}\right]^{\alpha_{r+1}/2} [(1+|u(t)|_r^2)^{2r/(2r-1)}]^{\alpha_{r+1}/2}\, dt$$
$$\leqq \left[\int_0^T \frac{|u(t)|_{r+1}^2}{(1+|u(t)|_r^2)^{2r/(2r-1)}}\, dt\right]^{\alpha_{r+1}/2} \left[\int_0^T (1+|u(t)|_r^2)^{\gamma}\, dt\right]^{1-(\alpha_{r+1}/2)}$$

(by Hölder's inequality),

where

$$\gamma=\frac{2r}{2r-1}\frac{\alpha_{r+1}}{2}\frac{2}{2-\alpha_{r+1}}=\frac{1}{2r-1}=\frac{\alpha_r}{2}.$$

Therefore (4.13) follows for $r+1$, due to (4.16) and the induction assumption. □

Remark 4.1. As indicated at the beginning of this section, all the results have been established in the periodic case. We do not know whether they are valid in the bounded case (although it is likely that they are) except for the following special results: (4.1) and (4.2) for $r=1$, which coincide with (3.24); in Theorem 4.1 the fact that the complement of $\mathcal{O}_1$ in $[0, T]$ has Lebesgue measure 0 and is closed; and (4.13) for $r=2(\alpha_r=\frac{2}{3})$ (same proofs)[1].

The following interesting consequence of Theorem 4.2, which relies only on (4.13) with $r=2$, is valid in both the periodic and the bounded cases.

THEOREM 4.3. *We assume that $n=3$ and we consider the periodic or bounded case. We assume that $u_0 \in H$ and $f\in L^\infty(0, T; H)$. Then any weak solution u of Problem 2.1 belongs to $L^1(0, T; \mathbb{L}^\infty(\mathcal{O}))$, $\mathcal{O}=\Omega$ or Q.*

Proof. Because of (2.26)

$$|u(t)|_{\mathbb{L}^\infty(\mathcal{O})} \leqq c\,\|u(t)\|^{1/2}\,|Au(t)|^{1/2},$$

and by Hölder's inequality

$$\int_0^T |u(t)|_{\mathbb{L}^\infty(\mathcal{O})} \leqq c\left(\int_0^T |Au(t)|^{2/3}\,dt\right)^{3/4}\left(\int_0^T \|u(t)\|^2\,dt\right)^{1/4},$$

and the right-hand side is finite, thanks to (4.13) ($r=2$, cf. Remark 4.1).

[1] Cf. also § 6, and in particular Remark 6.2, for a complete extension of Theorem 4.1 to the bounded case.

5 Regularity and Fractional Dimension

If the weak solutions of the Navier–Stokes equations develop singularities, as Leray has conjectured, then a natural problem is to study the nature of the singularities. B. Mandelbrot conjectured in [1]–[2] that the singularities are located on sets of Hausdorff dimension <4 (in space and time), and V. Scheffer has given an estimate of the dimension of the singularities which he successively improved in [1]–[4]. A more recent (and improved) estimate is due to L. Caffarelli–R. Kohn–L. Nirenberg [1]. In this section we present some results concerning the Hausdorff dimension of the singular set of weak solutions, following (for one of them) the presentation in C. Foias–R. Temam [5].

5.1. Hausdorff measure. Time singularities. We recall some basic definitions concerning the Hausdorff dimension (cf. H. Federer [1]). Let X be a metric space and let $D>0$. The D-dimensional Hausdorff measure of a subset Y of X is

$$\mu_D(Y)=\lim_{\varepsilon\searrow 0}\mu_{D,\varepsilon}(Y)=\sup_{\varepsilon>0}\mu_{D,\varepsilon}(Y), \tag{5.1}$$

where

$$\mu_{D,\varepsilon}(Y)=\inf\sum_j(\operatorname{diam}B_j)^D, \tag{5.2}$$

the infimum being taken over all the coverings of Y by balls B_j such that diam B_j (= diameter of B_j) $\leqq\varepsilon$.

It is clear that $\mu_{D,\varepsilon}(Y)\geqq\mu_{D,\varepsilon'}(Y)$ for $\varepsilon\leqq\varepsilon'$ and $\mu_D(Y)\in[0,+\infty]$. Since $\mu_{D,\varepsilon}(Y)\leqq\varepsilon^{D-D_0}\mu_{D_0,\varepsilon}(Y)$ for $D>D_0$, then if $\mu_{D_0}(Y)<\infty$ for some $D_0\in(0,\infty)$ then $\mu_D(Y)=0$ for all $D>D_0$. In this case the number

$$\inf\{D,\ \mu_D(Y)=0\}=\inf\{D,\ \mu_D(Y)<\infty\}$$

is called the Hausdorff dimension of Y. If the Hausdorff dimension of a set Y is finite, then Y is homeomorphic to a subset of a finite dimensional Euclidean space. Finally, let us also mention the useful fact that $\mu_D(\cdot)$ is countably additive on the Borel subsets of X (see Federer [1]).

The first result on the Hausdorff dimension of singularities concerns the set of t in $[0,T]$ on which $u(t)$ is singular, $u(t)\notin V$. Actually this is a restatement by V. Scheffer [1] of a result of J. Leray [3] (cf. also S. Kaniel–M. Shinbrot [1]).

THEOREM 5.1. *Let $n=3$, $\mathcal{O}=\Omega$ or Q, and let u be a weak solution to the Navier–Stokes equations (Problem 2.1). Then there exists a closed set $\mathscr{E}\subset[0,T]$ whose $\frac{1}{2}$-dimensional Hausdorff measure vanishes, and such that u is (at least) continuous from $[0,T]\backslash\mathscr{E}$ into V.*

Proof. i) The proof of this theorem is partly contained in that of Theorem 4.1 (cf. also Remark 4.1). We set $\mathscr{E}=[0,T]\backslash\mathcal{O}_1$, and we only have to prove that the $\frac{1}{2}$-Hausdorff measure of $\mathscr{E}$ is 0.

Let (α_i, β_i), $i \in I$, be the connected components of $\mathcal{O}_1$. A preliminary result, due to J. Leray [3], is

$$\sum_{i \in I} (\beta_i - \alpha_i)^{1/2} < \infty. \tag{5.3}$$

Indeed, let (α_i, β_i) be one of these intervals and let $t \in (\alpha_i, \beta_i)$. According to Theorem 3.2 and (3.29),

$$(\beta_i - t) \geqq T_1(\|u(t)\|) = \frac{K_6}{(1 + \|u(t)\|^2)^2},$$

where K_6 depends only on f, ν and $\mathcal{O}$ $(=\Omega$ or $Q)$. Thus

$$\frac{\sqrt{K_6}}{(\beta_i - t)^{1/2}} \leqq 1 + \|u(t)\|^2, \qquad t \in (\alpha_i, \beta_i).$$

Then we integrate on (α_i, β_i) to obtain

$$2\sqrt{K_6}(\beta_i - \alpha_i)^{1/2} \leqq (\beta_i - \alpha_i) + \int_{\alpha_i}^{\beta_i} \|u(t)\|^2 \, dt,$$

and we add all these relations for $i \in I$ to get

$$2\sqrt{K_6} \sum_{i \in I} (\beta_i - \alpha_i)^{1/2} \leqq T + \int_0^T \|u(t)\|^2 \, dt < \infty.$$

ii) The proof now follows that given in V. Scheffer [1, §3]. For every $\varepsilon > 0$, we can find a finite part I_ε of I such that

$$\sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i) \leqq \varepsilon, \qquad \sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i)^{1/2} \leqq \varepsilon. \tag{5.4}$$

The set $[0, T] \setminus \bigcup_{i \in I_\varepsilon} (\alpha_i, \beta_i)$ is the union of a finite number of mutually disjoint closed intervals, say B_j, $j = 1, \ldots, N$. It is clear that $\bigcup_{j=1}^N B_j \supset \mathscr{E}$, and since the intervals (α_i, β_i) are mutually disjoint, each interval (α_i, β_i), $i \notin I_\varepsilon$, is included in one, and only one, interval B_j. We denote by I_j the set of i's such that $B_j \supset (\alpha_i, \beta_i)$. It is clear that $I_\varepsilon, I_1, \ldots, I_N$, is a partition of I and that $B_j = (\bigcup_{i \in I_j} (\alpha_i, \beta_i)) \cup (\beta_j \cap \mathscr{E})$ for all $j = 1, \ldots, N$. Hence

$$\operatorname{diam} B_j = \sum_{i \in I_j} (\beta_i - \alpha_i) \leqq \varepsilon \qquad (\text{by } (5.3)),$$

and

$$\mu_{1/2,\varepsilon}(\mathscr{E}) \leqq \sum_{j=1}^N (\operatorname{diam} B_j)^{1/2} \leqq \sum_{j=1}^N \left(\sum_{i \in I_j} (\beta_i - \alpha_i) \right)^{1/2} \leqq \sum_{i \notin I_\varepsilon} (\beta_i - \alpha_i)^{1/2} \leqq \varepsilon.$$

Letting $\varepsilon \to 0$, we find $\mu_{1/2}(\mathscr{E}) = 0$. □

5.2. Space and time singularities. The second result concerns the Hausdorff dimension in space and time of the set of (possible) singularities of a weak solution (cf. V. Scheffer [1]–[4], C. Foias–R. Temam [5]):

THEOREM 5.2. *Let $n=3$, $\mathcal{O}=\Omega$ or Q; let u be a weak solution to the Navier–Stokes equations (Problem 2.1) and assume moreover that*

$$u_0 \in D(A). \tag{5.5}$$

Then there exists a subset $\mathcal{O}_0 \subset \mathcal{O}$ such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} |u(x,t)| < \infty \quad \textit{for all } x \in \mathcal{O}\setminus\mathcal{O}_0 \quad \textit{and} \quad T\in(0,\infty), \tag{5.6}$$

$$\textit{the Hausdorff dimension of } \mathcal{O}_0 \textit{ is } \leqq \tfrac{5}{2}. \tag{5.7}$$

Before proving Theorem 5.2 we present some preliminary material of intrinsic interest.

Lemma 5.1 is borrowed from V. Scheffer [1]; we reproduce it for the convenience of the reader[1]

LEMMA 5.1. *For $a>0$ and $f\in L^1(\mathbb{R}^n)$, let $A_a(f)$ be the set of those $x\in\mathbb{R}^n$ such that there exists m_x with*

$$\int_{|y-x|\leqq 2^{-m}} |f(y)|\,dy \leqq 2^{-am} \quad \textit{for all } m \geqq m_x. \tag{5.8}$$

Then the Hausdorff dimension of $\mathbb{R}^n\setminus A_a(f)$ is $\leqq a$.

Proof. By definition of $A_a(f)$, for any $\varepsilon>0$ and $x\in\mathbb{R}^n\setminus A_a(f)$ there exists a ball $B_\varepsilon(x)$ centered in x such that

$$\int_{B_\varepsilon(x)} |f(y)|\,dy > 2^{-a}(\operatorname{diam} B_\varepsilon(x))^a \quad \text{and} \quad \operatorname{diam} B_\varepsilon(x) \geqq \varepsilon. \tag{5.9}$$

By a Vitali covering lemma (see E. M. Stein [1]), there exists a system $\{B_\varepsilon(x_j): j\in J\}\subset\{B_\varepsilon(x): x\in\mathbb{R}^n\setminus A_a(f)\}$ such that the $B_\varepsilon(x_j)$'s are mutually disjoint, J is at most countable and

$$\bigcup_{x\in\mathbb{R}^n\setminus A_a(f)} B_\varepsilon(x) \subset \bigcup_{j\in J} B'_\varepsilon(x_j), \tag{5.10}$$

where $B'_\varepsilon(x_j)$ $(j\in J)$ denotes the ball centered in x_j and with diameter $5(\operatorname{diam} B_\varepsilon(x_j))$. By virtue of (5.1), (5.9), (5.10), it follows that

$$\begin{aligned}\mu_a(\mathbb{R}^n\setminus A_a(f)) &= \lim_{\varepsilon\to 0} \mu_{a,5\varepsilon}(\mathbb{R}^n\setminus A_a(f)) \leqq \liminf_{\varepsilon\to 0} \sum_{j\in J} (\operatorname{diam} B'_\varepsilon(x_j))^a \\ &\leqq \lim_{\varepsilon\to 0} 10^a \sum_{j\in J} \int_{B_\varepsilon(x_j)} |f|\,dx \leqq 10^a \int_{\mathbb{R}^n} |f|\,dx < +\infty,\end{aligned}$$

and the Hausdorff dimension of $\mathbb{R}^n\setminus A_a(f)$ does not exceed a.

---

[1] In Lemma 5.1 and Theorem 5.3 the dimension of space n is any integer $\geqq 1$ (and is not restricted as everywhere else to 2 or 3).

THEOREM 5.3. *Let G be an open subset in $\mathbb{R}^n$ and let $T \in (0, \infty)$, $1 < p < \infty$ and $f \in L^p(G \times (0, T))$ be given. Let moreover*

$$v \in L^\infty(0, T; L^2(G)) \cap L^2(0, T; H_0^1(G)) \tag{5.11}$$

be such that

$$\frac{\partial v}{\partial t} - \nu \, \Delta v = f \quad \textit{in the distribution sense in } G \times (0, T) \tag{5.12}$$

and

$$v(t) \in \mathscr{C}(\bar{G}) \quad \textit{for a.e. } t \in (0, T) \quad \textit{and}^2 \quad v(0) \in \mathscr{C}(\bar{G}). \tag{5.13}$$

Then there exists a subset $G_0 \subset G$ such that

$$\operatorname*{ess\,sup}_{t \in (0,T)} |v(x, t)| < \infty \quad \textit{if } x \in G \setminus G_0 \tag{5.14}$$

and the Hausdorff dimension of G_0 is $\leqq \max\{n + 2 - 2p, 0\}$.

Proof. We infer from (5.11)–(5.12) and the regularity theory for the heat equation (cf. O. A. Ladyzhenskaya–V. A. Solonnikov–N. N. Ural'ceva [11]) that:

$$\frac{\partial v}{\partial t} \in L^p(G \times (0, T)) \quad \text{and} \quad v \in L^p(0, T; W^{2,p}(G))$$

(where $W^{2,p}(G)$ is the usual Sobolev space of functions defined on G, which together with all their derivatives up to order 2 belong to $L^p(G)$). If $\phi \in C_0^\infty(G)$ and $v_1 = \phi v$, then v_1 will satisfy (5.11)–(5.13), and also

$$\frac{\partial v_1}{\partial t} - \nu \, \Delta v_1 = f_1 \quad \text{in the distribution sense in } G \times (0, T),$$

where f_1 is a suitable function which also belongs to $L^p(G \times (0, T))$.

Therefore, taking into account that the Hausdorff measures are Borel measures, we can consider only the special case

$$v(t) \in \mathscr{C}(G), \quad G = \mathbb{R}^n, \quad \operatorname{supp} v(t) \subset K \quad \text{for all } t \in (0, T) \setminus \omega, \quad 0 \notin \omega \tag{5.15}$$

where K is a certain compact set $\subset \mathbb{R}^n$ while ω is of Lebesgue measure $=0$. It is clear that in this case $v(x, t)$ coincides almost everywhere in $R^n \times (0, T)$ with the function

$$w = w_0 + w_1,$$

---

[2] It is well known (cf. for instance J. L. Lions [1] or R. Temam [RT, Chap. III, Lemma 1.2]) that if v satisfies (5.11) and (5.12) then v is equal for almost every $t \in (0, T)$ to a function in $\mathscr{C}([0, T]; L^2(G))$ and therefore $v(0)$ makes sense.

where

$$(5.16)\qquad \begin{aligned} w_0(x,t) &= \int_{\mathbb{R}_n} E(x-y,t)v(y,0)\,dy, \\ w_1(x,t) &= \int_0^t \int_{\mathbb{R}^n} E(x-y,\tau)f(y,\tau)\,dy\,d\tau \end{aligned}$$

and

$$E(x,t) = \frac{1}{(2\sqrt{\pi\nu t})^n}\exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Therefore we can assume that, for any $t\in(0,T)\backslash\omega$, $v(x,t)=w(x,t)$ everywhere on $\mathbb{R}^n$. It follows that for such t's and for all $x_0\in\mathbb{R}^n$, we have

$$(5.17)\qquad \begin{aligned} &\frac{1}{r^n}\int_{|x-x_0|\leq r} |v(x,t)-w_0(x,t)|\,dx \\ &\leq \frac{1}{r^n}\int_{|x-x_0|\leq r} |w_1(x,t)|\,dx \\ &\leq \int_0^t \int_{\mathbb{R}^n} E(x_0-x,t-\tau)\frac{1}{r^n}\int_{|y-x|\leq r} |f(y,\tau)|\,dy\,d\tau\,dx \\ &\leq \int_0^t \int_{\mathbb{R}^n} E(x_0-x,t-\tau)f^*(x,\tau)\,dx\,d\tau = w_1^*(x,t) \end{aligned}$$

(by (4.16) and the change of variables $x'=y-x+x_0$, $y'=y$)

where, for all $\{x,s\}\in\mathbb{R}^n\times(0,\infty)$,

$$f^*(x,s) = \sup_r \frac{1}{r^n}\int_{|y-x|\leq r} |f(y,s)|\,dy.$$

From the classical theorem on the maximal functions (see E. M. Stein [1]), we have

$$(5.18)\qquad f^*\in L^p(\mathbb{R}^n\times(0,T)).$$

Since for $t\in[0,T]\backslash\omega$, $v(x,t)-w_0(x,t)$ is continuous in x, from (5.17) we finally infer that

$$(5.19)\qquad |v(x,t)|\leq w_0(x,t)+w_1^*(x,t) \quad \text{for all } x\in\mathbb{R}^n \quad \text{and} \quad t\in(0,T)\backslash\omega.$$

But, setting

$$M_j = \{(y,\tau): |y-x|\leq 2^{-j},\ \tau\geq 0,\ t-\tau\leq 2^{-2j}\} \quad \text{for } j=1,2,\ldots,$$

we have for all $x \in \mathbb{R}^n$, $t \in (0, T)$, and because of standard estimates on E:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} E(x-y, t-\tau) f^*(y, \tau)\, dy\, d\tau \\
&\leqq c_n' \iint \frac{1}{(|x-y|^2 + 4\nu(t-\tau))^{n/2}} f^*(y, \tau)\, dy\, d\tau \\
&\leqq c_n' \Bigg(\iint_{(\mathbb{R}^n \times (0,T)) \setminus M_1} \frac{1}{|x-y|^n} f^*(y, \tau)\, dy\, d\tau \\
&\qquad + \sum_{j=1}^{\infty} \iint_{M_j \setminus M_{j+1}} \frac{1}{(|x-y|^2 + 4\nu(t-\tau))^{n/2}} f^*(y, \tau)\, dy\, d\tau \Bigg) \\
&\leqq c_n' \Bigg(c_n'' \left(\int_0^T \int_{\mathbb{R}^n} f^*(y, \tau)^p\, dy\, d\tau \right)^{1/p} \\
&\qquad + \sum_{j=1}^{\infty} 2^{n(j+1)} \left(\frac{1}{(1+4\nu)^{n/2}} \right) \iint_{M_j \setminus M_{j+1}} f^*(y, \tau)\, dy\, d\tau \Bigg) \\
&\leqq c_{n,v}' \Bigg(\left(\int_0^T \int_{\mathbb{R}^n} f^*(y, \tau)\, dy\, d\tau \right)^{1/p} \\
&\qquad + \sum_{j=1}^{\infty} 2^{nj} (2^{-2j} 2^{-nj})^{(p-1)/p} \left(\iint_{M_j} f^*(y, \tau)^p\, dy\, d\tau \right)^{1/p} \\
&\leqq c_{n,v,f}' + \sum_{j=1}^{\infty} 2^{(n-[(n+2)(p-1)/p])j} \left(\int_{|y-x| \leqq 2^{-j}} \int_0^T f^*(y, \tau)^p\, d\tau\, dy \right)^{1/p} \\
&= c_{n,v,f}' + c_{n,v} \sum_{j=1}^{\infty} 2^{[(n+2)/p]-2)j} \left(\int_{|y-x| \leqq 2^{-j}} g(y)\, dy \right)^{1/p}
\end{aligned}
\tag{5.20}
$$

where $c_n', c_n'', \ldots$ are suitable constants (with respect to $\{x, t\}$) and

$$g(\cdot) = \int_0^T f^*(\cdot, \tau)^p\, d\tau \in L^1(\mathbb{R}^n) \qquad \text{(see (5.18))}.$$

Therefore $A_a(g)$ (see Lemma 5.1) makes sense, and if

$$a > \max\{0, n+2-2p\},$$

obviously

$$\sup_{t \in (0,T)} w_1^*(x, t) < \infty \quad \text{for } x \in A_a(g).$$

Consequently (since w_0 is continuous on $\mathbb{R}^n \times (0, T)$), by virtue of (5.19) we have

$$\operatorname*{ess\,sup}_{t \in (0,T)} |v(x, t)| < \infty \quad \text{for } x \in A_a(g).$$

Therefore the set

$$G_0 : \left\{ x \in \mathbb{R}^n : \operatorname*{ess\,sup}_{t \in (0,T)} |v(x, t)| = \infty \right\}$$

is included in $\mathbb{R}^n \setminus A_a(g)$. From Lemma 5.1 we infer that the Hausdorff dimension of G_0 is $\leqq a$. The conclusion (5.14) is now obtained by letting $a \to \max\{0, n+2-2p\}$.

Remark 5.1. The proof of Theorem 5.2 shows that the result remains valid if $G = Q$ and we replace (5.11) by

$$v \in L^\infty(0, T; L^2(G)) \cap L^2(0, T; H^1_p(G)).$$

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. Obviously it is sufficient to prove the result for a fixed $T \in (0, \infty)$. Also it is clear that each component $v(x, t) = u_j(x, t)$ $(j = 1, 2, 3)$ of $u(t)$ satisfies (5.11)–(5.12). Since

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; L^6(\Omega)),$$

we find by interpolation that $u \in L^r(0, T; L^{6r/(3r-4)}(\Omega))$ for every $r \geqq 2$ (see J. L. Lions–J. Peetre [1]), whereupon, for $r = \frac{10}{3}$, we obtain

$$u \in L^{10/3}(0, T; L^{10/3}(\Omega)).$$

It follows that

$$\psi = \sum u_i D_i u \in L^{5/4}(0, T; L^{5/4}(\Omega)) \tag{5.21}$$

since

$$L^{10/3} \cdot L^2 \subset L^{5/4}.$$

We then get

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu \,\Delta u + \operatorname{grad} p = h \quad \text{in } \mathcal{O} \times (0, T), \\ &\operatorname{div} u = 0, \\ &u = 0 \quad \text{on}^3\ \Gamma \times (0, T), \\ &u = u_0 \in \mathbb{H}^2(\Omega) \quad \text{for } t = 0, \end{aligned} \tag{5.22}$$

where (see (5.21))

$$h = f - \psi \in L^{5/4}(\Omega \times (0, T)).$$

By referring to the regularity theory for the Stokes system (5.22) (cf. K. K. Golovkin–O. A. Ladyzhenskaya [1], K. K. Golovkin–V. A. Solonnikov [1], V. A. Solonnikov [2]), we see that

$$D_j p \in L^\alpha(\Omega \times (0, T))$$

[3] This boundary condition must be replaced by the periodicity condition when $\mathcal{O} = Q$; see Remark 5.1.

for any $1<\alpha<\frac{5}{4}$. Thus for $v(x, t)=u_j(x, t)$ we obtain

$$\frac{\partial v}{\partial t}-\nu\,\Delta v=h_j-D_jp\in L^\alpha(\Omega\times(0, T))$$

on $\Omega\times(0, T)$, so that by virtue of Theorem 5.3, we obtain that the set

$$\Omega_0=\left\{x\in\Omega\colon \operatorname*{ess\,sup}_{t\in(0,T)}|v(x, t)|=\infty\right\}$$

has the Hausdorff dimension $\leqq 5-2\alpha$. The conclusion follows by letting $\alpha\to\frac{5}{4}$. $\square$

6 Successive Regularity and Compatibility Conditions at $t = 0$ (Bounded Case)

In this section we assume that $n=2$ or 3 and consider the flow in a bounded domain $\Omega \subset \mathbb{R}^n$. We are interested in the study of higher regularity properties of strong solutions, assuming that the data u_0, f, Ω possess further regularity properties. While this question is essentially solved in § 4 for the space periodic case (through Lemmas 4.2 and 4.1), the situation in the bounded case is more involved. In particular, we derive in this section the so-called *compatibility conditions* for the Navier–Stokes equations, i.e., the necessary and sufficient conditions on the data (on $\partial\Omega$ at $t=0$) for the solution u to be smooth up to time $t=0$.

6.1. Further properties of the operators A and *B*. We recall that if Ω satisfies (1.7) with $r=m+2$, and if m is an integer $>n/2$ ($m \geqq 2$ for $n=3,2$), then $H^m(\Omega)$ is a multiplicative algebra:

(6.1) If $u, v \in H^m(\Omega)$, $m \geqq 2$, $n \leqq 3$ then $u \cdot v \in H^m(\Omega)$ and $|u \cdot v|_m \leqq c_m(\Omega)\, |u|_m\, |v|_m$.

For every integer m we define the space

$$E_m = \mathbb{H}^m(\Omega) \cap H, \tag{6.2}$$

which is a Hilbert space for the norm induced by $\mathbb{H}^m(\Omega) = H^m(\Omega)^n$. It is clear that $E_{m+1} \subset E_m$, for all m and that $E_0 = H$:

$$E_{m+1} \subset E_m \subset \cdots \subset E_1 \subset E_0 = H. \tag{6.3}$$

The orthogonal projection from $\mathbb{L}^2(\Omega)$ onto H being denoted by P as before, we introduce the operator $\mathcal{A}$

$$\mathcal{A}u = -P\,\Delta u, \tag{6.4}$$

which is linear continuous from E_m, $m \geqq 2$, into $E_0 = H$. Actually the operator P is linear continuous from $\mathbb{H}^m(\Omega)$ into $\mathbb{H}^m(\Omega) \cap H = E_m$ (cf. [RT, Chap. I, Remark 1.6]), and therefore

(6.5) $\mathcal{A}$ is linear continuous from E_{m+2} into E_m.

Now if $u \in E_{m+2} \cap V$, $m \geqq 0$, and $\mathcal{A}u = f \in E_m$, then $\Delta u + f = (I-P)\,\Delta u$ belongs to $H^\perp$, which amounts to saying (cf. § 2.5) that there exists $p \in H^1(\Omega)$ such that $\Delta u + f = \nabla p$. Hence

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega,\\ \operatorname{div} u &= 0 \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.6}$$

and u is the solution to the Stokes problem associated with f. The theorem on the regularity of the solutions of Stokes problem mentioned in § 2.5 implies then that

(6.7) $\qquad \mathscr{A}$ is an isomorphism from $E_{m+2} \cap V$ onto[1] $E_m, \qquad m \geqq 0.$

It is clear from (6.6) that[2]

$$\mathscr{A}u = Au \quad \forall\, u \in D(A). \tag{6.8}$$

Operator B. Lemma 2.1 shows us that $b(u, v, w)$ is a trilinear continuous form on $\mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega) \times H$ if $m_1 + m_2 > n/2$ (or $m_1 + m_2 \geqq n/2$ if $m_i \neq 0$ for all i). This amounts to saying that the operator B defined in (2.41) is bilinear continuous from $\mathbb{H}^{m_1}(\Omega) \times \mathbb{H}^{m_2+1}(\Omega)$ into H, under the same conditions on m_1, m_2. Clearly $(u \cdot \nabla)v$ belongs in this case to $\mathbb{L}^2(\Omega)$ and

$$B(u, v) = P[(u \cdot \nabla)v]. \tag{6.9}$$

We have

LEMMA 6.1.

$$B(E_{m+1} \times E_{m+1}) \in E_m \quad \textit{for } m \geqq 1. \tag{6.10}$$

Proof. If $m = 1$, $u, v \in \mathbb{H}^2(\Omega)$, then obviously $(u \cdot \nabla)v \in \mathbb{L}^2(\Omega)$ and

$$D_i((u \cdot \nabla)v) = (D_i u) \cdot (\nabla v) + u \cdot (D_i \nabla v)$$

belongs to $\mathbb{L}^2(\Omega)$, since the first derivatives of u and v are in $\mathbb{L}^6(\Omega)$ (at least) and u and v are continuous on $\bar{\Omega}$. The operator P mapping $\mathbb{H}^r(\Omega)$ into $\mathbb{H}^r(\Omega) \cap H$, $B(u, v) = P((u \cdot \nabla)v)$ is in $\mathbb{H}^1(\Omega) \cap H = E_1$.

If $m \geqq 2$, then u_i and $D_i v_j$ are in $H^2(\Omega)$, and their product is too, due to (6.1). Hence $(u \cdot \nabla)v \in \mathbb{H}^m(\Omega)$ and $P((u \cdot \nabla)v)$ as well[3].

6.2. Regularity results. If u_0 and f are given, satisfying

$$u_0 \in V, \qquad f \in L^2(0, T; H), \tag{6.11}$$

then Theorem 3.2 asserts the existence of a (strong) solution u of Problem 2.2 defined on some interval $[0, T']$, $T' = T$ if $n = 2$, $T' \leqq T$ if $n = 3$. Changing our notation for convenience, we write T instead of T', and

$$u \in L^2(0, T; D(A)) \cap \mathscr{C}([0, T]; V), \tag{6.12}$$

$$u' \in L^2(0, T; H). \tag{6.13}$$

[1] If $\mathscr{G}$ is the inverse of $\mathscr{A}|_{E_{m+2} \cap V}$, $\mathscr{G}\mathscr{A}u = u$, for all $u \in E_{m+2} \cap V$, but in general $\mathscr{G}\mathscr{A}u \neq u$ for an arbitrary $u \in E_{m+2}$, $m \geqq 0$.

[2] Because of (6.7) and (6.8), V_m is a closed subspace of E_m, for all $m \in N$, the norm induced by E_m being equivalent to that of V_m. In the space periodic case, the analogue of $E_m = \dot{\mathbb{H}}^m_p(Q) \cap H$ is equal to V_m.

[3] The proof shows also that $B(E_m \times E_{m+1}) \subset E_m$ for $m \geqq 2$.

Our goal is to establish further regularity properties on u, assuming more regularity on u_0, f.

We introduce the space

$$\mathcal{W}_m = \left\{v \in \mathscr{C}([0, T]; E_m),\ v^{(j)} = \frac{d^j v}{dt^j} \in \mathscr{C}([0, T]; E_{m-2j}),\ j = 1, \ldots, l\right\}, \tag{6.14}$$

where m is an integer and $l = [m/2]$ is the integer part of $m/2$, i.e. $m = 2l$ or $m = 2l+1$.

We begin by assuming that for some $m \geqq 2$

$$u_0 \in E_m \cap V, \qquad f \in \mathcal{W}_{m-2}, \tag{6.15}$$

and we make the following observation:

LEMMA 6.2. *If u is a strong solution to the Navier–Stokes equations and u_0 and f satisfy (6.15) for some $m \geqq 2$, then the derivatives at $t = 0$ of u, $u^{(j)}(0) = (d^j u/dt^j)(0)$, $j = 1, \ldots, l$, can be determined "explicitly" in terms of u_0 and $f(0)$, and*

$$u^{(j)}(0) \in E_{m-2j}, \qquad j = 1, \ldots, l = \left[\frac{m}{2}\right]. \tag{6.16}$$

Proof. By successive differentiation of (2.43), we find

$$\frac{du^{(j)}}{dt^j} + \nu\mathcal{A}u^{(j)} + \beta^{(j)} = f^{(j)}, \tag{6.17}$$

where we set $\phi^{(j)} = d^j\phi/dt^j$, $\beta(t) = Bu(t)$, so that

$$\beta^{(j)} = \sum_{i=0}^{j} \binom{j}{i} B(u^{(j-i)}, u^{(i)}). \tag{6.18}$$

Now by (6.5), (6.10), (6.15), $\beta(0) = Bu(0) \in E_{m-1}$, and $u'(0) = -\nu\mathcal{A}u(0) - B(u(0)) + f(0) \in E_{m-2}$. Similarly, if $m \geqq 4$,

$$u''(0) = -\nu\mathcal{A}u'(0) - B(u'(0), u(0)) - B(u(0), u'(0)) + f'(0) \in E_{m-4}.$$

The proof continues by induction on j using (6.5), (6.10), (6.15), and shows that

$$\begin{aligned} u^{(j)}(0) &= -\nu\mathcal{A}u^{(j-1)}(0) - \sum_{i=0}^{j-1} \binom{j-1}{i} B(u^{(j-1-i)}(0), u^{(i)}(0)) + f^{(j-1)}(0) \\ &\in E_{m-2j}, \end{aligned} \tag{6.19}$$

for $j = 1, \ldots, l$. □

It is known that for an initial and boundary value problem, the solutions may not be smooth near $t = 0$, even if the data are $\mathscr{C}^\infty$. For Navier–Stokes equations, the following theorem gives the necessary and sufficient conditions on the data for regularity up to $t = 0$, the compatibility conditions (cf. (6.21)).

THEOREM 6.1. *We assume that* $n=2$ *or* 3, Ω *satisfies* (1.7) *with* $r=m+2$, $m\geqq 2$, u_0 *and* f *satisfy* (6.15) *and*

$$\frac{d^l f}{dt^l}\in\begin{cases}L^2(0,T;H) & \text{if } m=2l+1,\\ L^2(0,T;V') & \text{if } m=2l,\end{cases}\tag{6.20}$$

and that u *is a (strong) solution of Problem* 2.2.

Then a necessary and sufficient condition for u *to belong to* $\mathcal{W}_m$ *is that*

$$\frac{d^j u}{dt^j}(0)\quad (\text{as given by (6.19)})\in V\tag{6.21}$$

$$\text{for } j=1,\dots,l \quad \text{if } m=2l+1,\quad j=1,\dots,l-1\quad \text{if } m=2l.$$

Proof. i) We first show that the condition (6.20) (which is void for $m=2$) is necessary if $m\geqq 3$.

We know that u is in $\mathscr{C}([0,T];V)$. If u belongs to $\mathcal{W}_m$, then $u\in\mathscr{C}([0,T];E_m)$, $u'\in\mathscr{C}([0,T];E_{m-2})$, and necessarily u' takes its values in $E_{m-2}\cap V$, so that $u'\in\mathscr{C}([0,T];E_{m-2}\cap V)$ and $u'(0)\in V$. The proof is the same for the other derivatives.

We now prove that under conditions (6.15), (6.20) and (6.21) u is in $\mathcal{W}_m$.

ii) For $j=1$, the equation

$$\frac{du^{(1)}}{dt}+\nu\mathscr{A}u^{(1)}+B(u,u^{(1)})+B(u^{(1)},u)=f^{(1)}\tag{6.22}$$

($u^{(1)}=u'$), together with $u^{(1)}(0)\in V$ given by (6.19), allows us to show that[4]

$$u^{(1)}\in L^2(0,T;D(A))\cap\mathscr{C}([0,T];V).\tag{6.23}$$

We continue by induction; once we establish that

$$u^{(j)}\in L^2(0,T;D(A))\cap\mathscr{C}([0,T];V),\qquad j=0,\dots,l-1,\tag{6.24}$$

we consider (6.17), (6.18) with $j=l$ and $u^{(l)}(0)$ given by (6.19) and belonging to V (if $m=2l+1$) or to H (if $m=2l$). In a similar manner, we show that

$$u^{(l)}\in\begin{cases}L^2(0,T;D(A))\cap\mathscr{C}([0,T];V) & \text{if } m=2l+1,\\ L^2(0,T;V)\cap\mathscr{C}([0,T];H) & \text{if } m=2l.\end{cases}\tag{6.25}$$

iii) The next step in the proof consists in showing that

$$u^{(j)}\in\mathscr{C}([0,T];D(A))\quad\text{for } j=0,\dots,l-1.\tag{6.26}$$

---

[4] The fact that $u^{(1)}\in L^2(0,T;D(A))\cap L^\infty(0,T;V)$ is obtained by deriving one more a priori estimate for the Galerkin approximation u_m of u (cf. (3.42)–(3.45)): that u_m' *belongs to a bounded set of* $L^2(0,T;D(A))\cap L^\infty(0,T;V)$. As (3.47), (3.52), this estimate is obtained by differentiating (3.45) with respect to t and taking the scalar product in H of the differentiated equation with u_m' and Au_m'.

The fact that $u\in\mathscr{C}([0,T];V)$ follows by interpolation, like (2.50), after we show that $u''=f'-\nu Au'-B(u',u)-B(u,u')\in L^2(0,T;H)$. It is clear that f' and $\nu Au'$ are in $L^2(0,T;H)$. For the quadratic terms, it follows from (2.47), (2.48) that B maps $(L^2(0,T;D(A))\cap L^\infty(0,T;V))^2$ into $L^4(0,T;H)$.

For that purpose we write (6.17), (6.18) in the form

$$\nu \mathcal{A} u^{(j)} = -u^{(j+1)} - \sum_{i=0}^{j} \binom{j}{i} B(u^{(j-i)}, u^{(i)}) + f^{(j)}; \tag{6.27}$$

$f^{(j)}$ and $u^{(j+1)}$ are in $\mathscr{C}([0, T]; H)$ (at least). Because of Lemma 2.1 and (2.36), B is bilinear continuous from $V \times V$ into $V_{-1/2}$. Hence by (6.24), for $i = 0, \ldots, j$, and $j = 0, \ldots, l-1$,

$$B(u^{(j-i)}, u^{(i)}) \in \mathscr{C}([0, T]; V_{-1/2}). \tag{6.28}$$

Since A is an isomorphism from $V_{3/2}$ onto $V_{-1/2}$, (6.27) implies then that $u^{(j)} \in \mathscr{C}([0, T]; V_{3/2})$, $j = 0, \ldots, l-1$. Using again Lemma 2.1, B is a bilinear continuous operator form $V_{3/2} \times V_{3/2}$ into H. Thus, for the same values of i and j as in (6.28),

$$B(u^{(j-i)}, u^{(i)}) \in \mathscr{C}([0, T]; H),$$

and (6.26) follows.

iv) Finally we show that $u \in \mathscr{W}_m$, i.e.,

$$u^{(j)} \in \mathscr{C}([0, T]; E_{m-2j}), \qquad j = 0, \ldots, l. \tag{6.29}$$

For $j = l$, this is included in (6.25). For $j = l-1$, using (6.10) and (6.26) we find that

$$\sum_{i=0}^{j} \binom{j}{i} B(u^{(j-i)}, u^{(i)}) \in \mathscr{C}([0, T]; E_1),$$

and then (6.5), (6.27) and (6.25) show that $u^{(l-1)} \in \mathscr{C}([0, T]; E_{m-2l+2})$. The proof continues by induction for $j = l-2, \ldots, 0$.

Theorem 6.1 is proved. □

6.3. Other results. We particularize Theorem 6.1 to the case $m = 3$.

THEOREM 6.2. *We assume that $n = 2$ or 3, Ω satisfies* (1.7) *with $r = 5$, $u_0 \in \mathbb{H}^3(\Omega) \cap V$, $f \in \mathscr{C}([0, T]; \mathbb{H}^1(\Omega) \cap H)$ and $df/dt \in L^2(0, T; H)$. Then the solution u of Problem* 2.2 *defined by Theorem* 3.2 *on some interval $[0, T_*]$, $0 < T_* \leq T$, belongs to $\mathscr{C}([0, T_*]; \mathbb{H}^3(\Omega) \cap V)$, if and only if*

$$-\nu P\, \Delta u_0 + P\left(\sum_{i=1}^{n} u_{0i} \frac{\partial u_0}{\partial x_i}\right) + f(0) = 0 \quad \textit{on } \Gamma. \tag{6.30}$$

The only condition left in (6.21) is (6.30). Since $u'(0) \in E_1 = \mathbb{H}^1(\Omega) \cap H$, $u'(0) \cdot \boldsymbol{\nu} = 0$ on Γ, and (6.30) is a restrictive assumption for the tangential value on Γ of

$$\Psi_0 = -\nu\, \Delta u_0 + \sum_{i=1}^{n} u_{0i} \frac{\partial u_0}{\partial x_i} + f(0) \in \mathbb{H}^1(\Omega). \tag{6.31}$$

Another way to formulate (6.30) is this: $\Psi_0 = P\Psi_0 + \nabla q_0$, where (cf. [RT, Chap.

I, Thm. 1.5]) q_0 is a solution of the Neumann problem:

$$\Delta q_0 = \operatorname{div} \Psi_0 \quad \text{in } \Omega, \qquad \frac{\partial q_0}{\partial \boldsymbol{\nu}} = \Psi_0 \cdot \boldsymbol{\nu} \quad \text{on } \Gamma; \tag{6.32}$$

Ψ_0 must be such that the tangential components on Γ of Ψ_0 and ∇q_0 coincide:

$$\nabla_\tau q_0 = (\Psi_0)_\tau \quad \text{on } \Gamma; \tag{6.33}$$

(6.32), (6.33) constitute an overdetermined boundary value problem (Cauchy problem on Γ for the Laplacian) which does possess a solution.

If the compatibility conditions (6.21) are not satisfied, the other hypotheses of Theorem 6.1 being verified, we get a similar result of regularity on $\Omega \times (0, T]$:

THEOREM 6.3. *We make the same hypotheses as in Theorem* 6.1, *but the conditions in* (6.21) *are not satisfied*[5]. *Then*

$$\frac{d^j u}{dt^j} \in \mathscr{C}((0, T]; E_{m-2j}), \qquad j = 1, \ldots, l. \tag{6.34}$$

Proof. Thanks to (3.39), there exists $0 < t_0 < T_*$, t_0 arbitrarily close to 0, such that $u(t_0) \in D(A)$. Equation (6.19) for t_0 and $j = 1$ shows that $u'(t_0) \in H$. We conclude as in Theorem 6.1 or 3.2 that $u' \in L^2(t_0, T_*; V) \cap \mathscr{C}([t_0, T_*]; H)$.

We choose t_1, $t_0 < t_1 < T_*$, t_1 arbitrarily close to t_0, such that $u'(t_1) \in V$, and conclude that $u' \in L^2(t_1, T_*; D(A)) \cap \mathscr{C}([t_1, T_*]; V)$. We then choose t_2, $t_1 < t_2 < T_*$, t_2 arbitrarily close to t_1, such that $u'(t_2) \in D(A)$, etc.

Finally we get that $u \in \mathscr{C}([t_l, T_*]; E_m)$, $u' \in \mathscr{C}([t_l, T_*]; E_{m-2}), \ldots$, for t_l arbitrarily close to 0, and the result is proved. □

Remark 6.1. If Γ is $\mathscr{C}^\infty$, $u_0 \in \mathscr{C}^\infty(\bar{\Omega})^n \cap H$, $f \in \mathscr{C}^\infty(\bar{\Omega} \times [0, T])^n$, then $u \in \mathscr{C}^\infty(\bar{\Omega} \times [0, T])^n$. The $\mathscr{C}^\infty$ regularity in $\bar{\Omega} \times (0, T]$ was proved in O. A. Ladyzhenskaya [1]. Of course, by combining Theorems 6.1 and 6.2 with the imbedding theorems of Sobolev spaces into spaces of continuously differentiable functions, one can get partial results of regularity in spaces of $\mathscr{C}^k$ functions.

Remark 6.2. Let u be a weak solution of the Navier–Stokes equations (Problem 2.1) in the three-dimensional case. Let $\mathscr{O}_1$ be the set of $\mathbb{H}^1$-regularity of u defined in Theorem 4.1 and Remark 4.1, and let $\mathscr{E} = [0, T] \backslash \mathscr{O}_1$. According to Theorem 6.3, if $u_0 \in H$, $f \in \mathscr{W}_{m-2}$ and hypothesis (6.20) is verified, $\mathscr{O}_1$ is also an interval of $\mathbb{H}^r$-regularity, $r \leq m$, i.e., $u \in \mathscr{C}(\mathscr{O}_1; \mathbb{H}^m(\Omega) \cap V)$. This is the analogue of Theorem 4.1 for the case of a bounded domain.

Remark 6.3. Let u be a weak solution of Navier–Stokes equations (Problem 2.1) in the three-dimensional case. We assume that $u_0 \in \mathbb{H}$, $f \in \mathbb{H}^{m-2}(\Omega) \cap H$, $m \geq 3$, is independent of t. Then for every $t \in \mathscr{O}$, $u'(t)$ makes sense and

[5] It suffices also to assume that $u_0 \in V$ instead of $u_0 \in E_m \cap V$.

belong to V. If we introduce the analogue $\psi(t)$ of (6.31)

$$\psi(t) = -\nu\, \Delta u(t) + \sum_{i=1}^{3} u_i(t) \frac{\partial u(t)}{\partial x_i} + f(t),$$

and write that the overdetermined boundary value problem similar to (6.32), (6.33) possesses a solution, we conclude that $u(t)$ belongs, for t in $\mathcal{O}_1$, to a complicated "manifold" of V: ψ depends on $u(t)$ (and f); $q(t) = \mathcal{N}(\operatorname{div} \psi(t), \psi(t) \cdot \boldsymbol{\nu})$, where $\mathcal{N}$ is the "Green's function" of the Neumann problem (6.32): the condition (6.33) is the "equation" of the manifold.

A similar remark holds for every $t > 0$ for a strong solution; similar remarks follow from the other conditions (6.21) if $m \geqq 5$.

Remark 6.4. If we introduce pressure, then we clearly get regularity results for pressure:

$$\nabla p = -\frac{\partial u}{\partial t} + \nu\, \Delta u - (u \cdot \nabla) u + f.$$

In the situation of Theorem 6.1,

$$p \in \mathscr{C}([0, T]; H^{m-1}(\Omega)), \qquad \frac{\partial^j p}{\partial t^j} \in \mathscr{C}([0, T]; H^{m-1-2j}(\Omega)), \quad j = 1, \ldots, l-1, \tag{6.35}$$

and in the situation of Theorem 6.3,

$$p \in \mathscr{C}((0, T]; H^{m-1}(\Omega)), \qquad \frac{\partial^j p}{\partial t^j} \in \mathscr{C}((0, T]; H^{m-1-2j}(\Omega)), \quad j = 1, \ldots, l-1. \tag{6.36}$$

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7 Analyticity in Time

In this section we prove that the strong solutions are analytic in time as $D(A)$-valued functions. We assume for simplicity that $f \in H$ is independent of t: we could as well assume that f is an H-valued analytic function in a neighborhood in $\mathbb{C}$ of the positive real axis. The interest of the proof given below is that it is quite simple and relies on the same type of method as that used for existence. The method applies also to more general nonlinear evolution equations with an analytic nonlinearity.

7.1. The analyticity result. The main result is the following one:

THEOREM 7.1. *Let there be given u_0 and f, $u_0 \in V$, $f \in H$ independent of t.*

If $n=2$, the (strong) solution u of Problem 2.2 given by Theorem 3.2 is analytic in time, in a neighborhood of the positive real axis, as a $D(A)$-valued function.

If $n=3$, the (strong) solution u of Problem 2.2 given by Theorem 3.2 is analytic in time, in a neighborhood in $\mathbb{C}$ of the interval $(0, T_*)$, as a $D(A)$-valued function.

Proof. i) Let $\mathbb{C}$ denote the complex plane and $H_{\mathbb{C}}$ the complexified space of H, whose elements are denoted $u+iv$, $u, v \in H$, $i=\sqrt{-1}$. Similarly $V_{\mathbb{C}}$, $V'_{\mathbb{C}}$ are the complexified V, V', and $X_{\mathbb{C}}$ is the complexified space of a real space X. By linearity, A (resp. P_m, resp. B) extends to a selfadjoint operator in $H_{\mathbb{C}}$ (resp. to the orthogonal projection in $H_{\mathbb{C}}$, $V_{\mathbb{C}}$, $V'_{\mathbb{C}}$, onto the space $\mathbb{C}w_1 + \cdots + \mathbb{C}w_m$, resp. a bilinear operator from $V_{\mathbb{C}} \times V_{\mathbb{C}}$ into $V'_{\mathbb{C}}$).

Consider now the complexified form of the Galerkin approximation of Navier–Stokes equations, i.e., (compare with (3.41)–(3.45)) the complex differential system in $P_m H_{\mathbb{C}}$:

$$\frac{du_m}{d\zeta}(\zeta) + \nu A u_m(\zeta) + P_m B(u_m(\zeta), u_m(\zeta)) = P_m f, \tag{7.1}$$

$$u_m(0) = P_m u_0, \tag{7.2}$$

where $\zeta \in \mathbb{C}$, and u_m maps $\mathbb{C}$ (or an open subset of $\mathbb{C}$) into $P_m H_{\mathbb{C}} = \mathbb{C}w_1 + \cdots + \mathbb{C}w_m$. The complex differential system (7.1), (7.2) possesses a unique solution u_m defined in $\mathbb{C}$ in some neighborhood of the origin.

It is clear that the restriction of $u_m(\zeta)$ to some interval $(0, T_m)$ of the real axis coincides with the Galerkin approximation $u_m(t)$ defined in the real field by (3.41)–(3.45).

ii) As in the real case, we now prove some a priori estimates on u_m. We take the scalar product in $H_{\mathbb{C}}$ of (7.1) with $Au_m(\zeta)$

$$\left(\left(\frac{du_m(\zeta)}{d\zeta}, u_m(\zeta)\right)\right) + \nu\,|Au_m(\zeta)|^2 + \langle B(u_m(\zeta), u_m(\zeta)), Au_m(\zeta)\rangle = \langle f, Au_m(\zeta)\rangle.$$

We multiply this relation by $e^{i\theta}$, with $\zeta = se^{i\theta}$, and for fixed $\theta, |\theta| < \pi/2$, we get:

$$\operatorname{Re} e^{i\theta}\left(\left(\frac{du_m(\zeta)}{d\zeta}, u_m(\zeta)\right)\right)\Big|_{\zeta=se^{i\theta}} = \operatorname{Re}\left(\left(\frac{du_m(se^{i\theta})}{ds}, u_m(se^{i\theta})\right)\right) = \frac{1}{2}\frac{d}{ds}\|u_m(se^{i\theta})\|^2.$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|u_m(se^{i\theta})\|^2 + \nu\cos\theta\,|Au_m(se^{i\theta})|^2 \\ = -\operatorname{Re} e^{i\theta}\langle B(u_m(se^{i\theta}), u_m(se^{i\theta})), Au_m(se^{i\theta})\rangle \\ + \operatorname{Re} e^{i\theta}\langle f, Au_m(se^{i\theta})\rangle.\end{aligned} \tag{7.3}$$

The absolute value of the term involving f is less than or equal to

$$|f|\cdot|Au_m(se^{i\theta})| \leqq \frac{\nu\cos\theta}{4}|Au_m(se^{i\theta})|^2 + \frac{1}{\nu\cos\theta}|f|^2. \tag{7.4}$$

For the term involving B, we use the inequality (2.32), which clearly extends to the complex case[1],

$$|b(u, v, w)| \leqq c_3^*\|u\|\,\|v\|^{1/2}|Av|^{1/2}|w| \quad \forall\, u \in D(A_{\mathbb{C}}),\ v \in V_{\mathbb{C}},\ w \in H_{\mathbb{C}}. \tag{7.5}$$

This allows us to write

$$\begin{aligned}&|\langle B(u_m(se^{i\theta}), u_m(se^{i\theta})), Au_m(se^{i\theta})\rangle| \\ &\leqq c_3^*\|u_m(se^{i\theta})\|^{3/2}|Au_m(se^{i\theta})|^{3/2} \\ &\leqq \frac{\nu\cos\theta}{4}|Au_m(se^{i\theta})|^2 + \frac{c'}{|\cos\theta|^3}\|u_m(se^{i\theta})\|^6\end{aligned}$$

(by Young's inequality, see (3.10)),

where c' depends on the data ν, u_0, Ω.

If we take into account this majorization and (7.4), then (7.3) becomes

$$\frac{d}{ds}\|u_m(se^{i\theta})\|^2 + \nu\cos\theta\,|Au_m(se^{i\theta})|^2 \leqq \frac{2}{\nu\cos\theta}|f|^2 + \frac{c'}{|\cos\theta|^3}\|u_m(se^{i\theta})\|^6. \tag{7.6}$$

The inequality (3.25) is still valid in the complex case

$$\|v\| \leqq \frac{1}{\sqrt{\lambda_1}}|Av| \quad \forall\, v \in V_{\mathbb{C}},$$

and thus

$$\begin{aligned}\frac{d}{ds}\|u_m(se^{i\theta})\|^2 + \nu\lambda_1\cos\theta\|u_m(se^{i\theta})\|^2 &\leqq \frac{2}{\nu\cos\theta}|f|^2 + \frac{c'}{|\cos\theta|^3}\|u_m(se^{i\theta})\|^6 \\ &\leqq \frac{1}{\cos^3\theta}\left(\frac{2}{\nu}|f|^2 + c'\right)\|u_m(se^{i\theta})\|^6.\end{aligned} \tag{7.7}$$

---

[1] This relation, which is not so good as the first relation (2.31), is valid for both dimensions, $n = 2$ and 3.

This differential inequality is of the same type as (3.27), setting $y(s) = \|u_m(se^{i\theta})\|^2$, $c_4' = (1/(\cos^3\theta))((2/\nu)|f|^2 + c')$. We conclude, as in Lemma 3.2, that there exists K_6' which depends only on f, ν and Ω (or Q), such that

$$1+\|u_m(se^{i\theta})\|^2 \leqq 2(1+\|u_{0m}\|^2) \leqq 2(1+\|u_0\|^2), \tag{7.8}$$

for

$$s \leqq |\cos\theta|^3 T_1'(\|u_0\|), \tag{7.9}$$

$$T_1'(\|u_0\|) = \frac{K_6'}{(1+\|u_0\|^2)^2}\left(\leqq T_1'(\|u_{0m}\|) = \frac{K_6'}{(1+\|u_{0m}\|^2)^2}\right). \tag{7.10}$$

This shows that the solution u_m of (7.1)–(7.2), which was defined and analytic in a neighborhood of $\zeta = 0$, actually extends to an analytic solution of this equation in an open set of $\mathbb{C}$ containing (see Fig. 7.1):

$$\Delta(u_0) = \left\{\zeta = se^{i\theta},\ 0 < s < |\cos\theta|^3 T_1'(\|u_0\|),\ |\theta| < \frac{\pi}{2}\right\}. \tag{7.11}$$

The estimate (7.8)–(7.10) shows that

$$\sup_{\zeta\in\Delta(u_0)} \|u_m(\zeta)\|^2 \leqq \text{const} = 2(1+\|u_0\|^2). \tag{7.12}$$

iii) The analyticity of u_m and Cauchy's formula allow us to deduce from (7.12) a priori estimates on the derivatives of u_m (with respect to ζ) on compact subsets of $\Delta(u_0)$. Indeed, for $\zeta \in \Delta(u_0)$ and $k \in \mathbb{N}$, $k \geqq 1$,

$$\frac{d^k u_m}{d\zeta^k}(\zeta) = \frac{k!}{2\pi i}\int_{|z-\zeta|=d/2} \frac{u_m(z)}{(z-\zeta)^{k+1}}\,dz, \tag{7.13}$$

where $d = d(\zeta, \partial\Delta(u_0))$ is the distance of ζ to the boundary $\partial\Delta(u_0)$ of $\Delta(u_0)$.

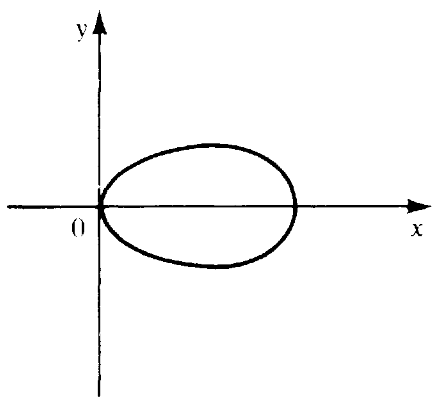

FIG. 7.1. *The region $\Delta(u_0)$ in $\mathbb{C}$.*

Therefore

$$\left\|\frac{d^k u_m}{d\zeta^k}(\zeta)\right\| \leqq \frac{2^k k!}{d^k} \cdot \sup_{z \in \Delta(u_0)} \|u_m(z)\|,$$

and by (7.12), for any compact set $K \subset \Delta(u_0)$

$$\text{(7.14)}^2 \qquad \sup_{\zeta \in K} \left\|\frac{d^k u_m(\zeta)}{d\zeta^k}\right\| \leqq \frac{2^{k+1} k!}{[d(K, \partial\Delta(u_0))]^k} (1+\|u_0\|^2)^{1/2}.$$

Considering (7.1), we find that

$$\begin{aligned} \nu\, |Au_m(\zeta)| &\leqq |f| + \left|\frac{du_m}{d\zeta}(\zeta)\right| + |B(u_m(\zeta), u_m(\zeta))| \\ &\leqq |f| + \left|\frac{du_m}{d\zeta}(\zeta)\right| + c_3 \|u_m(\zeta)\|^{3/2} |Au_m(\zeta)|^{1/2} \qquad \text{(by (2.32))} \\ &\leqq |f| + \left|\frac{du_m}{d\zeta}(\zeta)\right| + \frac{\nu}{2} |Au_m(\zeta)| + \frac{c_3^2}{2\nu} \|u_m(\zeta)\|^3. \end{aligned}$$

Thus

$$\text{(7.15)} \qquad |Au_m(\zeta)| \leqq \frac{2}{\nu} |f| + \frac{c_3^2}{\nu^2} \|u_m(\zeta)\|^3 + 2 \left|\frac{du_m}{d\zeta}(\zeta)\right|,$$

and it follows from (7.14) that for every compact subset K of $\Delta(u_0)$

$$\text{(7.16)} \qquad \sup_{\zeta \in K} |Au_m(\zeta)| \leqq c_4',$$

where $c_4' < \infty$, depends on K and the data,

$$\text{(7.17)} \quad c_4'(K) = \frac{2}{\nu} |f| + \frac{2^{3/2} c_3^2}{\nu^2} (1+\|u_0\|^2)^{3/2} + \frac{2^3}{\sqrt{\lambda_1}[d(K, \partial\Delta(u_0))]} (1+\|u_0\|^2)^{1/2}.$$

Using again Cauchy's formula (7.13) and (7.16), we obtain also for every $\zeta \in K$ and $k \in \mathbb{N}$:

$$\left|A \frac{d^k}{d\zeta^k} u_m(\zeta)\right| \leqq \frac{2^k k!}{[d(K, \partial\Delta(u_0))]^k} \sup_{z \in K'} |Au_m(\zeta)|,$$

$$\text{(7.18)} \qquad \sup_{\zeta \in K} \left|A \frac{d^k}{d\zeta^k} u_m(\zeta)\right| \leqq \frac{2^k k!\, c_4'(K')}{[d(K, \partial\Delta(u_0))]^k},$$

where K' (containing K) is the set

$$\text{(7.19)} \qquad \{z \in \Delta(u_0),\ d(z, \partial\Delta(u_0)) \geqq \tfrac{1}{2} d(K, \partial\Delta(u_0))\}.$$

iv) We now pass to the limit, $m \to \infty$. Since the set $\{v \in V_{\mathrm{C}}, \|v\| \leqq \rho\}$ is compact in H_{C} for any $0 < \rho < \infty$, we can apply to the sequence u_m the vector

[2] $d(X, Y) = \inf_{x \in X\, y \in Y} d(x, y)$.

version of the classical Vitali's theorem. Thus we can extract a subsequence $u_{m'}$ which converges in $H_{\mathbb{C}}$, uniformly on every compact subset of $\Delta(u_0)$ to an $H_{\mathbb{C}}$-valued function $u^*(\zeta)$ which is analytic in $\Delta(u_0)$ and satisfies obviously:

$$(7.20) \qquad \sup_{\zeta\in\Delta(u_0)} \|u(\zeta)\|^2 \leqq 2(1+\|u_0\|^2).$$

Since the restriction of u_m to the real axis coincides with the Galerkin approximation in $\mathbb{R}_+$ of the Navier–Stokes equation, it is clear that the restriction of $u^*(\zeta)$ to some interval $(0, T')$ of the real axis coincides with the *unique* (strong) solution u of the Navier–Stokes equations given by Theorem 3.2. Hence u^* is nothing else than the analytic continuation to $\Delta(u_0)$ (at least) of u, and we will denote the limit $u(\zeta)$ instead of $u^*(\zeta)$. Secondly the *whole* sequence $u_m(\cdot)$ converges to $u(\cdot)$ in the above sense (i.e. uniformly on compact subsets of $\Delta(u_0)$, for the norm of $H_{\mathbb{C}}$).

Since the injection of $D(A)$ in V is compact, it also follows from (7.16) and Vitali's theorem that the sequence u_m converges to u in V uniformly on every compact subset of $\Delta(u_0)$, and that

$$(7.21) \qquad \sup_{\zeta\in K} |Au(\zeta)| \leqq c_4'(K),$$

with the same constant $c_4'(K)$ as in (7.17). Finally the majorizations (7.14), (7.18) imply that $d^k u_m/d\zeta^k$ converges to $d^k u/d\zeta^k$ in V uniformly on every compact subset K of $\Delta(u_0)$, and

$$(7.22) \qquad \sup_{\zeta\in K} \left\| \frac{d^k u(\zeta)}{d\zeta^k} \right\| \leqq \frac{2^{k+1}k!}{[d(K, \partial\Delta(u_0))]^k} (1+\|u_0\|^2)^{1/2},$$

$$(7.23) \qquad \sup_{\zeta\in K} \left| \frac{A\, d^k u(\zeta)}{d\zeta^k} \right| \leqq \frac{2^k k!\, c_4'(K')}{[d(K, \partial\Delta(u_0))]^k},$$

K' defined in (7.19).

To conclude, we observe that the reasoning made at $t=0$ can be made at any other point $t_0 \in (0, \infty)$ such that $u(t_0) \in V$. We obtain that u is a $D(A)$-valued analytic function at least in the region of $\mathbb{C}$:

$$(7.24) \qquad \bigcup_{t_0} \{t_0 + \Delta(u(t_0))\},$$

where the union is for those t_0's for which ($t_0 \in (0, \infty)$ and $u(t_0) \in V$). Finally we observe that, actually, $\Delta(u_0) = \Delta(\|u_0\|)$ depends only on the norm in V of u_0 and decreases as $\|u_0\|$ increases. Therefore if u is bounded in V on some interval $[\alpha, \beta]$, $\sup_{t\in[\alpha,\beta]} \|u(t)\| \leqq R$, then

$$(7.25) \qquad \bigcup_{t_0\in[\alpha,\beta]} \{t_0 + \Delta(u(t_0))\} \supset \bigcup_{t_0\in[\alpha,\beta]} \{t_0 + \Delta(R)\},$$

and this guarantees that the domain of analyticity of u contains the region mentioned in the statement of Theorem 7.1 (See Fig. 7.2.) The proof of Theorem 7.1 is complete.

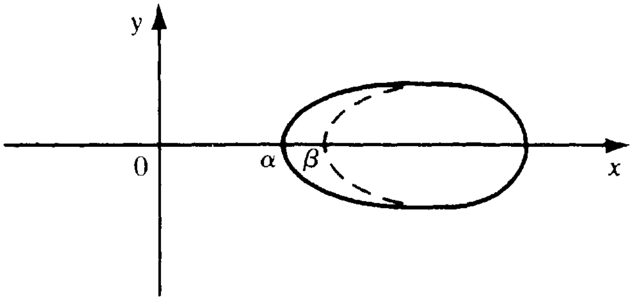

FIG. 7.2. $\bigcup_{t_0\in[\alpha,\beta]}\{t_0+\Delta(R)\}$.

7.2. Remarks.

Remark 7.1. Of course the result of analyticity given by Theorem 7.1 applies as well to a weak solution of the Navier–Stokes equations (Problem 2.1, $n=3$) near a point t_0 belonging to the set of $\mathbb{H}^1$-regularity (cf. § 4). If $f\in H$ is independent of t, and (α,β) is a maximal interval of $\mathbb{H}^1$-regularity of a weak solution u, then u is a $D(A)$-valued analytic function in

$$\bigcup_{t\in(\alpha,\beta)}\{t+\Delta(u(t))\}.$$

Remark 7.2. Theorem 7.1 and Remark 7.1 extend easily to the case where f is an $H_{\mathbb{C}}$-valued analytic function of time in a neighborhood of the positive real axis: it suffices to replace everywhere in the proof $\Delta(u_0)$ (or $t_0+\Delta(u(t_0))$) by its intersection with the domain of analyticity of f.

We conclude this section with the following:

PROPOSITION 7.1. *Under the same hypotheses as in Theorem* 7.1, *for* $0<t\leqq \frac{3}{4}T_1'(\|u_0\|)$, $T_1'(\|u_0\|)$ *given by* (7.10), *the following relations hold:*

$$\|u^{(k)}(t)\|=\left\|\frac{d^ku}{dt^k}(t)\right\|\leqq\frac{2^{k+1}k!}{t^k}(1+\|u_0\|^2)^{1/2}, \tag{7.26}$$

$$|Au^{(k)}(t)|=\left|A\frac{d^ku}{dt^k}(t)\right|\leqq\frac{c_5'(k)}{t^{k+1}}+\frac{c_6'(k)}{t^k},\qquad k\geqq 1, \tag{7.27}$$

where

$$c_5'(k)=2^{k+1}\left(\frac{k!}{\sqrt{\lambda_1}}(1+\|u_0\|^2)^{1/2},\ c_6'(k)\right)=2^{2k}k!\left(\frac{2}{\nu}|f|+\frac{2^{3/2}c_3^2}{\nu^2}(1+\|u_0\|^2)^{3/2}\right).$$

Proof. It suffices to apply (7.22), (7.23) with $K=\{t\}$. The distance $d(t,\partial\Delta(u_0))$ appears in the right-hand side of these inequalities, and an elementary calculation shows that for $0\leqq t\leqq\frac{1}{2}T_1'(\|u_0\|)$, this distance is $\geqq t/2$. $\square$

Remark 7.3. When the compatibility conditions (6.21) are not satisfied, u is not smooth near $t=0$ and the H-norm of the derivatives $u^{(k)}(t)$ tends to infinity as $t\to 0$. The relations (7.21), (7.22) give some indications on the way in which they tend to infinity. A similar result has been obtained by totally different methods by G. Iooss [1] and by D. Brézis [1].

8 Lagrangian Representation of the Flow

We assume that the fluid fills a bounded region Ω of $\mathbb{R}^3$. The Lagrangian representation of the flow of the fluid, mentioned in § 1, is determined by a function $\Phi:\{a, t\}\in\Omega\times(0, T)\to\Phi(a, t)\in\Omega$, where $\Phi(a, t)$ represents the position at time t of the particle of fluid which was at point a at time $t=0$ (flow studied for time t, $0\leqq t\leqq T$). This also amounts to saying that $\{t\mapsto\Phi(a, t)\}$ is the parametric representation of the trajectory of this particle.

The Lagrangian representation of the flow is not used too often because the Navier–Stokes equations in Lagrangian coordinates are highly nonlinear. It plays an important role, however, in two cases at least: it is used in the numerical computation of a flow with a free boundary, and, in the mathematical theory of the Navier–Stokes equations, it is the starting point of the geometrical approach developed by V. I. Arnold [1], D. Ebin–J. Marsden [1], among others.

If we are given u, a strong solution of the Navier–Stokes equations, then it is easy to determine the trajectories of the particles of fluid for the corresponding flow: for every $a\in\Omega$, the function $t\mapsto\Phi(a, t)$, also denoted ξ or ξ_a, is a solution of the ordinary differential equation

$$\frac{d\xi(t)}{dt}=u(\xi(t), t) \tag{8.1}$$

with initial (or Cauchy) data

$$\xi(0)=a. \tag{8.2}$$

Our goal in this section is to show that, using one of the new a priori estimates mentioned in § 4, it is possible to determine the trajectories of the fluid (in some weak sense), *even if u is a weak solution* of the Navier–Stokes equations.

8.1. The main result. We assume for convenience that

$$u_0\in V, \qquad f\in\mathscr{C}([0, T]; V), \tag{8.3}$$

and that we are given u, a weak solution of the Navier–Stokes equations (Problem 2.1) associated with u_0 and f, and that Ω satisfies (1.7) with $r=2$.

We have first to give meaning to (8.1), (8.2) since u is not regular.

LEMMA 8.1. *If ξ is a continuous function from* $[0, T]$ *into $\bar{\Omega}$ and u is a weak solution of Problem* 2.1, *then $u(\xi(t), t)$ is defined for almost every $t\in[0, T]$, the function $t\mapsto u(\xi(t), t)$ belongs to $L^1(0, T;\mathbb{R}^3)$ and* (8.1) *and* (8.2) *make sense.*

Proof. It follows from Theorem 4.1, Remark 4.1 and Theorem 5.1, that u is continuous from $[0, T]\setminus\mathscr{E}$ into $D(A)$, where $\mathscr{E}$ has Lebesgue measure 0. Due to (1.7), $H^2(\Omega)$ is included in $\mathscr{C}(\bar{\Omega})$ (cf. §§ 2.3 and 2.5, dimension $n=3$), and $D(A)\subset\mathscr{C}(\bar{\Omega})^3$. Hence $u(\xi(t), t)$ is well defined for every $t\in[0, T]\setminus\mathscr{E}$ and is continuous from $[0, T]\backslash\mathscr{E}$ into $\mathbb{R}^3$. The function $t \mapsto u(\xi(t), t)$ is measurable. If we show that this function is integrable, then (8.1) will make sense in the distribution sense and (8.2) obviously makes sense. But,

$$(8.4) \qquad |u(\xi(t), t)| \leqq |u(t)|_{\mathbb{L}^\infty(\Omega)} \quad \forall\, t \in [0, T]\backslash\mathscr{E},$$

and according to Theorem 4.3, $t \mapsto |u(t)|_{\mathbb{L}^\infty(\Omega)}$ is L^1. □

The main result can be stated:

THEOREM 8.1. *Let Ω be an open bounded set of $\mathbb{R}^3$ which satisfies* (1.7) *with $r = 2$, and let u be a weak solution of the Navier–Stokes equations corresponding to the data u_0, f, which satisfy* (8.3).

Then, for every $a \in \Omega$, the equations (8.1), (8.2) *possess (at least) one continuous solution from $[0, T]$ into $\bar{\Omega}$. Moreover, we can choose this solution in such a way that the mapping $\Phi: \{a, t\} \mapsto \xi_a(t)$, belongs to $L^\infty(\Omega \times (0, T))^3$* [1] *and $\partial\phi/\partial t \in L^1(\Omega \times (0, T))^3$.*

8.2. Proof of Theorem 8.1. i) Let w_j, $j \in \mathbb{N}$, denote the eigenfunctions of the operator A (cf. (2.17)) and, as in (3.44) we denote by P_m the orthogonal projector in H on the space spanned by $w_1, \ldots, w_m$. The function u being continuous from $[0, T]$ into V' with $u' \in L^{4/3}(0, T; V')$, u_m is, in particular, continuous from $[0, T]$ into $D(A)$, with $u_m' \in L^{4/3}(0, T; D(A))$.

Due to Agmon's inequality (2.21) (cf. § 2.5), there exists a constant c_1' depending only on Ω, and such that

$$(8.5) \qquad |v|_{\mathbb{L}^\infty(\Omega)} \leqq c_1' \, \|v\|^{1/2} \, |Av|^{1/2} \quad \forall\, v \in D(A);$$

P_m being a projector in V and $D(A)$,

$$(8.6) \qquad |u_m(t)|_{\mathbb{L}^\infty(\Omega)} \leqq c_1' \, \|u_m(t)\|^{1/2} \, |Au_m(t)|^{1/2} \leqq \theta(t),$$

for every $t \in [0, T]\backslash\mathscr{E}$, where

$$(8.7) \qquad \theta(t) = c_1' \, \|u(t)\|^{1/2} \, |Au(t)|^{1/2},$$

$\theta \in L^1(0, T)$ by Theorem 4.2 and Remark 4.1.

Now as $m \to \infty$

$$(8.8) \qquad u_m \to u \quad \text{in } L^2(0, T; V) \text{ and } L^1(0, T; \mathbb{L}^\infty(\Omega)).$$

Indeed the convergence in $L^2(0, T; V)$ is a straightforward consequence of the properties of P_m and the Lebesgue dominated convergence theorem. The convergence in $L^1(0, T; \mathbb{L}^\infty(\Omega))$ is proved in a similar manner, using the following consequence of (8.5):

$$|u_m(t) - u(t)|_{\mathbb{L}^\infty(\Omega)} \leqq c_1' \, \|u_m(t) - u(t)\|^{1/2} \, |Au_m(t) - Au(t)|^{1/2} \quad \forall\, t \notin \mathscr{E},$$

---

[1] and even $\Phi \in L^\infty(\Omega; \mathscr{C}([0, T]; \mathbb{R}^3))$.

and by Hölder's inequality[2]

$$
(8.9)\qquad \begin{aligned}&\int_0^T |u_m(t)-u(t)|_{\mathbb{L}^\infty(\Omega)}\,dt\\ &\quad\leq c_1'\left(\int_0^T \|u_m(t)-u(t)\|^2\,dt\right)^{1/4}\left(\int_0^T |Au_m(t)-Au(t)|^{2/3}\,dt\right)^{3/4}\end{aligned}
$$

ii) For every m, we define in a trivial manner an approximation ξ_m of ξ, where ξ_m is solution of (6.1), (6.2) with u replaced by u_m. The usual theorem on the Cauchy problem for ordinary differential equations (O.D.E.'s) asserts that ξ_m is defined on some interval of time $[0, T(m, a)]$, $0<T(m, a)\leq T$. But $u_m:\bar\Omega\to\mathbb{R}^3$ vanishes on $\partial\Omega$, and if we extend u_m by 0 outside $\bar\Omega$, we see that a trajectory $\xi_m(t)$ starting at time $t=0$ from $a\in\bar\Omega$ (condition (8.2)) cannot leave $\bar\Omega$ at a time $<T$. Hence $T(m, A)=T$, and $\xi_{ma}(t)=\xi_m(t)\in\bar\Omega$ for all $t\in[0, T]$, $(a\in\bar\Omega)$.

The theorem on the continuous dependence on a parameter of the solution of an O.D.E., shows that the function $\Phi_m:\Omega\times[0, T]\to\bar\Omega$ defined by

$$
(8.10)\qquad \Phi_m(a, t)=\xi_{ma}(t)\qquad (\xi_m=\xi_{ma})
$$

is in $\mathscr{C}^1(\bar\Omega\times[0, T])^3$. Since $\operatorname{div} u_m=0$, the Jacobian of Φ_m, $\det(D\Phi_m(\cdot, t)/Da)$ is equal to one for every t and therefore $\Phi_m(\cdot, t):\Omega\mapsto\Omega$ is locally invertible for every t. Furthermore, if $a\in\partial\Omega$, $\xi_m(t)\equiv a$ is a trivial solution of (8.1), (8.2) (with u replaced by u_m), and hence $\Phi_m(a, t)=a$ for all $a\in\partial\Omega$ and for all t. The classical theorems on the global invertibility of $\mathscr{C}^1$ mappings (cf. F. Browder [1]) imply that, for every t, $\Phi_m(\cdot, t)$ is a $\mathscr{C}^1$ diffeomorphism from $\bar\Omega$ onto itself. Finally, as $\operatorname{div} u_m=0$, $\Phi_m(\cdot, t)$ preserves the areas.

iii) We now pass to the limit $m\to\infty$, *for a fixed* $a\in\bar\Omega$. Due to (8.6),

$$
\left|\frac{d\xi_m(t)}{dt}\right|=|u_m(\xi_m(t), t)|\leq\theta(t)\quad\forall\, t\in\mathscr{E},
$$

and since $\theta\in L^1(0, T)$, the functions $\xi_m=\xi_{ma}$ are equicontinuous. Thus, there exist a subsequence m' (depending on a) and a continuous function $\xi=\xi_a$ from $[0, T]$ into $\bar\Omega$, such that $\xi_{m'}\to\xi$, as $m'\to\infty$, uniformly on $[0, T]$. We will conclude that ξ is a solution of (8.1), (8.2), provided we establish that

$$
(8.11)\qquad u_{m'}(\xi_{m'}(\cdot), \cdot)\to u(\xi(\cdot), \cdot)\quad \text{in } \mathbb{L}^1(0, T).
$$

But, for $t\notin\mathscr{E}$,

$$
\begin{aligned}&|u_{m'}(\xi_{m'}(t), t)-u(\xi(t), t)|\\ &\quad\leq|u_{m'}(\xi_{m'}(t), t)-u(\xi_{m'}(t), t)|+|u(\xi_{m'}(t), t)-u(\xi(t), t)|\\ &\quad\leq|u_{m'}(t)-u(t)|_{\mathbb{L}^\infty(\Omega)}+|u(\xi_{m'}(t), t)-u(\xi(t), t)|.\end{aligned}
$$

[2] Since the smooth functions are not dense in L^∞, and $L^{2/3}(0, T; D(A))$ is not a normed space, there is not much flexibility in the construction of a sequence of smooth functions u_m approximating u in the norm of $L^1(0, T; \mathbb{L}^\infty(\Omega))$.

Because of (8.8), (8.9), it remains, to prove (8.11), to show that

$$u(\xi_{m'}(\cdot), \cdot) \to u(\xi(\cdot), \cdot) \quad \text{in } \mathbb{L}^1(0, T). \tag{8.12}$$

For every $t \notin \mathscr{E}$, $u(\cdot, t) \in D(A)$ is a continuous function on $\bar{\Omega}$; hence $u(\xi_{m'}(t), t) \to u(\xi(t), t)$, and $|u(\xi_{m'}(t), t)| \leqq |u(t)|_{\mathbb{L}^\infty(\Omega)}$, and Lebesgue's theorem gives us the result (8.12).

iv) We infer from iii) that, for every fixed $a \in \bar{\Omega}$, there exists a solution $\xi = \xi_a$ (*not necessarily unique*) of (8.1), (8.2). We denote by $\Lambda(a)$ the set of continuous functions from $[0, T]$ into $\bar{\Omega}$ which satisfy (8.1), (8.2). According to (8.1), (8.5), a function ξ in $\Lambda(a)$ belongs to the Sobolev space $\mathbb{W}^{1,1}(0, T) = W^{1,1}(0, T)^3$, and $|\xi'(t)| \leqq \theta(t)$ almost everywhere (for all $t \notin \mathscr{E}$). We consider now the set-valued mapping

$$\Lambda : a \in \bar{\Omega} \mapsto \Lambda(a) \subset Y,$$

where Y is the subset of $\mathbb{W}^{1,1}(0, T)$ of functions η taking their values in $\bar{\Omega}$ and such that $|\eta'(t)| \leqq \theta(t)$ almost everywhere; Y is closed in $\mathbb{W}^{1,1}(0, T)$: it is a complete metric space. For every $a \in \bar{\Omega}$, $\Lambda(a)$ is not empty because of part iii) of the proof. As for (8.12), one can show easily that $\Lambda(a)$ is closed in Y and also that the graph of Λ is closed.

By the von Neumann measurable selection theorem (recalled in the Appendix to this section), Λ admits a measurable section L, i.e., a measurable mapping $L : \Omega \mapsto Y$, with $L(a) \in \Lambda(a)$ for all $a \in \bar{\Omega}$. Thus $\xi_a = L(a)$ is a solution of (6.1), (6.2), and $\xi_a(t) \in \bar{\Omega}$ for all $t \in [0, T]$. We set $\Phi(a, t) = \xi_a(t)$, and clearly, Φ possesses the desired properties. □

Remark 8.1. It would be interesting to prove further properties of Φ: for instance that Φ preserves the areas, that $\Phi(a, \cdot)$ is unique at least for almost every $a \in \bar{\Omega}$, etc.

8.3. Appendix. For the convenience of the reader, we recall here the measurable selection theorem (cf. C. Castaing [1]):

THEOREM 8.A. *Let X and Y be two separable Banach spaces and Λ a multiple-valued mapping from X to the set of nonempty closed subsets of Y, the graph of Λ being closed.*

Then Λ admits a universally Radon measurable section, i.e., there exists a mapping L from X into Y, such that

$$L(x) \in \Lambda(x) \quad \forall x \in X,$$

and L is measurable for any Radon measure defined on the Borel sets of X.

PART II

Questions Related to Stationary Solutions and Functional Invariant Sets (Attractors)

Orientation. The question studied in Part II pertains to the study of the behavior of the solutions of the Navier–Stokes equations when $t \to \infty$, and to the understanding of turbulence. Of course, before raising such questions, we have to be sure that a well-defined solution exists for arbitrarily large time; since this question is not solved if $n=3$, we will often restrict ourselves to the case $n=2$ or, for $n=3$, we will make the (maybe restrictive) assumption that a strong solution exists for all time for the problem considered.

In § 9 we start by describing the classical Couette–Taylor experiment of hydrodynamics, in which there has recently been a resurgence of interest. We present here our motivations for the problems studied in Part II. In § 10 we study the time independent solutions of the Navier–Stokes equations. In § 11, which is somewhat technical, we present a squeezing property of the trajectories of the flow (in the function space) which seems important. In § 12 we show that, if the dimension of space is $n=2$, then every trajectory converges (in the function space) to a bounded functional invariant set (which may or may not be an attracting set), and that the Hausdorff dimension of any bounded functional invariant set is finite. This supports the physical idea that, in a turbulent flow, all but a finite number of modes are damped.

Throughout §§ 10 to 12 we are considering the flow in a bounded domain (Ω or Q) with indifferently either 0 or periodic boundary conditions. We will not recall this point in the statements of the theorems.

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9 The Couette–Taylor Experiment

We recall what the Couette flow between rotating cylinders is: the fluid fills a domain Ω of $\mathbb{R}^3$ which is limited by two vertical axisymmetric cylinders with the same axis (radii r_1, r_2, $0<r_1<r_2$) and by two horizontal planes separated by a distance h. The outer cylinder is at rest and the inner one is rotating with an angular velocity ω_1. This corresponds to the flow in a fixed bounded Lipschitz domain Ω of $\mathbb{R}^3$ with vanishing forces per unit volume[1] and nonzero boundary conditions on $\Gamma=\partial\Omega: \boldsymbol{\phi}\neq 0$ in (1.9) (but $\boldsymbol{\phi}\cdot\boldsymbol{\nu}=0$). This case was not studied in Part I to avoid purely technical difficulties, but its mathematical treatment is very similar to that considered in Part I (cf., e.g., C. Foias–R. Temam [3], [4], [5]; J. C. Saut–R. Temam [2]). When $\omega_1(t)$ is constant, $\omega_1(t)=\bar{\omega}_1$, a Reynolds number can be defined for this flow (cf. (1.6)), $\mathrm{Re}=\bar{\omega}_1 r_1 d/\nu$, $d=r_2-r_1$, which corresponds for instance to the choice of $L_*=d$ as a characteristic length for the flow, and to $U_*=\bar{\omega}_1 r_1=$ the velocity on the inner cylinder as a characteristic velocity.

The Taylor experiment on the Couette flow is as follows. We start with the fluid at rest and we increase the angular velocity of the inner cylinder from 0 to some value $\bar{\omega}_1$:

i) If $\bar{\omega}_1$ (more precisely Re, which has no dimension) is sufficiently small, then after a very short transient period we observe a steady axisymmetric flow. The trajectories of the particles of fluid are essentially circles with the same axis as the cylinders.

ii) If $\bar{\omega}_1$(Re) is larger than some threshold λ', but not too large, then, after the transient period, another steady state appears, different from that in i). A cellular mode now appears: the trajectory of the particles of fluid is now the superposition of the motion around the axis and a roll-like motion in the azimuthal plane. The steady flow (see Fig. 9.1) remains axisymmetric, and in adjacent cells the fluid particles move in counterrotating spiral paths.

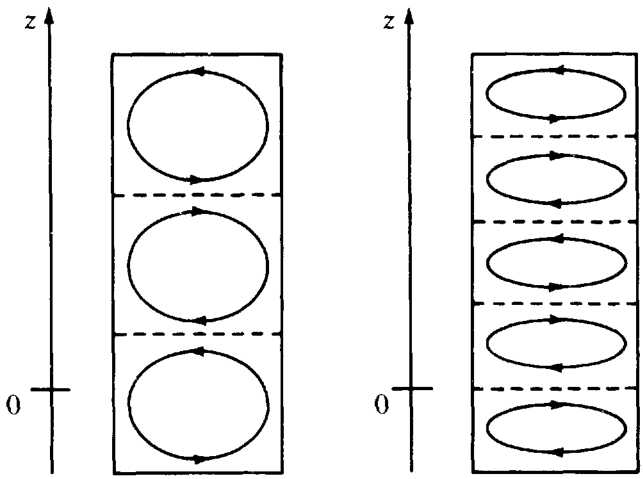
With changing (i.e., increasing) $\bar{\omega}_1$, a cellular structure may lose its stability and another cellular steady motion may appear, with more cells.

iii) When $\bar{\omega}_1$(Re) is larger than a higher threshold value, the flow observed after the transient period becomes unsteady with its cellular structure perturbed by circumferential travelling waves. At this point the flow is periodic in time.

As Re increases, the temporal variations of the flow become more complex, quasiperiodic perhaps, with two or more incommensurate frequencies. Finally the flow becomes totally turbulent with no apparent structure at all.

This is a somewhat simplified description of the phenomena, the flow observed depending in a sensitive way on the ratio d/l and/or on how the

[1] The gravity forces are of the form grad $(-\rho g x_3)$, where $0x_3$ is a vertical axis pointing upward, and we can incorporate them in the pressure term.



3-cells 5-cells

FIG. 9.1

constant angular velocity $\bar{\omega}_1$ is attained; different situations may also appear if the flow does not start from rest. We refer the reader to T. B. Benjamin [1], T. B. Benjamin–T. Mullin [1], [2], P. R. Fenstermacher–H. L. Swinney–J. P. Gollub [1], H. L. Swinney [1], J. P. Gollub [1]. Let us also mention the remarkable fact reported in T. B. Benjamin [1] that different stable steady motions have been observed for the same geometry and the same boundary velocity (same values of ν, r_1, r_2, h, $\bar{\omega}_1$).

Although the above description of the experiment is over-simplified, it is quite typical, and similar phenomena are observed for other experiments, such as Benard flow and flow past a sphere. Let us interpret, for the flows studied in Part I, what the corresponding mathematical problems are. We start from rest ($u_0 = 0$), and we can imagine that $f(t)$ is "increased" from 0 to some value $\bar{f} \in H$ through some complicated path, or through a linear one: $f(t) = \lambda(t)\, f_0$, $f_0 \in H$, $\lambda(t) \in \mathbb{R}$, increasing from 0 to $\bar{\lambda}$. The parameter[2]

$$R = \nu^{-2} L^{3-n/2}\, |\bar{f}|, \tag{9.1}$$

can play the role of the Reynolds number, L being a characteristic length of the domain (Ω or Q), n the dimension of space and $|\bar{f}|$ the H-norm of $\bar{f} \in H$. Then the conjectures on bifurcation and onset of turbulence are as follows (cf. D. Ruelle [1], D. Ruelle–F. Takens [1]):

a) If R is sufficiently small, the solution $u(t)$ of Problem 2.2 tends, for $t \to \infty$, to $\bar{u}$ a time independent solution of the Navier–Stokes equations.

b) For larger values of R, $\bar{u}$ may lose its stability and $u(t)$ converges for

[2] R has no dimension if $\rho = 1$; if $\rho \neq 1$, the number without dimension is $R = \rho\mu^{-2}\, |\bar{f}|\, L^{3-n/2}$, $\mu = \nu$.

$t \to \infty$ to another stationary solution $\bar{u}'$; this can be repeated for higher values of R, $u(t) \to \bar{u}''$, ...

c) After a higher threshold, $u(t)$ converges, for $t \to \infty$, to a time periodic solution of the Navier–Stokes equations $u_{\Pi}(t)$, or to a quasiperiodic solution.

d) For higher values of R, $u(t)$ as $t \to \infty$ tends to lie on a "strange attractor," such as the product of a Cantor set with an interval. This would explain the chaotic behavior of turbulence.

The small contributions presented below to this list of outstanding problems are:

- The proof of a), which is elementary and has been known for a long time;
- Some properties of the set of stationary solutions of the Navier–Stokes equations in connection with b) (§ 10);
- Existence and a property of the limit set if $n = 2$ in the case d) (§ 12).

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10 Stationary Solutions of the Navier–Stokes Equations

Assuming that the forces are independent of time, we are looking for time independent solutions of the Navier–Stokes equations, i.e., a function $u = u(x)$ (and a function $p = p(x)$) which satisfies (1.4), (1.5), and either (1.9) with $\phi = 0$, or (1.10):

$$-\nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \mathcal{O} \ (= \Omega \text{ or } Q), \tag{10.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{O}, \tag{10.2}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega \quad \text{if } \mathcal{O} = \Omega, \tag{10.3}$$

or

$$u(x + Le_i) = u(x), \quad i = 1, \dots, n, \quad x \in Q \quad \text{if } \mathcal{O} = Q. \tag{10.4}$$

In the functional setting of § 2, the problem is

(10.5) Given f in H (or V'), to find $u \in V$ which satisfies

$$\nu((u, v)) + b(u, u, v) = (f, v) \quad \forall\, v \in V, \tag{10.6}$$

or

$$\nu Au + Bu = f \quad \text{in } H \text{ (or } V'). \tag{10.7}$$

10.1. Behavior for $t \to \infty$. The trivial case. We start by recalling briefly the results of existence and uniqueness of solutions for (10.5)–(10.7).

THEOREM 10.1. *We consider the flow in a bounded domain with periodic or zero boundary conditions ($\mathcal{O} = \Omega$ or Q), and $n = 2$ or 3. Then:*

i) *For every f given in V' and $\nu > 0$, there exists at least one solution of* (10.5)–(10.7).

ii) *If f belongs to H, all the solutions belong to $D(A)$.*

iii) *Finally, if*

$$\nu^2 > c_1 \|f\|_{V'}, \tag{10.8}$$

where c_1 is a constant depending only on $\mathcal{O}$[1], *then the solution of* (10.5)–(10.7) *is unique.*

Proof. We give only the principle of the proof and refer the reader to the literature for further details.

For existence we implement a Galerkin method (cf. (3.41)–(3.45)) and look,

[1] c_1 is the constant in (2.30) when $m_1 = m_3 = 1$, $m_2 = 0$. Since $|f| \geqq \sqrt{\lambda_1}\, \|f\|_{V'}$ if $f \in H$, a sufficient condition for (10.8) is $\nu^2 > c_1\sqrt{\lambda_1}\, |f|$, which can be compared to (9.1).

for every $m \in \mathbb{N}$, for an approximate solution u_m,

$$u_m = \sum_{j=1}^{m} \xi_{jm} w_j, \qquad \xi_{jm} \in \mathbb{R}, \tag{10.9}$$

such that

$$\nu((u_m, v)) + b(u_m, u_m, v) = \langle f, v\rangle \tag{10.10}$$

for every v in W_m = the space spanned by $w_1, \ldots, w_m$. Equation (10.10) is also equivalent to

$$\nu A u_m + P_m B u_m = P_m f. \tag{10.11}$$

The existence of a solution u_m of (10.10)–(10.11) follows from the Brouwer fixed point theorem (cf. [RT, Chap. II, § 1] for the details). Taking $v = u_m$ in (10.10) and taking into account (2.34), we get

$$\nu \|u_m\|^2 = \langle f, u_m\rangle \leqq \|f\|_{V'} \|u_m\|, \tag{10.12}$$

and therefore

$$\|u_m\| \leqq \frac{1}{\nu} \|f\|_{V'}. \tag{10.13}$$

We extract from u_m a sequence $u_{m'}$, which converges weakly in V to some limit u, and since the injection of V in H is compact, this convergence holds also in the norm of H:

$$u_{m'} \to u \quad \text{weakly in } V, \quad \text{strongly in } H. \tag{10.14}$$

Passing to the limit in (10.10) with the sequence m', we find that u is a solution of (10.6).

To prove ii), we note that if $u \in V$, then $Bu \in V_{-1/2}$ (instead of V') because of (2.36) (applied with $m_1 = 1$, $m_2 = 0$, $m_3 = \frac{1}{2}$). Hence $u = \nu^{-1} A^{-1}(f - Bu)$ is in $V_{3/2}$. Applying again Lemma 2.1 (with $m_1 = \frac{3}{2}$, $m_2 = \frac{1}{2}$, $m_3 = 0$), we conclude now that $Bu \in H$, and thus u is in $D(A)$.

We can provide useful a priori estimates for the norm of u in V and in $D(A)$. Setting $v = u$ in (10.6) we obtain (compare to (10.12)–(10.13))

$$\nu \|u\|^2 = \langle f, u\rangle \leqq \|f\|_{V'} \|u\|, \tag{10.15}$$

$$\|u\| \leqq \frac{1}{\nu} \|f\|_{V'} \qquad \left(\leqq \frac{1}{\nu\sqrt{\lambda_1}} |f| \text{ if } f \in H\right). \tag{10.16}$$

For the norm in $D(A)$ we infer from (10.7) that

$$\begin{aligned}
\nu |Au| &\leqq |f| + |Bu| \leqq |f| + c_2 \|u\|^{3/2} |Au|^{1/2} && \text{(by the first inequality (2.32))}\\
&\leqq |f| + \frac{\nu}{2} |Au| + \frac{c_2^2}{2\nu} \|u\|^3 && \text{(by the Schwarz inequality)}\\
&\leqq |f| + \frac{\nu}{2} |Au| + \frac{c_2^2}{2\nu^4 \lambda_1^{3/2}} |f|^3 && \text{(with (10.6)).}
\end{aligned}$$

Finally

$$(10.17)\qquad |Au| \leqq \frac{2}{\nu}|f| + \frac{c_2^2}{\nu^5 \lambda_1^{3/2}}|f|^3.$$

For the uniqueness result iii), let us assume that u_1 and u_2 are two solutions of (10.6):

$$\nu((u_1, v)) + b(u_1, u_1, v) = \langle f, v\rangle,$$
$$\nu((u_2, v)) + b(u_2, u_2, v) = \langle f, v\rangle.$$

Setting $v = u_1 - u_2$, we obtain by subtracting the second relation from the first one:

$$\begin{aligned}\nu\,\|u_1 - u_2\|^2 &= -b(u_1, u_1, u_1 - u_2) + b(u_2, u_2, u_1 - u_2)\\ &= -b(u_1 - u_2, u_2, u_1 - u_2) \qquad \text{(with (2.34))}\\ &\leqq c_1\,\|u_1 - u_2\|^2\,\|u_2\| \qquad \text{(using (2.30) with } m_1 = m_3 = 1,\ m_2 = 0)\\ &\leqq \frac{c_1}{\nu}\|f\|_{V'}\,\|u_1 - u_2\|^2 \qquad \text{(with (10.16) applied to } u_2).\end{aligned}$$

Therefore

$$\left(\nu - \frac{c_1}{\nu}\|f\|_{V'}\right)\|u_1 - u_2\|^2 \leqq 0$$

and $u_1 - u_2 = 0$ if (10.8) holds. ☐

Concerning the behavior for $t \to \infty$ of the solutions of the time-dependent Navier–Stokes equations, the easy case (which corresponds to point a) in § 9) is the following:

THEOREM 10.2. *We consider the flow in a bounded domain with periodic or zero boundary conditions ($\mathcal{O} = \Omega$ or Q) and $n = 2$ or 3. We are given $f \in H$, $\nu > 0$ and we assume that*

$$(10.18)\qquad \nu\left(\frac{\lambda_1}{c_2'}\right)^{3/4} > \left(\frac{2}{\nu}|f| + \frac{c_2^2|f|^3}{\nu^5\lambda_1^{3/2}}\right)$$

where c_2', c_2 depend only on $\mathcal{O}$.

Then the solution of (10.7) (denoted u_∞) is unique. If $u(\cdot)$ is any weak solution[2] *of Problem 2.1 with $u_0 \in H$ arbitrary and $f(t) \equiv f$ for all t, then*

$$(10.19)\qquad u(t) \to u_\infty \quad \text{in } H \quad \text{as } t \to \infty.$$

Proof. Let $w(t) = u(t) - u_\infty$. We have, by differences,

$$\frac{dw(t)}{dt} + \nu A w(t) + Bu(t) - Bu_\infty = 0,$$

[2] If $n = 2$, $u(\cdot)$ is unique and is a strong solution (at least for $t > 0$). If $n = 3$, $u(\cdot)$ is not necessarily unique, and we must assume that $u(\cdot)$ satisfies the energy inequality (see Remark 3.2).

and, taking the scalar product with $w(t)$,

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2+b(u(t),u(t),w(t))-b(u_\infty,u_\infty,w(t))=0^3. \tag{10.20}$$

Hence, with (2.34),

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2=-b(w(t),u_\infty,w(t)).$$

Using (2.30) with $m_1=\frac{1}{2}$, $m_2=1$, $m_3=0$, we can majorize the right-hand side of this equality by

$$c_1\,|w(t)|_{1/2}\,|w(t)|\,|Au_\infty|,$$

which, because of (2.20), is less than or equal to

$$c_1'\,|w(t)|^{3/2}\,\|w(t)\|^{1/2}\,|Au_\infty|;$$

with (3.10) this is bounded by

$$\frac{\nu}{2}\|w(t)\|^2+\frac{c_2'}{\nu^{1/3}}|w(t)|^2\,|Au_\infty|^{4/3}$$

where $c_2'=3(c_1'/2)^{4/3}$. Therefore,

$$\frac{d}{dt}|w(t)|^2+\nu\,\|w(t)\|^2\leqq\frac{c_2'}{\nu^{1/3}}|w(t)|^2\,|Au_\infty|^{4/3}, \tag{10.21}$$

$$\frac{d}{dt}|w(t)|^2+\left(\nu\lambda_1-\frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3}\right)|w(t)|^2\leqq 0. \tag{10.22}$$

If

$$\bar\nu=\nu\lambda_1-\frac{c_2'}{\nu^{1/3}}|Au_\infty|^{4/3}>0, \tag{10.23}$$

then (10.22) shows that $|w(t)|$ decays exponentially towards 0 when $t\to\infty$:

$$|w(t)|\leqq|w(0)|e^{-\bar\nu t},$$

$w(0)=u_0-u_\infty$. Using the estimation (10.17) for u_∞, we obtain a sufficient condition for (10.23), which is exactly (10.18).

If we replace $u(t)$ by another stationary solution u_∞^* of (10.7) in the computations leading to (10.22), we obtain instead of (10.22)

$$\nu\,|u_\infty^*-u_\infty|^2\leqq 0.$$

Thus (10–23) and (10.18) ensure that $u_\infty^*=u_\infty$; i.e., they are sufficient conditions for uniqueness of a stationary solution, like (10.8).

The proof is complete. □

---

[3] The proof that $\langle w'(t),w(t)\rangle=\frac{1}{2}(d/dt)\,|w(t)|^2$ is not totally easy when $n=2$, and relies on [RT, Chap. III, Lemma 1.2]. If $n=3$ we do not even have an equality in (10.20), but an inequality $\leqq$, which is sufficient for our purposes: a technically similar situation arises in [RT, Chap. III, § 3.6].

Remark 10.1. Under the assumptions of Theorem 10.2, consider the linear operator $\mathcal{A}$ from $D(A)$ into H defined by

$$(\mathcal{A}\phi, \psi) = \nu(A\phi, \psi) + b(\phi, u_\infty, \psi) + b(u_\infty, \phi, \psi) \quad \forall \phi, \psi \in D(A),$$

whose adjoint $\mathcal{A}^*$ from $D(A)$ into H is given by

$$(\mathcal{A}^*\phi, \psi) = \nu(A\phi, \psi) + b(\psi, u_\infty, \phi) + b(u_\infty, \psi, \phi) \quad \forall \phi, \psi \in D(A).$$

Then with the same notation as in the proof of Theorem 10.2, we have

$$\frac{dw(t)}{dt} + \mathcal{A}w(t) + Bw(t) = 0,$$

and (10.20) is the same as

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu(\mathcal{A}w(t), w(t)) = 0$$

or

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\left(\frac{\mathcal{A}+\mathcal{A}^*}{2} w(t), w(t)\right) = 0.$$

The operator $((\mathcal{A}+\mathcal{A}^*)/2)^{-1}$ is selfadjoint and compact from H into itself, and the conclusions of Theorem 10.2 will still hold if we replace (10.18) by the conditions that the *eigenvalues of* $(\mathcal{A}+\mathcal{A}^*)/2$ *are* >0.

10.2. An abstract theorem on stationary solutions. In this section we derive an abstract theorem on the structure of the set of solutions of a general equation

$$N(u) = f; \tag{10.24}$$

this theorem will then be applied to (10.7).

Nonlinear Fredholm operators. If X and Y are two real Banach spaces, a linear continuous operator L from X into Y is called a Fredholm operator if

i) $\dim \ker L < \infty$,
ii) range L is closed,
iii) coker $L = Y/_{\text{range } L}$ has finite dimension.

In such a case the *index* of L is the integer

$$i(L) = \dim \ker L - \dim \operatorname{coker} L. \tag{10.25}$$

For instance, if $L = L_1 + L_2$ where L_1 is compact from X into Y and L_2 is an isomorphism (resp. is surjective and $\dim \ker L_2 = q$), then L is Fredholm of index 0 (resp. of index q). For the properties of Fredholm operators, see for instance R. Palais [1], S. Smale [1].

Now let ω be a connected open set of X, and N a nonlinear operator from ω into Y; N is a nonlinear Fredholm map if N is of class $\mathscr{C}^1$ and its differential $N'(u)$ is a Fredholm operator from X into Y, at every point $u \in \omega$. In this case it follows from the properties of Fredholm operators that the index of $N'(u)$ is independent of u; we define the index of N as the number $i(N'(u))$.

Let N be a $\mathscr{C}^1$ mapping from an open set ω of X into Y, X, Y being again two real Banach spaces. We recall that $u \in X$ is called a regular point of N if $N'(u)$ is onto, and a singular point of N otherwise. The image given by N of the set of singular points of N constitutes the set of singular values of N. Its complement in Y constitutes the set of regular values of N. Thus a *regular value of* N is a point $f \in Y$ which does not belong to the image $N(\omega)$, or such that $N'(u)$ is onto at every point u in the preimage $N^{-1}(f)$.

Finally we recall that a mapping N of the preceding type is *proper* if the preimage $N^{-1}(K)$ of any compact set K of Y is compact in X.

We will make use of the following infinite dimensional version of Sard's theorem due to S. Smale [1] (see also K. Geba [1]).

THEOREM. *Let X and Y be two real Banach spaces and ω a connected open set of X. If $N: \omega \to Y$ is a proper $\mathscr{C}^k$ Fredholm map with $k > \max(\text{index } N, 0)$, then the set of regular values of N is a dense open set of Y.*

We deduce easily from this theorem:

THEOREM 10.3. *Let X and Y be two real Banach spaces and ω a connected open set of X, and let $N: \omega \to Y$ be a proper $\mathscr{C}^k$ Fredholm map, $k \geqq 1$, of index* 0.

Then there exists a dense open set ω_1 in Y and, for every $f \in \omega_1$, $N^{-1}(f)$ is a finite set.

If index $N = q > 0$ and $k \geqq q$, then there exists a dense open set ω_1 in Y and, for every $f \in \omega_1$, $N^{-1}(f)$ is empty or is a manifold in ω of class $\mathscr{C}^k$ and dimension q.

Proof. We just take $\omega_1 =$ the set of regular values of N which is dense and open by Smale's theorem. For every $f \in \omega_1$, the set $N^{-1}(f)$ is compact since N is proper. If index $(N) = 0$, then for every $f \in \omega_1$ and $u \in N^{-1}(f)$, $N'(u)$ is onto (by definition of ω_1) and is one-to-one since

$$\dim \ker N'(u) = \dim \operatorname{coker} N'(u) = 0.$$

Thus $N'(u)$ is an isomorphism, and by the implicit function theorem, u is an isolated solution of $N(v) = f$. We conclude that $N^{-1}(f)$ is compact and made of isolated points: this set is discrete.

If index $(N) = q \leqq k$, for every $f \in \omega_1$, and every $u \in N^{-1}(f)$, $N'(u)$ is onto and the dimension of its kernel is q: it follows that $N^{-1}(f)$ is a manifold of dimension q, of class $\mathscr{C}^k$ like N. $\square$

Applications of this theorem to the stationary Navier–Stokes equation (and to other equations) will be given below.

10.3. Application to the Navier–Stokes equations. We are going to show that Theorem 10.3 applies to the stationary Navier–Stokes equation (10.7) in the following manner:

$$X = D(A), \quad Y = H, \quad N(u) = \nu A u + B u. \tag{10.26}$$

It follows from (2.36) that $B(\cdot, \cdot)$ is continuous from $D(A) \times D(A)$ and even $V_{3/2} \times V_{3/2}$ into H and N makes sense as a mapping from $D(A)$ into H.

LEMMA 10.1. *$N: D(A) \to H$ is proper.*

Proof. Let K denote a compact set of H. Since K is bounded in H, it follows from the a priori estimation (10.17) that $N^{-1}(K)$ is bounded in $D(A)$ and thus compact in $V_{3/2}$. As observed before, $B(\cdot, \cdot)$ is continuous from $V_{3/2} \times V_{3/2}$ into H, and thus $B(N^{-1}(K))$ is compact in H.

We conclude that the set $N^{-1}(K)$ is included in

$$\nu^{-1} A^{-1}(K - B(N^{-1}(K))),$$

which is relatively compact in $D(A)$, and the result follows. □

We have the following *generic* properties of the set of stationary solutions to the Navier–Stokes equations.

THEOREM 10.4. *We consider the stationary Navier–Stokes equations in a bounded domain (Ω or Q) with periodic or zero boundary conditions, and $n = 2$ or 3.*

Then, for every $\nu > 0$, there exists a dense open set $\mathcal{O}_\nu \subset H$ such that for every $f \in \mathcal{O}_\nu$, the set of solution of (10.5)–(10.7) *is finite and odd in number.*

On every connected component of $\mathcal{O}_\nu$, the number of solutions is constant, and each solution is a $\mathscr{C}^\infty$ function of f.

Proof. i) We apply Theorem 10.3 with the choice of X, Y, N indicated in (10.26) (and $\omega = X$). It is clear that N is a $\mathscr{C}^\infty$ mapping from $D(A)$ into H and that

$$N'(u) \cdot v = \nu A v + B(u, v) + B(v, u) \quad \forall\, u, v \in D(A). \tag{10.27}$$

It follows from (3.20) that for every $u \in D(A)$, the linear mappings

$$v \mapsto B(u, v), \qquad v \mapsto B(v, u)$$

are continuous from V into H and they are therefore compact from $D(A)$ into H. Since A is an isomorphism from $D(A)$ onto H, it follows from the properties of Fredholm operators (recalled in § 10.2) that $N'(u)$ is a Fredholm operator of index 0.

We have shown in Lemma 10.1 that N is proper: all the assumptions of Theorem 10.3 are satisfied. Setting $\mathcal{O}_\nu =$ the set of regular values of $N = N_\nu$, we conclude that $\mathcal{O}_\nu$ is open and dense in H and, for every $f \in \mathcal{O}_\nu$, $N^{-1}(f)$, which is the set of solutions of (10.7), is finite.

ii) Let $(\mathcal{O}_i)_{i \in I}$ be the connected components of $\mathcal{O}_\nu$ (which are open), and let f_0, f_1 be two points of $\mathcal{O}_i$ for some i. Let $u_0 \in N^{-1}(f_0)$. There exists a continuous curve

$$s \in [0, 1] \mapsto f(s) \in \mathcal{O}_i, \qquad f(0) = f_0, \quad f(1) = f_1,$$

and the implicit function theorem shows the existence and uniqueness of a continuous curve $s \mapsto u(s)$ with

$$N(u(s)) = f(s), \qquad u(0) = u_0.$$

Since $f(s)$ is a regular value of N, for all $s \in [0, 1]$, $u(s)$ is defined for every s, $0 \leqq s \leqq 1$, and therefore $u(1) \in N^{-1}(f_1)$. Such a curve $\{s \mapsto u(s)\}$ can be constructed, starting from any $u_k \in N^{-1}(f_0)$. Two different curves cannot reach the same point $u_* \in N^{-1}(f_1)$ and cannot intersect at all, since this would not be consistent with the implicit function theorem around u_* or around the intersection point. Hence there are at least as many points in $N^{-1}(f_1)$ as in $N^{-1}(f_0)$. By symmetry the number of points is the same.

It is clear that each solution $u_k = u_k(f)$ is a $\mathscr{C}^\infty$ function of f on every $\mathcal{O}$.

iii) It remains to show that the number of solutions is odd. This is an easy application of the Leray–Schauder degree theory.

For fixed $\nu > 0$ and $f \in \mathcal{O}_\nu$, we rewrite (1.7) in the form

$$T_\nu(u) = \nu u + A^{-1}Bu = A^{-1}f = g, \tag{10.28}$$

$u, g \in D(A)$. By (10.17), every solution u_λ of $T_\nu(u_\lambda) = \lambda g$, $0 \leqq \lambda \leqq 1$, satisfies

$$|Au_\lambda| < R, \qquad R = 1 + \frac{2}{\nu}|f| + \frac{c_2^2}{\nu^5 \lambda_1^{3/2}}|f|^3.$$

Therefore the Leray–Schauder degree $d(T_\nu, \lambda g, B_R)$ is well defined, with B_R the ball of $D(A)$ of radius R. Also, when λg is a regular value of T_ν, i.e., λf is a regular value of N, the set $T_\nu^{-1}(\lambda g)$ is discrete $= \{u_1, \dots, u_k\}$, and $d(T_\nu, \lambda g, B_R) = \sum_{j=1}^k i(u_j)$ where $i(u_j) =$ index u_j.

It follows from Theorem 10.1 that there exists $\lambda_* \in [0, 1]$, and for $0 \leqq \lambda \leqq \lambda_*$, $N^{-1}(\lambda f)$ contains only one point u_λ. By arguments similar to that used in the proof of Theorem 10.1, one can show that $N'(u_\lambda)$ is an isomorphism, and hence for these values of λ, $d(T_\nu, \lambda g, B_R) = \pm 1$. By the homotopy invariance property of degree, $d(T_\nu, g, B_R) = \pm 1$ and consequently k must be an odd number.

The proof of Theorem 10.4 is complete. □

Remark 10.2.

i) The set $\mathcal{O}$ is actually unbounded in H; cf. C. Foias–R. Temam [13].

ii) Similar generic results have been proved for the flow in a bounded domain with a nonhomogeneous boundary condition (i.e., $\phi \neq 0$ in (1.9)): generic finiteness with respect to f for ϕ fixed, with respect to ϕ for f fixed and with respect to the pair f, ϕ; see C. Foias–R. Temam [3], [4], J. C. Saut–R. Temam [2], and for the case of time periodic solutions, J. C. Saut–R. Temam [2], R. Temam [7].

iii) When Ω is unbounded, we lack a compactness theorem for the Sobolev spaces $H^m(\Omega)$ (and lack the Fredholm property). We do not know whether results similar to that in Theorem 10.4 are valid in that case; cf. D. Serre [1] where a line of stationary solutions of Navier–Stokes equations is constructed for an unbounded domain Ω.

We now present another application of Theorem 10.3, with an operator of index 1, leading to a generic result in bifurcation. We denote by $S(f, \nu) \subset D(A)$ the set of solutions of (1.5)–(1.7) and

$$S(f) = \bigcup_{\nu > 0} S(f, \nu). \tag{10.29}$$

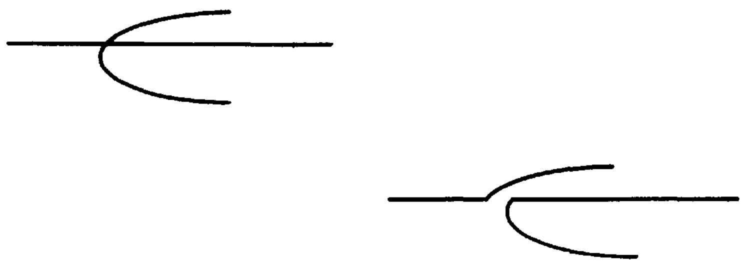

FIG. 10.1

THEOREM 10.5. *Under the same hypotheses as in Theorem* 10.4, *there exists a* G_δ*-dense set* $\mathcal{O} \subset H$, *such that for every* $f \in \mathcal{O}$, *the set* $S(f)$ *defined in* (10.29) *is a* $\mathscr{C}^\infty$ *manifold of dimension* 1.

Proof. We apply Theorem 10.3 with $X = D(A) \times \mathbb{R}$, $Y = H$, $\omega = \omega_m = D(A) \times (1/m, \infty) \subset X$, $m \in \mathbb{N}$, $N(u, \nu) = \nu Au + Bu$, for all $(u, \nu) \in X$. It is clear that N is $\mathscr{C}^\infty$ from ω into Y and

$$N'(u, \nu) \cdot (v, \mu) = \nu Av + B(u, v) + B(v, u) + \mu Au.$$

For $(u, \nu) \in \omega$, $N'(u, \nu)$ is the sum of the operator

$$(v, \mu) \to B(u, v) + B(v, u) + \mu Au,$$

which is compact[4], and the operator

$$(v, \mu) \to \nu Av,$$

which is onto and has a kernel of dimension 1. Thus $N'(u, \nu)$ is a Fredholm operator of index 1 and N is a nonlinear Fredholm mapping of index 1.

The proof of Lemma 10.1 and (10.17) shows that N is proper on $D(A) \times (\nu_0, \infty)$, for all $\nu_0 > 0$ and in particular $\nu_0 = 1/m$. Hence Theorem 10.3 shows that there exists an open dense set $\mathcal{O}_m \subset H$, and for every $f \in \mathcal{O}_m$, $N_m^{-1}(f)$ is a manifold of dimension 1, where N_m is the restriction of N to ω_m. We set $\mathcal{O} = \bigcap_{m \geq 1} \mathcal{O}_m$, which is a dense G_δ set in H, and, for every $f \in \mathcal{O}$, $S(f) = \bigcup_{m \geq 1} N_m^{-1}(f)$ is a manifold of dimension 1.

Remark 10.3. i) By the uniqueness result in Theorem 10.1, $S(f)$ contains an infinite branch corresponding to the large values of ν.

ii) Since $S(f)$ is a $\mathscr{C}^\infty$ manifold of dimension 1, it is made of the union of curves which *cannot intersect.* Hence the usual bifurcation picture (Fig. 10.1) is nongeneric and is a schematization (perfectly legitimate of course!) of generic situations of the type shown in Fig. 10.2.

Remark 10.4. Other properties of the set $S(f, \nu)$ are given in C. Foias–R. Temam [3], [4] and J. C. Saut–R. Temam [2]. In particular, for every $\nu, f, S(f, \nu)$ is a real compact analytic set of finite dimension.

[4] Same proof essentially as in Theorem 10.4.

10.4. Counterexamples. A natural question concerning Theorem 10.4 is whether $\mathcal{O}_\nu$ is the whole space H or just a subset. Since Theorem 10.4 is a straightforward consequence of Theorem 10.3, this question has to be raised at the level of Theorem 10.3. Unfortunately, the following examples show that we cannot answer this question at the level of generality of the abstract Theorem 10.3, since for one of the two examples presented $\mathcal{O}_\nu = H$, while for the second one $\mathcal{O}_\nu \neq H$.

Example 1. The first example is the one-dimensional Burgers equation which has been sometimes considered in the past as a model for the Navier–Stokes equations: consider a given $\nu > 0$ and a function $f:[0,1] \to \mathbb{R}$ which satisfies

$$-\nu \frac{d^2u}{dx^2} + u\frac{du}{dx} = f \quad \text{on } (0,1), \tag{10.30}$$

$$u(0) = u(1) = 0. \tag{10.31}$$

For the functional setting we take $H = L^2(0,1)$, $V = H_0^1(0,1)$, $D(A) = H_0^1(0,1) \cap H^2(0,1)$, $Au = -d^2u/dx^2$, for all $u \in D(A)$,

$$B(u,v) = u\frac{dv}{dx}, \quad \forall\, u, v \in V, \quad Bu = B(u,u).$$

Then, given f in H (or V'), the problem is to find $u \in D(A)$ (or V) which satisfies

$$\nu Au + Bu = f. \tag{10.32}$$

We can apply Theorem 10.3 with $X = D(A)$, $Y = H$, $N(u) = \nu Au + Bu$. The mapping N is obviously $\mathscr{C}^\infty$ and

$$N'(u)\cdot v = \nu Av + B(u,v) + B(v,u);$$

$N'(u)$, as the sum of an isomorphism and a compact operator, is a Fredholm operator of index 0. Now we claim that every $u \in D(A)$ is a regular point, so that $\omega_1 = H$.

In order to prove that u is a regular point, we have to show that the kernel of $N'(u)$ is 0 (which is equivalent to proving that $N'(u)$ is onto, as index $N'(u) = 0$). Let v belong to $N'(u)$; v satisfies

$$-\nu\frac{d^2v}{dx^2} + u\frac{dv}{dx} + v\frac{du}{dx} = 0 \quad \text{on } (0,1), \qquad v(0) = v(1) = 0.$$

By integration $-\nu v' + uv = \text{constant} = a$, and by a second integration taking into account the boundary conditions we find that $v = 0$.

Since the solution is unique if $f = 0$ $(u = 0)$, we conclude that

(10.33) (10.30)–(10.31) possesses a unique solution $\forall\, \nu$, $\forall f$.

Example 2. The second example is due to Gh. Minea [1] and corresponds to a space H of finite dimension; actually $H = \mathbb{R}^3$.

Theorem 10.3 is applied with $X = Y = \mathbb{R}^3$, $N(u) = \nu Au + Bu$ for all $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. The linear operator A is just the identity, and the nonlinear (quadratic) operator B is defined by $Bu = (\delta(u_2^2 + u_3^2), -\delta u_1 u_2, -\delta u_1 u_3)$ and possesses the orthogonality property $Bu \cdot u = 0$. The equation $N(u) = f$ reads

$$\nu u_1 + \delta(u_2^2 + u_3^2) = f_1,$$
$$\nu u_2 - \delta u_1 u_2 = f_2,$$
$$\nu u_3 - \delta u_1 u_3 = f_3.$$

It is elementary to solve this equation explicitly. There are one or three solutions if $|f_2| + |f_3| \neq 0$. In the "nongeneric case", $f_2 = f_3 = 0$, we get either one solution, or one solution and a whole circle of solutions: $\omega_1 \neq Y$ in this case.

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11 The Squeezing Property

In this section we establish a squeezing property of the semigroup of operators associated with the time-dependent Navier–Stokes equations. This property, which will be useful for the next section, is also interesting by itself and has found other applications (see in particular Foias–Temam [7]): the squeezing property shows that, up to some error which can be made arbitrarily small, the flow is essentially characterized by a finite number of parameters.

11.1. An a priori estimate on strong solutions. The following a priori estimate is valid for *all time*; it is a particular case of the more general results stated in § 12.3 (Lemma 12.2); we refer the reader to that section for more details.

LEMMA 11.1. *Assume that $u_0 \in H$, and f satisfies*

$$\text{(11.1)} \qquad \begin{aligned} &f \textit{ is continuous and bounded from } [0, \infty) \textit{ into } H,\\ &f' \textit{ is continuous and bounded from } [0, \infty) \textit{ into } V', \end{aligned}$$

and let u be the strong solution to the Navier–Stokes equations given by Theorem 3.2, defined on $[0, \infty)$ if $n = 2$, on $[0, T_1(u_0)]$ if $n = 3$.

Then u is bounded in V on $[0, \infty)$ if $n = 2$, on $[0, T_1(u_0)]$ if $n = 3$, and for every $\alpha > 0$, Au is bounded on $[\alpha, \infty)$ if $n = 2$, on $[\alpha, T_1(u_0)]$ if $n = 3$,

$$\text{(11.2)} \qquad \sup_{t \geqq 0} \|u(t)\| \leqq c'(\|u_0\|, f, \nu, \Omega),$$

$$\text{(11.3)}^{1} \qquad \sup_{t \geqq \alpha} |Au(t)| \leqq c''(\alpha, \|u_0\|, f, \nu, \Omega)$$

where c' depends on $\|u_0\|, f, \nu, \Omega$ (or Q), and c'' depends on the same data and furthermore α.

Remark 11.1. i) If $n = 3$, (11.2) follows from (3.28)–(3.29), but (11.2) must be proved for large time if $n = 2$.

ii) If u is defined on some interval (τ_0, τ_1) and $\sup_{\tau_0 \leqq t \leqq \tau_1} \|u(t)\| \leqq c_1' < \infty$ then, by inspection of the proof of (11.3), we can see that, for every $\alpha > 0$, $\sup_{\tau_0 + \alpha \leqq t \leqq \tau_1} |Au(t)| \leqq c_1''$, where c_1'' depends on c_1' and the data u_0, f, ν, Ω (or Q).

11.2. The squeezing property. We assume that $n = 2$ or 3. Let u_0 and v_0 be given in V, $\|u_0\| \leqq R$, $\|v_0\| \leqq R$, and let f be given satisfying

$$\text{(11.4)} \qquad \begin{aligned} &f \textit{ is continuous and bounded from } [0, \infty) \textit{ into } H,\\ &f' \textit{ is continuous and bounded from } [0, \infty) \textit{ into } V'. \end{aligned}$$

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[1] $0 < \alpha \leqq T_1(u_0)$ if $n = 3$, and in this case the supremum is for $0 \leqq t \leqq T_1(u_0)$ in (11.2) and for $\alpha \leqq t \leqq T_1(u_0)$ in (11.3).

We denote by u and v the strong solutions of Navier–Stokes equations corresponding respectively to (u_0, f) and to (v_0, f), which are given by Theorem 3.2, and are defined on $(0, \infty)$ if $n=2$ and on $[0, T_1(R)]$ if $n=3$, where

$$T_1(R)=\frac{K_6}{(1+R^2)^2} \tag{11.5}$$

(see (3.29)).

We recall that w_m, λ_m denote the eigenfunctions and eigenvalues of A and that P_m is the projector in H (or V, V', $D(A)$) on the space W_m spanned by $w_1, \ldots, w_m$.

THEOREM 11.1. *Under the above hypotheses, for every $\alpha>0$, there exist two constants c_5, c_6, which depend on α, R, f, ν, Ω (or Q) and such that, for every $t \geqq \alpha$, for every m sufficiently large, i.e. satisfying*

$$\lambda_{m+1} \geqq c_6, \tag{11.6}$$

we have either

$$|u(t)-v(t)| \leqq \sqrt{2}\,|P_m(u(t)-v(t))| \tag{11.7}$$

or

$$|u(t)-v(t)| \leqq e^{-c_5\lambda_{m+1}}. \tag{11.8}$$

Proof. We assume in this proof that $t \geqq \alpha/2$ if $n=2$, $\alpha/2 \leqq t \leqq T_1(R)$ if $n=3$.

i) We set

$$\begin{aligned} &w(t)=u(t)-v(t),\\ &p(t)=|P_m w(t)|, \qquad q(t)=|(I-P_m)w(t)|. \end{aligned} \tag{11.9}$$

We have, by taking the difference of the equations (2.43) for u and v,

$$\frac{dw}{dt}+Aw+B(u, w)+B(w, v)=0. \tag{11.10}$$

Taking the scalar product, in H, of (11.10) with $P_m w(t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}p^2+\nu\,\|P_m w\|^2\\ &\quad=-b(u, w, P_m w)-b(w, v, P_m w)\\ &\quad=b(u, P_m w, (I-P_m)w)-b(P_m w, v, P_m w)\\ &\qquad\quad -b((I-P_m)w, v, P_m w) \qquad \text{(with (2.33)–(2.34))}. \end{aligned} \tag{11.11}$$

We apply Lemma 2.1 (i.e. (2.29)) respectively with $m_1, m_2, m_3=2, 0, 0$; $\frac{1}{4}, 1, \frac{1}{4}$; $0, 1, 1$, and we use (11.3) for u and v, with α replaced by $\alpha/2$:

$$\begin{aligned} &b(u, P_m w, (I-P_m)w) \geqq -c_1 c''\,\|P_m w\|\,q,\\ &-b(P_m w, v, P_m w) \geqq -c_1 c''\,|P_m w|_{1/4}^2\\ &\qquad\qquad \geqq -c_1' p^{3/2}\,\|P_m w\|^{1/2} \qquad \text{(with (2.20))},\\ &-b((I-P_m)w, v, P_m w) \geqq -c_1 c'' q\,\|P_m w\|. \end{aligned} \tag{11.12}$$

With these lower bounds, (11.11) becomes

$$\frac{1}{2}\frac{d}{dt}p^2 \geqq -\nu\,\|P_m w\|^2 - c_2' q\,\|P_m w\| - c_1' p^{3/2}\,\|P_m w\|^{1/2}. \tag{11.13}$$

Since $\lambda_1 \leqq \lambda_{m+1}$ and

$$\|P_m w\| \leqq \lambda_m^{1/2}\,|P_m w| \leqq \lambda_{m+1}^{1/2} p,$$

we have

$$p\frac{dp}{dt} \geqq -p(\nu\lambda_{m+1}p + c_1'\lambda_{m+1}^{1/4}\lambda_1^{-1/4}p + c_2'\lambda_{m+1}^{1/2}q). \tag{11.14}$$

In a similar manner, taking the scalar product in H of (11.10) with $(I-P_m)w$, we obtain

$$\frac{1}{2}\frac{d}{dt}q^2 + \nu\,\|(I-P_m)w\|^2 \leqq c_3' p\,\|(I-P_m)w\| + c_4'\,\|(I-P_m)w\|^{1/2}\,q^{3/2}.$$

But

$$\|(I-P_m)w\| \geqq \lambda_{m+1}^{1/2}\,|(I-P_m)w| \geqq \lambda_1^{1/2}, \tag{11.15}$$

and thus

$$q\frac{dq}{dt} \leqq \|(I-P_m)w\|\,(-\nu\lambda_{m+1}^{1/2}q + c_3'p + c_4'\lambda_1^{-1/4}q). \tag{11.16}$$

We set $\lambda = \lambda_{m+1}$, $\rho = \max(c_1'\lambda_1^{-1/4},\ c_2',\ c_3',\ c_4'\lambda_1^{-1/4})$. Thus (11.14)–(11.16) take the form:

$$\begin{aligned} p\frac{dp}{dt} &\geqq -p(\nu\lambda p + \rho p + \rho\lambda^{1/2}q), \\ q\frac{dq}{dt} &\leqq \|(I-P_m)w\|(-\nu\lambda^{1/2}q + \rho p + \rho q), \end{aligned} \tag{11.17}$$

and if

$$-(\nu\lambda^{1/2} - \rho)q + \rho p \leqq 0, \tag{11.18}$$

then for almost every t[2]

$$\begin{aligned} \frac{dp}{dt} &\geqq -\lambda^{1/2}[(\nu\lambda^{1/2} + \rho)p + \rho q], \\ \frac{dq}{dt} &\leqq \lambda^{1/2}[-(\nu\lambda^{1/2} - \rho)q + \rho p]. \end{aligned} \tag{11.19}$$

ii) We now want to prove the alternative (11.7)–(11.8). Let us consider a

---

[2] The first relation of (11.19) follows easily from the first relation of (11.17) at a point t_0 where $p(t_0) > 0$. If $p(t_0) = 0$ then either p is not differentiable at t_0 (which can only occur on a set of points t_0 of measure 0) or $dp(t_0)/dt \geqq 0$. The differentiability almost everywhere of p follows trivially from the differentiability almost everywhere in V' of u and v, and thus $w = u - v$.

specific point t_0, $t_0 \geqq \alpha$ if $n=2$, $\alpha \leqq t_0 \leqq T_1(R)$ if $n=3$. If $q(t_0) \leqq p(t_0)$ then (11.7) is satisfied at t_0 and we have nothing to prove. Therefore we assume that

$$q(t_0) > p(t_0), \tag{11.20}$$

and on the other hand we assume that m is sufficiently large so that

$$\nu\lambda^{1/2} - \rho > 2\rho. \tag{11.21}$$

This implies that

$$(\nu\lambda^{1/2} - \rho) q(t) > 2\rho p(t), \tag{11.22}$$

in a neighborhood of t_0.

At this point two possibilities can occur: either (11.22) is valid on the whole interval $[t_0 - \alpha/2, t_0]$, or (11.22) is valid for $t \in (t_1, t_0)$, with $t_1 \geqq t_0 - \alpha/2$ and

$$(\nu\lambda^{1/2} - \rho) q(t_1) = 2\rho p(t_1). \tag{11.23}$$

The second possibility is discussed in point iii) of the proof; in the first case, since (11.18) and (11.19) are satisfied on $[t_0 - \alpha/2, t_0]$, we have

$$\frac{dq(t)}{dt} \leqq -\frac{\lambda^{1/2}}{2} (\nu\lambda^{1/2} - \rho) q(t), \qquad t \in \left[t_0 - \frac{\alpha}{2}, t_0\right],$$

and consequently

$$q(t_0) \leqq \exp\left(-\frac{1}{2} \lambda^{1/2} (\nu\lambda^{1/2} - \rho)\alpha\right) q\left(t_0 - \frac{\alpha}{2}\right). \tag{11.24}$$

Due to (11.2),

$$q\left(t_0 - \frac{\alpha}{2}\right) \leqq \left| w\left(t_0 - \frac{\alpha}{2}\right)\right| \leqq \frac{c'}{\sqrt{\lambda_1}}, \tag{11.25}$$

and (11.8) follows at t_0 with appropriate constants c_5, c_6.

iii) The last step consists in proving (11.7) at $t = t_0$, $\alpha \leqq t_0 (\leqq T_1(R)$ if $n=3)$, assuming that (11.22) is satisfied on an interval (t_1, t_0), and (11.23) is satisfied at t_1 $(t_0 - \alpha/2 \leqq t_1 < t_0)$.

We denote by (11.19)′ the differential system obtained by replacing the inequalities in (11.19) by equality signs. It is natural to formally associate with (11.19)′ the differential system

$$\frac{dp}{dq} = \frac{(\nu\lambda^{1/2} + \rho)p + \rho q}{(\nu\lambda^{1/2} - \rho)q - \rho p}.$$

We prove by elementary calculations that $\Phi(p(t), q(t))$ remains constant when $p(\cdot)$, $q(\cdot)$ are solutions of (11.19)′ and

$$\Phi(p, q) = (p+q) \cdot \exp\left(\frac{\nu\lambda^{1/2} q}{\rho(p+q)}\right). \tag{11.26}$$

By elementary calculations we also check that

$$\frac{d}{dt}\Phi(p(t), q(t)) \leqq 0$$

when $p(\cdot)$, $q(\cdot)$ are solutions of (11.19), in particular on $[t_1, t_0]$. Thus

$$\Phi(p(t_0), q(t_0)) \leqq \Phi(p(t_1), q(t_1)).$$

By (11.23),

$$\Phi(p(t_1), q(t_1)) = \frac{\nu\lambda^{1/2}+\rho}{2\rho}\, q(t_1) \cdot \exp\left(\frac{2\nu\lambda_1}{\nu\lambda^{1/2}+1}\right),$$

and on the other hand, with (11.20),

$$\Phi(p(t_0), q(t_0)) \geqq q(t_0) \cdot \exp\left(\frac{\nu\lambda^{1/2}}{2\rho}\right).$$

Since

$$q(t_1) \leqq |w(t_1)| \leqq c'/\sqrt{\lambda_1} \tag{11.27}$$

and $p(t_0) < q(t_0)$, we obtain an estimate of the type (11.8) for $|w(t_0)| = (p^2(t_0)+q^2(t_0))^{1/2} \leqq \sqrt{2}q(t_0)$.

The proof is complete. □

Remark 11.2. i) As in Remark 11.1 ii), the squeezing property is valid for any pair of strong solutions u, v of the Navier–Stokes equations, defined on some interval (τ_0, τ_1), and satisfying

$$\sup_{\tau_0 \leqq t \leqq \tau_1} \|u(t)\| \leqq c_1' < \infty, \qquad \sup_{\tau_0 \leqq t \leqq \tau_1} \|v(t)\| \leqq c_1' < \infty,$$

so that

$$\sup_{\tau_0+\alpha/2 \leqq t \leqq \tau_1} \|Au(t)\| \leqq c_1'' < \infty, \qquad \sup_{\tau_0+\alpha/2 \leqq t \leqq \tau_1} |Av(t)| \leqq c_1'' < \infty.$$

ii) Of course we can replace the coefficient $\sqrt{2}$ in the right-hand side of (11.6) by any constant $\beta > 1$. It is clear that in such case c_5 and c_6 depend also on β.

Remark 11.3. It is reasonable to consider as physically "undistinguishable" two strong solutions u, v, which satisfy (11.8) for some m sufficiently large. Mathematically, we can consider an equivalence relation among the strong solutions which are defined on (τ_0, τ_1) (and bounded in V on this interval): two solutions u, v are said to be equivalent if there exist two constants c_5, c_6 such that (11.6)–(11.8) hold on (τ_0, τ_1). It is easy to check that this relation is indeed an equivalence one. *The squeezing property implies that $P_m u$ characterizes the equivalence class of u*, i.e., characterizes u up to an error $\exp(-c\lambda_{m+1}^{1/2})$.

Remark 11.4. Let us mention an alternate form of the squeezing property which is useful for later purposes.

Let u and v be two strong solutions of the Navier–Stokes equations, defined and bounded in V on some *finite* interval $\tau_0 \leqq t \leqq \tau_1 < \infty$. For every $\alpha > 0$, there exist two constants c_5^*, c_6^* which depend on τ_0, τ_1, α, R, f, ν, Ω[3], such that, on $[\tau_0 + \alpha, \tau_1]$, we have either (11.7) or

$$|u(t) - v(t)| \leqq |u(t_0) - v(t_0)| \exp(-c_5^* \lambda_{m+1}^{1/2}) \tag{11.28}$$

for every m satisfying

$$\lambda_{m+1} \geqq c_6^*. \tag{11.29}$$

The proof is exactly the same, except that in (11.25) and (11.27) we replace the majorizations

$$\left| u\left(\tau_0 - \frac{\alpha}{2}\right) - v\left(\tau_0 - \frac{\alpha}{2}\right) \right| \leqq \frac{c'}{\sqrt{\lambda_1}}, \qquad |u(t_1) - v(t_1)| \leqq \frac{c'}{\sqrt{\lambda_1}}$$

by the estimate ($\leqq c'' |w(\tau_0)|$) which follows from the next lemma.

LEMMA 11.2. *If u and v are two strong solutions defined and bounded in V on a finite interval $\tau_0 \leqq t \leqq \tau_1$, $n = 2$ or 3, then there exists a constant c'' which depends on τ_0, τ_1, f, ν, Ω (or Q), and R, with*

$$R \geqq \sup_{t \in [\tau_0, \tau_1]} \|u(t)\|, \qquad R \geqq \sup_{t \in [\tau_0, \tau_1]} \|v(t)\|, \tag{11.30}$$

such that

$$|u(t) - v(t)| \leqq c'' \cdot |u(\tau_0) - v(\tau_0)| \quad \forall\, t \in [\tau_0, \tau_1]. \tag{11.31}$$

Proof. It follows easily from the energy inequality (3.24), valid for strong solutions, and from the assumptions on u and v, that Au and Av belong to $L^2(\tau_0, \tau_1; H)$ with their norm in this space bounded by a constant depending on τ_0, τ_1, f, ν, Ω (or Q) and R.

We take the scalar product in H of (11.10) with $w(t)$, and using (2.29), (2.33), (2.34) and (11.30) and Remark 11.1 ii), we get, as for (11.13):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |w|^2 + \nu \|w\|^2 &= -b(w, v, w) \leqq c_1 |Av|\, |w|^{3/2} \|w\|^{1/2} \\ &\leqq \frac{\nu}{2} \|w\|^2 + c_5' |w|^2 |Av|^{4/3}, \end{aligned}$$

$$\frac{d}{dt} |w|^2 \leqq 2c_5' |Av|^{4/3} |w|^2, \tag{11.32}$$

and since $|Av|^{4/3} \in L^1(0, T)$, (11.31) follows from Gronwall's lemma. □

[3] Here $R \geqq \sup_{t \in [\tau_0, \tau_1]} \|u(t)\|$, $R \geqq \sup_{t \in [\tau_0, \tau_1]} \|v(t)\|$.

12 Hausdorff Dimension of an Attractor

In this section we derive an important property of functional invariant sets and attractors which are bounded in V, i.e., in the H^1-norm. *We show that these sets have a finite Hausdorff dimension.*

Definitions are given below but it is well known that attractors encompass the long-time behavior of solutions of the Navier–Stokes equations. As explained below, attractors are always bounded in the H^1-norm in space dimension $n = 2$, but it is not known whether this is true in space dimension $n = 3$.

The main result of this section was first proved in C. Foias–R. Temam [5] and the proof presented below is based on this article. Although this result was subsequently improved in different ways, the proof in this article relies mostly on the Navier–Stokes equation techniques presented in this book, while the subsequent proofs depend more heavily on dynamical systems techniques (cf. the comments in the Comments and Bibliography section).

This result on the Navier–Stokes attractors shows that the long-time behavior of the solutions of these equations is *finite dimensional* although these equations have *infinite dimension*. As indicated before, this is true without restriction for two-dimensional flows and, in space dimension three, this is true for flows which remain smooth for all time. Other aspects of finite dimensionality of flows are mentioned in the Comments and Bibliography section.

12.1. Functional invariant sets and attractors. Throughout this section, we assume that $f(t)=f$ is independent of t and belongs to H. Let $S(t)$ be the semigroup associated with the *strong* solution of the Navier–Stokes equation

$$\frac{du}{dt}+\nu Au+Bu=f,$$

i.e., the mapping $u_0=u(0)\mapsto u(t)$, which is defined on V for all $t\geqq 0$ if $n=2$, and for $t\in[0, T_1(u_0)]$ (at least), if $n=3$.

DEFINITION 12.1. A *functional invariant set* for the Navier–Stokes equations is a subset X of V which satisfies the following properties:

i) For every $u_0\in X, S(t)u_0$ is defined for every $t>0$.

ii) $S(t)X=X$ for all $t>0$.

We recall that an attractor (in V or H) is a functional invariant set X which satisfies furthermore the condition:

iii) X possesses an open neighborhood ω (in V or H), and for every $u_0\in\omega$, $S(t)u_0$ tends to X, in V or H, as $t\to\infty$.

If $n=2$, we have the following:

LEMMA 12.1. *Let us assume that $n=2$, and that u_0 and f are given in H. Then there exists a functional invariant set $X\subset V$, compact in H, such that the distance in H of $u(t)$ to X tends to 0 as $t\to\infty$, where $u(\cdot)$ is the solution of Problem* 2.1.

Proof. We set

$$X=\bigcap_{\tau\geqq 0}\overline{\{u(s),\, s>\tau\}} \tag{12.1}$$

where the closures are taken in H. An element $\phi\in H$ is in X if and only if

$$\forall\,\varepsilon>0,\quad \forall\,\tau>0,\quad \exists\, s>\tau\quad \text{such that } |u(s)-\phi|\leqq\varepsilon. \tag{12.2}$$

It is easy to see that X is also equal to

$$\bigcap_{\tau\geqq t_0}\overline{\{u(s),\, s>\tau\}}\qquad (t_0\geqq 0). \tag{12.3}$$

Since for $t_0>0$, the set

$$\{u(s),\, s\geqq t_0\} \tag{12.4}$$

is bounded in V (cf. Lemma 11.1), so is its closure in V and so is X. By Theorem 3.2, since $n=2$, $S(t)\phi$ is defined for all $t>0$, for every $\phi\in X$, and the first condition i) in Definition 12.1 is satisfied. For condition ii) we observe that $S(t)X\subset X$ follows easily from the fact that $S(t)u(s)=u(s+t)$, for all $s, t\geqq 0$. In order to show that $S(t)X=X$, we note that $S(t)$ is *injective because of the time analyticity property* (see § 7) and therefore since the set (12.4) is relatively compact in $H(t_0>0)$, the sequence $u(s_j), j\in\mathbb{N}$, converges in H whenever the sequence $S(t)u(s_j)$ converges. Indeed, if $S(t)u(s_j)$ converges in H to some limit ϕ, as $j\to\infty$ (and $s_j\to\infty$), then by (12.2), $\phi\in X$. The sequence $u(s_j)$, being bounded in V, is relatively compact in H and possesses cluster points in H; if ψ and ψ' are two cluster points, $S(t)\psi=S(t)\psi'=\phi$ so that $\psi'=\psi$, the cluster point is unique and the whole sequence $u(s_j)$ converges to it in H as $j\to\infty$.

Finally we show that the distance in H of $u(t)$ to X tends to 0 as $t\to\infty$. We argue by contradiction. If this were not true, we could find $\varepsilon_0>0$, and a sequence $s_j\to\infty$, such that

$$\inf_{\phi\in X}|u(s_j)-\phi|\geqq\varepsilon_0>0\quad \forall\, j.$$

Since $u(s_j)$ is bounded in V we can, by extracting a sequence, assume that $u(s_j)$ converges in H to some limit ψ. By (12.2), $\psi\in X$, in contradiction with

$$|u(s_j)-\psi|\geqq\varepsilon_0>0\quad \forall\, j.$$

The lemma is proved. □

12.2. Hausdorff dimension of functional invariant sets.

THEOREM 12.1. *Let $n=2$ or 3 and let $f\in H$. We assume that X is a functional invariant set bounded in V. Then the Hausdorff dimension of X is finite and this dimension is bounded by a number c, which depends on Ω (or Q), ν, $|f|$ and*[1] $R=\sup_{\phi\in X}\|\phi\|$.

Proof. i) We are going to make use of the squeezing property, in the conditions indicated in Remark 11.4: we fix $\alpha>0$ and choose $\tau_0=0$, $\tau_1=2\alpha$,

[1] The bound given in the proof is, in fact, an increasing function of ν^{-1}, $|f|$ and R.

and $t \in [\alpha, 2\alpha]$[2]. We then choose m sufficiently large so that (11.29) is verified and (see (11.28))

$$\exp(-c_5^* \lambda_{m+1}^{1/2}) \leqq \eta, \tag{12.5}$$

and the value (or one value) of η which is appropriate below is

$$\eta = \frac{2-\sqrt{2}}{8}. \tag{12.6}$$

We set $S(t) = S$, then Theorem 11.1 and Remark 11.4 show us that for all $\phi, \psi \in X$, we have either

$$|S\phi - S\psi| \leqq \sqrt{2}\,|P_m(S\phi - S\psi)| \quad \text{or} \quad |S\phi - S\psi| \leqq \eta\,|\phi - \psi|. \tag{12.7}$$

Given $r > 0$, since X is relatively compact in H, it can be covered by a finite number of balls of H of radius $< r$, say $B_1, \ldots, B_M$. By definition, $SX = X$; hence

$$X \subset \bigcup_{k=1}^{M} S(B_k \cap X) \tag{12.8}$$

and (12.7) implies that for all $k = 1, \ldots, M$, and for all $u, v \in S(B_k \cap X)$ we have either

$$|u - v| \leqq \sqrt{2}\,|P_m(u - v)|, \quad \text{or} \quad |u - v| \leqq \eta \operatorname{diam} B_k. \tag{12.9}$$

Let ϕ be an arbitrary point of $S(B_k \cap X)$ and let Y_k be the largest subset of $S(B_k \cap X)$, containing ϕ and such that

$$|\psi - \theta| \leqq \sqrt{2}\,|P_m(\psi - \theta)| \quad \forall\, \psi, \theta \in Y_k;$$

of course Y_k may be reduced to ϕ[3].

ii) We now define a reiterated covering of X.

Let $B_{k1}^m, \ldots, B_{kM_k}^m$ be a covering of $P_m S(B_k \cap X)$ by balls of $P_m H$ of radius $\leqq r_k' = \frac{1}{4} \operatorname{diam} B_k$. For those j's such that $P_m Y_k \cap B_{kj}^m \neq \varnothing$, we choose arbitrarily an $u_{kj} \in Y_k$, such that $P_m u_{kj} \in P_m Y_k \cap B_{kj}^m$, and we denote by B_{kj} the ball of H centered at u_{kj} and of radius

$$r_k'' = \sqrt{2}\, r_k' + \eta \operatorname{diam} B_k = \left(\frac{\sqrt{2}}{4} + \eta\right) \operatorname{diam} B_k.$$

Setting $B_{kj} = \varnothing$ if $B_{kj}^m \cap P_m Y_k = \varnothing$, we have

$$S(B_k \cap X) \subset \bigcup_{j=1}^{M_k} B_{kj}. \tag{12.10}$$

Indeed, if $u \in S(B_k \cap X)$, then there exists $u' \in Y_k$ ($u' = u$ if $u \in Y_k$) such that, by (12.9) and the definition of Y_k,

$$|u - u'| \leqq \eta \operatorname{diam} B_k.$$

---

[2] Since, by the definition of a functional invariant set, $S(t)\phi$ is defined for every $t \geqq 0$, for every $\phi \in X$ we are free to choose τ_0, τ_1 in Remark 11.4, $0 \leqq \tau_0 \leqq \tau_1 < +\infty$.

[3] If $Y_k \neq S(B_k \cap X)$ then $\sup_{\psi \in S(B_k \cap X) \setminus Y_k} \operatorname{dist}(\psi, Y_k) \leqq \eta \operatorname{diam} B_k$.

Then

$$|u-u_{kj}| \leqq |u-u'|+|u'-u_{kj}| \leqq \eta \operatorname{diam} B_k + \sqrt{2}\,|P_m(u'-u_{kj})|$$
$$\leqq \eta \operatorname{diam} B_k + \sqrt{2}\, r_k'.$$

Before proceeding to the next step, let us observe that we can derive an upper bound for M_k.

By Lemma 11.2,

$$|S\phi - S\psi| \leqq c''\,|\phi-\psi| \quad \forall\, \phi, \psi \in B_k \cap X;$$

hence

$$|u-v| \leqq c'' \operatorname{diam} B_k \quad \forall\, u, v \in S(B_k \cap X),$$

(12.11)
$$\operatorname{diam} S(B_k \cap X) \leqq c'' \operatorname{diam} B_k,$$
$$\operatorname{diam} P_m S(B_k \cap X) \leqq c'' \operatorname{diam} B_k.$$

Since $P_m S(B_k \cap X)$ is included in a ball of $P_m H$ of radius $c'' \operatorname{diam} B_k$, we have

(12.12)
$$M_k \leqq l_m\left(\frac{1}{4c''}\right),$$

where $l_m(\sigma)$ is, in $\mathbb{R}^m$, the minimum number of balls of radius $\leqq \sigma$ which is necessary to cover a ball of radius 1, $l_m(\sigma) \leqq 2^{-m/2}\sigma^{-m}$.

iii) We have

$$\operatorname{diam} B_{kj} \leqq \left(\frac{\sqrt{2}}{2} + 2\eta\right) \operatorname{diam} B_k$$
$$\leqq \varepsilon \operatorname{diam} B_k \qquad \text{(with the choice (12.6) of } \eta)$$
$$\leqq \varepsilon r$$

$(\varepsilon = (2-\sqrt{2})/4)$; with (12.10)–(12.12), for any $\gamma > 0$,

$$\sum_{j=1}^{M_k} (\operatorname{diam} B_{kj})^\gamma \leqq M_k \varepsilon^\gamma (\operatorname{diam} B_k)^\gamma \leqq l_m\left(\frac{1}{4c''}\right) \varepsilon^\gamma (\operatorname{diam} B_k)^\gamma.$$

Since the B_{kj}, $k=1,\dots,M$, $j=1,\dots,M_k$, for all k, constitute a covering of X by balls of radius $\leqq \varepsilon r$, we infer from the definition of $\mu_{\gamma,s}(X)$ in (5.2) that

$$\mu_{\gamma,\varepsilon r}(X) \leqq \sum_{k=1}^{M} \sum_{j=1}^{M_k} (\operatorname{diam} B_{kj})^\gamma \leqq \mu_{\gamma,\varepsilon r}(X) \leqq l_m\left(\frac{1}{4c''}\right) \varepsilon^\gamma \sum_{k=1}^{M} (\operatorname{diam} B_k)^\gamma.$$

This last inequality, valid for any covering of X by balls of radius $\leqq r$, implies in its turn

(12.13)
$$\mu_{\gamma,\varepsilon r}(X) \leqq \lambda \mu_{\gamma,r}(X),$$

where $\lambda = l_m(1/4c'')\varepsilon^\gamma$. By reiteration,

(12.14)
$$\mu_{\gamma,\varepsilon^j r}(X) \leqq \lambda^j \mu_{\gamma,r}(X).$$

Since $\varepsilon < 1$, if γ is sufficiently large, then $\lambda < 1$ and $\lambda^j \to 0$ as $j \to \infty$, so that

$$\mu_\gamma(X) = \sup_{\rho > 0} \mu_{\gamma,\rho}(X) = \lim_{j\to\infty} \mu_{\gamma,\varepsilon^j r}(X) = 0.$$

Therefore, provided

$$\gamma \geqq -\frac{\log l_m(1/4c'')}{\log((2-\sqrt{2}))/4}, \tag{12.15}$$

the Hausdorff dimension of X is $\leqq \gamma$.

The proof is complete. □

Remark 12.1. We do not know, even if $n = 2$, whether every functional invariant set is bounded in V. But Lemma 12.2 shows us that this is the case for those associated with the limit of $u(t)$ as $t \to \infty$.

The question of the boundedness of X in V is totally open if $n = 3$.

12.3. Other properties of functional invariant sets. We conclude this section by indicating some regularity results and a priori estimates which are valid for long times, and we infer from these estimates some properties of functional invariant sets; in particular, if f is sufficiently regular and $n = 2$, a functional invariant set is necessarily carried by the space of $\mathscr{C}^\infty$ functions.

LEMMA 12.2. *Let $n = 2$ or 3, and consider the time-dependent Navier–Stokes equations with the data u_0, f satisfying*

$$u_0 \in H, \tag{12.16}$$

$$\frac{d^j f}{dt^j} \text{ is continuous and bounded from } [0,\infty) \text{ into } \mathbb{H}^{m-2j-2}(\mathcal{O}) \cap H, \quad j = 0, \dots, l-1, \tag{12.17}$$

$$\frac{d^l f}{dt^l} \text{ is continuous and bounded from } [0,\infty) \text{ into } V_{m-2l-1}, \tag{12.18}$$

where m is an integer $\geqq 1$ and $l = [m/2]$[4].

We assume that the solution u to Problem 2.2 satisfies

$$u \in L^\infty(t_0, \infty; V) \quad \text{for some } t_0 \geqq 0. \tag{12.19}$$

Then u is continuous and bounded from $[\eta, \infty)$ into $\mathbb{H}^m(\mathcal{O})$, $d^j u/dt^j$ is continuous and bounded from $[\eta,\infty)$ into $\mathbb{H}^{m-2j}(\mathcal{O})$, $j = 1, \dots, l$, $\forall\, \eta > 0$, and the corresponding norms of u and $d^j u/dt^j$ are bounded by constants which depend only on the data, t_0, η, and $|u|_{L^\infty(t_0,\infty;V)}$.

We do not give the proof of these a priori estimates, which is rather long; we refer the reader to C. Guillopé [1].

We recall that, if $n = 2$, then (12.19) is automatically satisfied for every $t_0 > 0$ and the result holds for $\eta > 0$ arbitrarily small. When $n = 3$, we do not know whether (12.19) is true; however, the boundedness of $\|u(t)\|$ for $t \to \infty$ is closely related to Leray's conjecture on turbulence.

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[4] $\mathcal{O} = \Omega$ or Q; $V_{m-2l-1} = V'$ if $m = 2l$, and $= H$ if $m = 2l+1$.

Let us recall that Leray's hypothesis, and his motivation in introducing the concept of weak solution was that singularities may develop spontaneously over a finite interval. We know that $|u(t)|$ remains bounded even for weak solutions but the enstrophy $\|u(t)\|$ may perhaps become infinite. We formulate Leray's assumption in a more precise way:

(12.20) There exist $\Omega \subset \mathbb{R}^3$, $T>0$, $\nu>0$, $u_0 \in V$, $f \in L^\infty(0, T; H)$ such that $\|u(t)\|$ becomes infinite at some time $t \in (0, T)$, u being the weak solution of the Navier-Stokes equations.

Let us assume now that $f \in L^\infty(0, \infty; H)$ and is not "too chaotic" at infinity, i.e.,

(12.21) There exists $\alpha>0$ such that for any sequence $t_j \to +\infty$, the sequence of functions $f_j(t)=f(t+t_j), 0<t<\alpha$ possesses at least one cluster point in $L^2(0, \alpha; H)$.

This property is trivially satisfied if f is independent of time, or more generally if

$$f \in L^2_{\mathrm{loc}}(0, \infty; V),\ f' \in L^2_{\mathrm{loc}}(0, \infty; V') \tag{12.22}$$

with

$$|f|_{L^2(s,s+\alpha;V)} + |f|_{L^2(s,s+\alpha;V')} \leqq c(f)\alpha,$$

for all $s>0, \alpha>0$, where $c(f)$ is independent of s. In such a case we have the following result proved elsewhere (cf. C. Foias–R. Temam [13]):

THEOREM 12.2. *Given* $\Omega \in \mathbb{R}^3, T, \nu$ *and* $f \in L^\infty(0, \infty; H)$ *satisfying* (12.21), *then either* (12.19) *or* (12.20) *holds; i.e., there exists* $u_0 \in V$ *for which* $\|u(t)\|$ *becomes infinite on the interval* $[0, \alpha]$, $\alpha>0$ *given in* (12.21).

Finally we infer from Lemma 12.2 the following *regularity result* for a functional invariant set (or attractor):

THEOREM 12.3. *Let us assume that f is independent of t, and* $f \in \mathscr{C}^\infty(\bar{\mathcal{O}})^n \cap H^5$. *Then any functional invariant set X, bounded in V, for the corresponding Navier–Stokes equations, is contained in* $\mathscr{C}^\infty(\bar{\mathcal{O}})^n$.

Proof. We consider the semigroup operator $S(t)$ defined at the beginning of § 12.1. Since $S(t)X=X$, for all $t>0$, and $S(t)$ is one-to-one by the analyticity property, every $\phi \in X$ is of the form $\phi = S(t)u_0$, with $u_0 \in X$, and therefore by Lemma 12.2, ϕ is in $\mathbb{H}^m(\bar{\mathcal{O}})$, for all m.

Remark 12.2. i) This result is comparable to regularity results for the solutions of stationary Navier–Stokes equations, but it applies also to periodic, and quasi periodic solutions, among others.

ii) The conclusion of Theorem 12.2 is valid if we assume only that X is bounded in $V_{1/2+\varepsilon}$ for some $\varepsilon>0$. See C. Guillopé [1] for the details.

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[5] If we consider the flow in $\Omega (\mathcal{O}=\Omega)$, we also need (1.7) with $r=\infty$.

PART III

Questions Related to the Numerical Approximation

Orientation. In Part III, containing §§ 13 and 14, we develop some results related to the numerical approximation of the Navier–Stokes equations. This is far from being an exhaustive treatment of a subject which is developing rapidly, due to the needs of modern sophisticated technologies and the appearance of more and more powerful computers.

Part III does not discuss any practical developments concerning implementation of computational fluid dynamics algorithms[1] (though these are important). It is limited to the presentation of two questions related to the numerical analysis of the Navier–Stokes equations. The first is the question of stability and convergence of a particular nonlinear scheme; this is developed in § 13. The particular scheme was chosen almost arbitrarily among several schemes presently used in large scale computations and for which convergence has been proved. The second question, considered in § 14, is that of the approximation of the Navier–Stokes equations for large time: in the context of the questions studied in § 12 (see also § 9), the problem of numerical computation of turbulent flows is connected with the computation of $u(t)$ for t large (while the force, or more generally the excitation, is independent of time). Section 14 contains some useful remarks pertaining to this question.

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[1] See the comments following § 14.

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13 Finite Time Approximation

In this section we study the convergence of a space and time discretization scheme for the evolution Navier–Stokes equations. This scheme combines a discretization in time by an alternating direction (or decomposition) method with a discretization in space by finite elements. It belongs to a series of efficient schemes which have been introduced and studied from a theoretical point of view in the past (cf. A. J. Chorin [2], [3], [4], R. Temam [1], [2], [5], [6]), and which have recently regained interest and have been implemented in large scale computations for industrial applications (cf. e.g., H. O. Bristeau–R. Glowinski–B. Mantel–J. Periaux–P. Perrier–O. Pironneau [1], R. Glowinski–B. Mantel–J. Periaux [1]).

The proof of convergence is the same as that appearing in [RT, Chap. III, §§ 5, 6, 7] and in R. Temam [1], [2], [4], [5] for closely related schemes. However, for the proof of the strong convergence result which is necessary for the passage to the limit, we use a compactness argument different and slightly simpler than that considered in those references. This compactness result is given in § 13.3.

13.1. An example of space-time discretization. We consider the flow in a bounded domain with periodic or zero boundary condition (i.e., $\mathcal{O} = \Omega$ or Q), and $n = 2$ or 3. We are given u_0 and f as in Theorem 3.2 and we want to approximate the strong solution u to Problem 2.2 defined in Theorem 3.2, on $[0, T]$ if $n = 2$, on $[0, T_*]$ if $n = 3$.

We are given a family of finite dimensional subspaces V_h, $h \in \mathcal{H}$, of V, such that

$$\bigcup_{h \in \mathcal{H}} V_h \quad \text{is dense in } V. \tag{13.1}$$

For instance we can take $\mathcal{H} = \mathbb{N}$, V_h = the space spanned by the eigenvectors $w_1, \ldots, w_h$, and the discretization would correspond to a Galerkin method based on the spectral functions. Also V_h may be a more general subspace corresponding to a more general Galerkin method, but the most interesting case we have in mind is that in which V_h is a space of finite element functions with the set $\mathcal{H}$ properly defined: we refer to [RT, Chap. I, § 4] for the precise definition of a possible finite element space (Approximation APX4).

Actually, in order to be able to treat more finite element schemes we will consider the following more general situation:

(13.2) W_h, $h \in \mathcal{H}$, is a family of finite dimensional subspaces of $W = \mathbb{H}_0^1(\Omega)$ or $\mathbb{H}_p^1(Q)$, such that $\bigcup_{h \in \mathcal{H}} W_k$ is dense in W.

(13.3) For every h, V_h is a subspace of W_h, such that the family V_h, $h \in \mathcal{H}$, constitutes an external approximation[1] of V.

[1] For the precise definition of an external approximation cf. [RT, Chap. I, § 3.1]. The reader who so wishes may concentrate on the first, simpler case: $V_h \subset V$ with (13.1).

The cases covered by (13.2), (13.3) contain the finite element approximations of V called APX2, APX3, APX3′ in [RT, Chap. I, § 4] and others published in the literature (cf. M. Bercovier–M. Engelman [1], V. Girault–P. A. Raviart [1], among others).

For every h, let u_{0h} be the projection of u_0 in W, on V_h, i.e.,

$$
(13.4)\qquad \begin{aligned} &u_{0h} \in V_h,\\ &((u_{0h}, v_h)) = ((u_0, v_h)) \quad \forall\, v_h \in V_h. \end{aligned}
$$

Let N be an integer[2], $k = T'/N$. For every h and k we recursively define a family $u_h^{m+1/2}$ of elements of V_h, $m=0,\dots,N-1$, $i=1,2$. We start with

$$
(13.5)\qquad u_h^0 = u_{0h}.
$$

Assuming that u_h^0 (or more generally u_h^m, $m \geqq 0$) is known, we define $u_h^{m+1/2}$ and then u_h^{m+1} as follows:

$$
(13.6)\qquad \begin{aligned} &u_h^{m+1/2} \in V_h,\\ &\frac{1}{k}(u_h^{m+1/2} - u_h^m, v_h) + \frac{\nu}{2}((u_h^{m+1/2}, v_h)) = (f^m, v_h) \quad \forall\, v_h \in V_h, \end{aligned}
$$

where

$$
(13.7)\qquad f^m = \frac{1}{k}\int_{mk}^{(m+1)k} f(t)\, dt,
$$

and

$$
(13.8)\qquad \begin{aligned} &u_h^{m+1} \in W_h,\\ &\frac{1}{k}(u_h^{m+1} - u_h^{m+1/2}, v_h) + \frac{\nu}{2}((u_h^{m+1}, v_h)) + \tilde{b}(u_h^{m+1}, u_h^{m+1}, v_h) = 0\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, v_h \in W_h. \end{aligned}
$$

Here[3] $\tilde{b}(u, v, w)$ is the modification of the nonlinear term which was introduced in R. Temam [1], [2] and which guarantees the existence of a solution to (13.8), and the stability of the approximation for (13.8) or similar schemes:

$$
(13.9)\qquad \tilde{b}(u, v, w) = \sum_{i,j=1}^{n} \frac{1}{2}\int_{\mathcal{O}} \{u_i[(D_i v_j) w_j - v_j (D_i w_j)]\}\, dx.
$$

The existence and uniqueness of a solution $u_h^{m+1/2}$ for (13.6) follow from the Riesz representation theorem (or Lax–Milgram theorem). The existence of a solution $u_h^{m+1} \in W_h$ of (13.8) (a nonlinear finite dimensional problem) follows from the Brouwer fixed point theorem, using [RT, Chap. II, (2.34) and Lemma 1.4]. Problem (13.6) is just a variant of the linear stationary Stokes problem, and (13.8) is a relatively standard nonlinear Dirichlet problem. We refer to Glowinski, Mantel and Periaux [1] and to F. Thomasset [1], for the practical

[2] With the notation of Theorem 3.2, $T' = T$ if $n = 2$, $T' = T_* = \min(T, T_1(u_0))$ if $n = 3$.

[3] $b(u, v, w)$ may be $\neq 0$, while $\tilde{b}(u, v, w) = 0$, for $u, v \in \mathbb{H}^1_p(\Omega)$ (or $\mathbb{H}^1_p(Q)$), by lack of the free divergence property.

resolution of these problems. We note that, as usual (cf. R. Temam [3], [RT, Chap. III, § 7]), the alternating direction or splitting-up method allows us to decompose, and treat separately, the difficulties related to nonlinearity and to the incompressibility constraint $\operatorname{div} u = 0$ (contained of course in (13.6)).

We associate with this family of elements $u_h^{m+i/2}$ of W_h the following functions defined on $[0, T']$ (cf. R. Temam [3], [RT]):

- $u_k^{(i)}$ is the piecewise constant function which is equal to $u_h^{m+i/2}$ on $[mk, (m+1)k)$, $i = 1, 2$, $m = 0, \ldots, N-1$.
- $\tilde{u}_k^{(i)}$ is the continuous function from $[0, T']$ into W_h, which is linear on $(mk, (m+1)k)$ and equal to $u_h^{m+i/2}$ at mk, $i = 1, 2$, $m = 0, \ldots, N-1$.

By adding (13.6) and (13.8) we obtain a relation which can be reinterpreted in terms of these functions as

$$\left(\frac{d\tilde{u}_k^{(2)}}{dt}(t-k), v_h\right) + \frac{\nu}{2}((u_k^{(1)}(t) + u_k^{(2)}(t), v_h)) + \tilde{b}(u_k^{(2)}(t), u_k^{(2)}(t), v_h) = (f_k(t), v_h) \quad \forall\, v_h \in V_h \tag{13.10}$$

where

$$f_k(t) = f^m \quad \text{for } t \in [mk, (m+1)k). \tag{13.11}$$

Similarly, by adding (13.8) to the relation (13.6) for $m+1$, we arrive at an equation which is equivalent to

$$\left(\frac{d\tilde{u}_k^{(1)}}{dt}(t), v_h\right) + \frac{\nu}{2}((u_k^{(1)}(t+k) + u_k^{(2)}(t), v_h)) + \tilde{b}(u_k^{(2)}(t), u_k^{(2)}(t), v_h) = (f_k(t+k), v_h) \quad \forall\, v_h \in V_h. \tag{13.12}$$

13.2. The convergence theorem. We now discuss the behavior of these functions $u_k^{(i)}$, $\tilde{u}_k^{(i)}$, as h and $k \to 0$.

THEOREM 13.1. *Under the above assumptions, the functions $u_k^{(i)}$, $\tilde{u}_k^{(i)}$, $i = 1, 2$, belong to a bounded set of $L^2(0, T'; W) \cap L^\infty(0, T'; G)$, $G = \mathbb{L}^2(\mathcal{O})$.*

As k and $h \to 0$, $u_k^{(i)}$ and $\tilde{u}_k^{(i)}$ converge to the solution u of Problem 2.2 in $L^2(0, T'; W)$ and $L^q(0, T'; G)$ for all $1 \leqq q < \infty$.

Proof. i) We first derive the a priori estimates on the functions. We set $v_h = u_h^{m+1/2}$ in (13.6) and, observing that

$$(a-b, a) = \tfrac{1}{2}(|a|^2 - |b|^2 + |a-b|^2) \quad \forall\, a, b \in H,$$

we get

$$\begin{aligned} |u_h^{m+1/2}|^2 - |u_h^m|^2 + |u_h^{m+1/2} - u_h^m|^2 + \nu\, \|u_h^{m+1/2}\|^2 &= 2k(f^m, u_h^{m+1/2}) \leqq 2k\, |f^m|\, |u_h^{m+1/2}| \\ &\leqq \frac{2k}{\sqrt{\lambda_1}} |f^m|\, \|u_h^{m+1/2}\| \\ &\leqq \frac{\nu k}{2} \|u_h^{m+1/2}\|^2 + \frac{2k}{\nu\lambda_1} |f^m|^2; \end{aligned} \tag{13.13}$$

whence

$$|u_h^{m+1/2}|^2-|u_h^m|^2+|u_h^{m+1/2}-u_h^m|^2+\frac{\nu k}{2}\|u_h^{m+1/2}\|^2\leqq\frac{2k}{\nu\lambda_1}|f^m|^2. \tag{13.14}$$

Similarly, taking $v_h=u_h^{m+1}$ in (13.8) and taking into account (2.34), we obtain

$$|u_h^{m+1}|^2-|u_h^{m+1/2}|^2+|u_h^{m+1}-u_h^{m+1/2}|^2+\nu k\|u_h^{m+1}\|^2=0. \tag{13.15}$$

By adding all the relations (13.14), (13.15) for $m=0,\dots,N-1$, we find

$$\begin{aligned}
&|u_h^N|^2+\sum_{m=0}^{N-1}(|u_h^{m+1}-u_h^{m+1/2}|^2+|u_h^{m+1/2}-u_h^m|^2)\\
&\quad+\frac{k\nu}{2}\sum_{m=0}^{N-1}(\|u_h^{m+1/2}\|^2+2\|u_h^{m+1}\|^2)\\
&\qquad\leqq|u_{0h}|^2+\frac{2k}{\nu\lambda_1}\sum_{m=0}^{N-1}|f^m|^2\\
&\qquad\leqq|u_{0h}|^2+\frac{2k}{\nu\lambda_1}\sum_{m=0}^{N-1}\left|\int_{mk}^{(m+1)k}f(s)\,ds\right|^2\\
&\qquad\leq|u_{0h}|^2+\frac{2}{\nu\lambda_1}\int_0^T|f(s)|^2\,ds \qquad \text{(by the Schwarz inequality).}
\end{aligned} \tag{13.16}$$

Due to (13.4), $\|u_{0h}\|\leqq\|u_0\|$, and therefore the sequence u_{0h} is bounded in H. We then conclude that

$$\sum_{m=0}^{N-1}(|u_h^{m+1}-u_h^{m+1/2}|^2+|u_h^{m+1/2}-u_h^m|^2)\leqq L_1, \tag{13.17}$$

$$k\sum_{m=0}^{N-1}(\|u_h^{m+1/2}\|^2+2\|u_h^{m+1}\|^2)\leqq\frac{2L_1}{\nu}, \tag{13.18}$$

where

$$L_1=\frac{1}{\lambda_1}\|u_0\|^2+\frac{2}{\nu\lambda_1}\int_0^T|f(s)|^2\,ds. \tag{13.19}$$

By adding the relations (13.14) and (13.16) for $m=0,\dots,p$, we obtain, after simplification,

$$|u_h^{p+1}|^2\leqq L_1,\qquad p=0,\dots,N-1. \tag{13.20}$$

Adding the relations (13.14) for $m=0,\dots,p$ and (13.16) for $m=0,\dots,p-1$, we find, after dropping unnecessary terms,

$$|u_h^{p+1/2}|^2\leqq L_1,\qquad p=0,\dots,N-1. \tag{13.21}$$

The relations (13.18)–(13.21) amount to saying that the functions $u_k^{(i)}$, $\tilde{u}_k^{(i)}$, $i=1,2$, belong to a bounded set of $L^\infty(0,T';G)$, and that $u_k^{(1)}$, $u_k^{(2)}$, belong to a bounded set of $L^2(0,T';W)$. In order to show that $\tilde{u}_k^{(i)}$ also belongs to a bounded set of $L^2(0,T';W)$, we observe by direct calculation (cf. [RT, Chap.

III, Lemma 4.8]) that

$$(13.22)\qquad \begin{aligned} |u_k^{(i)} - \tilde{u}_k^{(i)}|^2_{L^2(0,T';G)} &= \frac{k}{3} \sum_{m=1}^{N} |u_h^{m+i/2} - u_h^{m-1+i/2}|^2 \\ &\leq \frac{kL_1}{3} \qquad \text{(by (13.17))}. \end{aligned}$$

ii) We want to pass to the limit as $k \to 0$, $h \to 0$. Due to the previous a priori estimates, there exist a subsequence (denoted k, h) and $u^{(1)}, u^{(2)}$ in $L^2(0, T'; W) \cap L^\infty(0, T'; G)$ such that

$$(13.23)\qquad u_k^{(i)} \to u^{(i)} \begin{cases} \text{weakly in } L^2(0, T; W), \\ \text{weak-star in } L^\infty(0, T'; H), \quad i = 1, 2. \end{cases}$$

The relations (13.17) and (13.22) imply that the same is true for $\tilde{u}_k^{(i)}$, $i = 1, 2$. Now we infer from (13.17) that

$$(13.24)\qquad |u_k^{(2)} - u_k^{(1)}|_{L^2(0,T';H)} \leq kL_1$$

and therefore

$$u_k^{(2)} - u_k^{(1)} \to 0 \quad \text{in } L^2(0, T'; H),$$

so that $u^{(2)} = u^{(1)}$. Also we infer from the properties of the V_h's (which constitute an external approximation[4] of V) that $u^{(1)}$ belongs in fact to $L^2(0, T'; V) \cap L^\infty(0, T'; H)$:

$$(13.25)\qquad u^{(1)} = u^{(2)} = u_* \in L^2(0, T; V) \cap L^\infty(0, T; H).$$

The last (and main) step in the proof is to show that u_* is a solution to Problem 2.2. We need for that purpose a result of strong convergence which will be proved with the help of the compactness theorem in § 13.3.

13.3. A compactness theorem.

THEOREM 13.2. *Let X and Y be two (not necessarily reflexive) Banach spaces with*

$$(13.26)\qquad Y \subset X, \quad \textit{the injection being compact.}$$

Let $\mathcal{G}$ be a set of functions in $L^1(\mathbb{R}; Y) \cap L^p(\mathbb{R}; X)$, $p > 1$, with

$$(13.27)\qquad \mathcal{G} \textit{ is bounded in } L^p(\mathbb{R}; X) \quad \textit{and} \quad L^1(\mathbb{R}; Y);$$

$$(13.28)\qquad \int_{-\infty}^{+\infty} |g(t+s) - g(s)|_X^p \, ds \to 0 \quad \textit{as } t \to 0,$$

uniformly for $g \in \mathcal{G}$;

(13.29) *the support of the functions $g \in \mathcal{G}$ is included in a fixed compact of $\mathbb{R}$, say $[-L, +L]$.*

Then the set $\mathcal{G}$ is relatively compact in $L^p(\mathbb{R}; X)$.

---

[4] For the details see [RT].

Proof. i) For every $a>0$, and for every $g\in L^p(\mathbb{R};X)$ we define the function $J_a g:\mathbb{R}\mapsto X$, by setting

$$(J_a g)(s)=\frac{1}{2a}\int_{s-a}^{s+a} g(t)\,dt=\frac{1}{2a}\int_{-a}^{+a} g(s+t)\,dt. \tag{13.30}$$

It follows from the first relation in (13.30) that the function $s\mapsto(J_a g)(s)$ is continuous from $\mathbb{R}$ into X; on the other hand $J_a g\in L^p(\mathbb{R};X)$, and furthermore

$$|(J_a g)(s)|_X\leqq\frac{1}{2a}\left(\int_{-a}^{+a}|g(s+t)|_X^p\,dt\right)^{1/p}(2a)^{1/p'}$$

(by the Hölder inequality),

$$\int_{-\infty}^{+\infty}|(J_a g)(s)|_X^p\,ds\leqq\int_{-\infty}^{+\infty}\frac{1}{2a}\int_{-a}^{+a}|g(s+t)|_X^p\,dt\,ds$$
$$\leqq\int_{-\infty}^{+\infty}|g(t)|_X^p\,dt,$$

$$|J_a g|_{L^p(\mathbb{R};X)}\leqq|g|_{L^p(\mathbb{R};X)}\quad\forall\,g\in L^p(\mathbb{R};X). \tag{13.31}$$

The same reasoning shows that $J_a g\in L^1(\mathbb{R};Y)$ and

$$|J_a g|_{L^1(\mathbb{R};Y)}\leqq|\,|_{L^1(\mathbb{R};Y)}\quad\forall\,g\in L^1(\mathbb{R};Y). \tag{13.32}$$

ii) It is easy to see that $J_a g\to g$ in $L^p(\mathbb{R};X)$ (resp. $L^1(\mathbb{R};Y)$) as $a\to 0$ for every $g\in L^p(\mathbb{R};X)$ (resp. $L^1(\mathbb{R};Y)$). Due to (13.31), it suffices to prove this point for functions g which are $\mathscr{C}^\infty$ with values in X (resp. Y) and have a compact support, and this is obvious.

We then observe that, due to (13.28),

$$J_a g\to g\quad\text{in } L^p(\mathbb{R};X)\quad\text{as } a\to 0\quad\text{uniformly for } g\in\mathscr{G}. \tag{13.33}$$

In order to prove (13.33), we write

$$J_a g(s)-g(s)=\frac{1}{2a}\int_{-a}^{+a}(g(s+t)-g(s))\,dt,$$

$$|J_a g(s)-g(s)|_X\leqq\frac{(2a)^{1/p'}}{2a}\left(\int_{-a}^{+a}|(s+t)-g(s)|_X^p\,dt\right)^{1/p},$$

$$\int_{-\infty}^{+\infty}|J_a g(s)-g(s)|_X^p\,ds\leqq\int_{-\infty}^{+\infty}\frac{1}{2a}\int_{-a}^{+a}|g(s+t)-g(s)|_X^p\,dt\,ds$$
$$\leqq\frac{1}{2a}\int_{-a}^{+a}\int_{-\infty}^{+\infty}|g(s+t)-g(s)|_X^p\,ds\,dt$$
$$\leqq\sup_{-a\leqq t\leqq a}\int_{-\infty}^{+\infty}|g(s+t)-g(s)|_X^p\,ds;$$

and (13.28) implies that this last quantity converges to 0, uniformly for $g\in\mathscr{G}$ as $a\to 0$.

iii) We now show that

(13.34) For every $a>0$ fixed, the set $\mathscr{G}_a=\{J_a g, g\in\mathscr{G}\}$ is uniformly equicontinuous in $\mathscr{C}(\mathbb{R};X)$.

Indeed, for every $s, t\in\mathbb{R}$

$$
\begin{aligned}
|J_a g(s)-J_a g(t)|_X &= \frac{1}{2a}\left|\int_{-a}^{+a}(g(s+z)-g(t+z))\,dz\right|_X\\
&= \frac{1}{2a}\left|\int_{s-a}^{s+a} g(z)\,dz-\int_{t-a}^{t+a} g(z)\,dz\right|_X\\
&= \frac{1}{2a}\left|\int_{s-a}^{t-a} g(z)\,dz-\int_{s+a}^{t+a} g(z)\,dz\right|_X\\
&\leqq \frac{2\,|s-t|^{1/p'}}{2a}|g|_{L^p(\mathbb{R};X)}\\
&\leqq c\,|s-t|^{1/p'}\quad \forall\, g\in\mathscr{G}.
\end{aligned}
$$

Hence we have (13.34).

We then apply Ascoli's theorem (cf. Bourbaki [1]) to show that the set $\mathscr{G}_a$ is relatively compact in $\mathscr{C}(\mathbb{R};X)$. All the functions $J_a g$ have their support included in a fixed compact, say $[-L-1, L+1]$ (assuming that $|a|\leqq 1$). Due to (13.34), the only assumption of Ascoli's theorem which remains to be checked is that

(13.35) For every $s\in\mathbb{R}$, the set $\{J_a g(s), g\in\mathscr{G}\}$ is relatively compact in X.

But according to (13.26), it suffices to show that this set is bounded in Y, and we have with (13.27)

$$
\begin{aligned}
|J_a g(s)|_Y=\frac{1}{2a}\left|\int_{-a}^{+a} g(s+t)\,dt\right|_Y &\leqq \frac{1}{2a}\int_{-\infty}^{+\infty}|g(s+t)|_Y\,dt\\
&\leqq \frac{1}{2a}|g|_{L^1(\mathbb{R};Y)}\leqq \text{const}\quad \forall\, g\in\mathscr{G}.
\end{aligned}
$$

The set $\mathscr{G}_a$ is therefore relatively compact in $\mathscr{C}(\mathbb{R};X)$, and due to (13.29) this set is, for every fixed a, relatively compact in $L^p(\mathbb{R};X)$.

iv) Finally we prove that the set $\mathscr{G}$ itself is relatively compact in $L^p(\mathbb{R};X)$. For instance, we have to prove (cf. Bourbaki [1]) that for every $\varepsilon>0$ there exist a finite number of points $g_1,\dots,g_N$, in $L^p(\mathbb{R};X)$, such that $\mathscr{G}$ is included in the union of the balls centered at g_i and of radius ε.

According to (13.33), for every $\varepsilon>0$ there exists a such that

$$
|J_a g-g|_{L^p(\mathbb{R};X)}\leqq\frac{\varepsilon}{3}\quad \forall\, g\in\mathscr{G}. \tag{13.36}
$$

Since $\mathscr{G}_a$ is compact, it can be covered by a finite number of balls of $L^p(\mathbb{R};X)$, centered at $J_a g_1,\dots,J_a g_N$, and of radius $\varepsilon/3$; finally it is clear with

(13.36) that the balls of $L^p(\mathbb{R}; X)$ centered at $g_1, \dots, g_N$, and of radius ε cover $\mathscr{G}$.

The proof of Theorem 13.2 is complete. $\square$

We have a similar result for functions g defined on a bounded interval, say $[0, T]$.

THEOREM 13.3. *We assume that X and Y are two Banach spaces which satisfy* (13.26). *Let $\mathscr{G}$ be a set bounded in $L^1(0, T; Y)$ and $L^p(0, T; X)$, $T > 0$, $p > 1$, such that*

$$\int_0^{T-a} |g(a+s)-g(s)|_X^p \, ds \to 0 \quad \text{as } a \to 0, \text{ uniformly for } g \in \mathscr{G}. \tag{13.37}$$

Then $\mathscr{G}$ is relatively compact in $L^q(0, T; X)$ for any q, $1 \leq q < p$.

Proof. This is proved by applying Theorem 13.2 in the following manner. We consider the set $\tilde{\mathscr{G}}$ of functions $\tilde{g} : \tilde{g}(s) = g(s)$ if $s \in [0, T]$ and $=0$ otherwise. The set $\tilde{\mathscr{G}}$ is bounded in $L^1(\mathbb{R}; Y) \cap L^q(\mathbb{R}; X)$ and satisfies (13.28) for p replaced by q, as

$$\begin{aligned}
\int_{-\infty}^{+\infty} |\tilde{g}(a+s)-\tilde{g}(s)|_X^q \, ds &= \int_0^{T-a} |g(a+s)-g(s)|_Z^q \, ds + \int_{T-t}^{T} |g(s)|_X^q \, ds + \int_0^t |g(s)|_X^q \, ds \\
&\leq T^{1-q/p} \left(\int_0^{T-a} |g(a+s)-g(s)|_X^p \, ds \right)^{p/q} \\
&\quad + 2a^{1-q/p} \left(\int_0^T |g(s)|_X^p \, ds \right)^{q/p} \\
&\leq T^{1-q/p} \left(\int_0^{T-a} |g(a+s)-g(s)|_X^p \, ds \right)^{q/p} + ca^{1-q/p},
\end{aligned}$$

and this goes to 0 uniformly with respect to $g \in \mathscr{G}$ as $a \to 0$.

Remark 13.1. Of course, under the assumptions of Theorem 13.2 we can obtain that $\mathscr{G}$ is relatively compact in $L^p(0, T; X)$ itself if we assume, instead of (13.37), that

$$\int_0^{T-a} |g(a+s)-g(s)|_X^p \, ds + \int_{T-a}^{T} |g(s)|_X^p \, ds + \int_0^a |g(s)|_X^p \, ds \to 0 \tag{13.38}$$
$$\text{as } a \to 0, \text{ uniformly for } g \in \mathscr{G}.$$

13.4. Proof of Theorem 13.1 (conclusion). i) By application of Theorem 13.3, we show that the functions $u_k^{(i)}, \tilde{u}_k^{(i)}$ converge to u strongly in $L^2(0, T'; G)$ or even $L^q(0, T'; G)$, $1 \leq q < \infty$.

By integration of (13.12),

$$\begin{aligned}
(\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t), v_h) = -\int_t^{t+\alpha} \Big\{ &\frac{\nu}{2} ((u_k^{(1)}(s+k)+u_k^{(2)}(s), v_h)) \\
&+ b(u_k^{(2)}(s), u_k^{(2)}(s), v_h) - (f_k(s+k), v_h) \Big\} \, ds
\end{aligned}$$
$$\forall v_h \in V_h.$$

We majorize the absolute value of the right-hand side of this equality as follows:

$$\left|\frac{\nu}{2}\int_t^{t+a}((u_k^{(1)}(s+k), v_h))\,ds\right| \leqq \frac{\nu}{2}a^{1/2}\|v_h\|\left(\int_t^{t+a}\|u_k^{(1)}(s+k)\|^2\,ds\right)^{1/2}$$
$$\leqq \frac{\nu}{2}a^{1/2}\|v_h\|\left(\int_0^{T'}\|u_k^{(1)}(s+k)\|^2\,ds\right)^{1/2}$$
$$\leqq c_1' a^{1/2}\|v_h\|,$$

where c_1' depends only on the data, u_0, f, ν, Ω (or Q), T'. Similarly,

$$\left|\frac{\nu}{2}\int_t^{t+a}((u_k^{(2)}(s), v_h))\,ds\right| \leqq c_2' a^{1/2}\|v_h\|,$$

$$\left|\int_t^{t+a}(f_k(s+k), v_h)\,ds\right| \leqq c_3' a^{1/2}\|v_h\|.$$

For the term involving b, we use Lemma 2.1 with $m_1=\frac{1}{2}$, $m_2=m_3=1$:

$$\left|\int_t^{t+a} b(u_k^{(2)}(s), u_k^{(2)}(s), v_h)\,ds\right| \leqq c_1\int_t^{t+a}|u_k^{(2)}(s)|_{1/2}\|u_k^{(2)}(s)\|\,\|v_h\|\,ds$$
$$\leqq c_4'\|v_h\|\int_t^{t+a}|u_k^{(2)}(s)|^{1/2}\|u_k^{(2)}(s)\|^{3/2}\,ds$$
$$\leqq c_5'\|v_h\|\,a^{1/4}\left(\int_t^{t+a}\|u_k^{(2)}(s)\|^2\,ds\right)^{1/2}$$

(by Hölder's inequality and since $u_k^{(2)}$ is bounded in $L^\infty(0, T'; G)$)

$$\leqq c_6' a^{1/4}\|v_h\|.$$

Finally we obtain the majorization

$$|(\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t), v_h)| \leqq c_7' a^{1/4}\|v_h\| \quad \forall\, v_h\in V_h. \tag{13.39}$$

Since $\tilde{u}_k^{(1)}$ is a V_h-valued function[5], we can take $v_h=\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t)$, and we find after integration with respect to t:

$$\int_0^{T'-a}|\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t)|^2\,dt \leqq c_7' a^{1/4}\int_0^{T'-a}\|\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t)\|\,dt$$
$$\leqq c_7'(T')^{1/2}a^{1/4}\left(\int_0^{T'-a}\|\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t)\|^2\,dt\right)^{1/2},$$

$$\int_0^{T'-a}|\tilde{u}_k^{(1)}(t+a)-\tilde{u}_k^{(1)}(t)|^2\,dt \leqq c_8' a^{1/4}. \tag{13.40}$$

[5] This is not the case for $\tilde{u}_k^{(2)}$ (a W_h-valued function).

We apply Theorem 13.3, as follows. $\mathscr{G}$ is the family of functions $\tilde{u}_k^{(1)}$ which is bounded in $L^2(0, T; W) \cap L^\infty(0, T; G)$, $(Y = W, X = G)$, $p = 2$, so that (13.37) follows from (13.40). We conclude that $\tilde{u}_k^{(1)}$ is relatively compact in $L^q(0, T; G)$, $1 \leqq q < 2$, and since $\tilde{u}_k^{(1)}$ converges weakly in $L^q(0, T'; G)$ to u_* (see (13.23)–(13.25)), this means that $\tilde{u}_k^{(1)}$ converges to u_*, strongly in $L^q(0, T'; G)$, $1 \leqq q < 2$. The sequence $\tilde{u}_k^{(1)}$ being bounded in $L^\infty(0, T'; G)$, we conclude from the Lebesgue dominated convergence theorem that we actually have

$$\tilde{u}_k^{(1)} \to u \quad \text{strongly in } L^q(0, T'; G) \quad \forall\, q, \quad 1 \leqq q < \infty. \tag{13.41}$$

As for (13.22), it is easy to check by direct calculation that

$$|\tilde{u}_k^{(i)} - u_k^{(i)}|^2_{L^2(0,T';G)} = \frac{k}{3} \sum_{m=1}^{N} |u_h^{m+i/2} - u_h^{m-1+i/2}|^2, \tag{13.42}$$

and with (13.17)

$$|\tilde{u}_k^{(i)} - u_k^{(i)}|^2_{L^2(0,T';G)} \leqq c\frac{k}{3}, \qquad i = 1, 2. \tag{13.43}$$

This, together with (13.24) and (13.41), shows that

$$u_k^{(i)} \text{ and } \tilde{u}_k^{(i)} \quad \text{converge to } u \text{ strongly in } L^q(0, T'; G) \quad \forall\, q, 1 \leqq q < \infty. \tag{13.44}$$

ii) Using (13.44) and the weak convergences obtained in § 13.2 (in particular (13.23)), the passage to the limit in (13.12) is standard. We will just give the main lines of the proof. Let ψ be any $\mathscr{C}^1$ scalar function on $[0, T']$ which vanishes near T'; we multiply (13.11) by $\psi(t)$, integrate in t and integrate by parts the first term to get

$$\begin{aligned} \int_0^{T'} \Big\{ &-(\tilde{u}_k^{(1)}(t), v_h\psi'(t)) + \frac{\nu}{2}((u_k^{(1)}(t+k) + u_k^{(2)}(t), v_h\psi(t))) \\ &+ b(u_k^{(2)}(t), u_k^{(2)}(t), v_h\psi(t)) \Big\}\, dt \\ &= \int_0^{T'} (f_k(t+k), \psi(t)v_h)\, dt + (u_{0h}, v_h)\psi(0). \end{aligned} \tag{13.45}$$

Let v be an arbitrary element of V. We choose for v_h an approximation of v, and passing to the limit in (13.45) we get

$$\begin{aligned} \int_0^{T'} &\{-(u_*(t), v\psi'(t)) + \nu((u_*(t), v\psi(t))) + b(u_*(t), u_*(t), v\psi(t))\}\, dt \\ &= \int_0^{T'} (f(t), v\psi(t))\, dt + (u_0, v)\psi(0). \end{aligned} \tag{13.46}$$

(The passage to the limit in the nonlinear term necessitates the strong convergence (13.41); cf. [RT].) We deduce from (13.46) that u_* is a solution to Problems 2.1–2.2, and since this solution is unique in the present situation, we conclude that $u_* = u$. By uniqueness, the entire sequences $u_k^{(i)}$, $\tilde{u}_k^{(i)}$ converge to u as k and h tent to 0. The last step of the proof is to obtain strong convergence in $L^2(0, T'; W)$.

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14 Long Time Approximation of the Navier–Stokes Equations

The study of the long time behavior of the solutions to the Navier–Stokes equations is directly related to an understanding of turbulent flow, as can be seen from Part II. In this section we present two results which tend to indicate, as do those in §§ 11 and 12, that the flow has, for time large, a finite dimensional structure. The first result is related to the flow itself (§ 14.1); the second to its Galerkin approximation (§ 14.2).

14.1. Long time finite dimensional approximation. Let f and g be two continuous bounded functions from $[0,\infty)$ into H, let u_0 and v_0 be given in V, and let u and v denote the corresponding solutions to Problem 2.2:

$$\frac{du}{dt}+\nu Au+Bu=f, \qquad u(0)=u_0, \tag{14.1}$$

$$\frac{dv}{dt}+\nu Av+Bv=g, \qquad v(0)=v_0. \tag{14.2}$$

We assume that u and v are defined and bounded for all time in V:

$$\sup_{t\geqq 0}\|u(t)\|\leqq c', \qquad \sup_{t\geqq 0}\|v(t)\|\leqq c'. \tag{14.3}$$

We recall that this property is automatically satisfied if $n=2$ (cf. Lemma 12.2). We recall also that for every $\alpha>0$

$$\sup_{t\geqq\alpha}|Au(t)|\leqq c'', \qquad \sup_{t\geqq\alpha}|Av(t)|\leqq c'', \tag{14.4}$$

where c'' depends on α, c' and the data (Lemma 12.2).

The space E. Let us now consider a finite dimensional subspace E of V. We denote by $P(E)$ the orthogonal projector in H onto E, and $Q(E)=I-P(E)$. Since $P(E)$ is not a projector in V, it may happen that $((\phi,\psi))\neq 0$, if $\phi\in E, \psi\in V$ and $P(E)\psi=0$. However (see Lemma 14.1 below), there exists $\rho(E), 0\leqq\rho(E)<1$, such that

$$|((\phi,\psi))|\leqq\rho(E)\,\|\phi\|\,\|\psi\| \quad \forall\,\phi\in E, \quad \forall\,\psi\in V, \quad P(E)\psi=0. \tag{14.5}$$

We also associate with E the two numbers $\lambda(E), \mu(E)$,

$$\lambda(E)=\inf\{\|\phi\|^2, \phi\in V, P(E)\phi=0, |\phi|=1\},$$
$$\mu(E)=\sup\{\|\psi\|^2, \psi\in E, |\psi|=1\},$$

so that

$$|\phi| \leqq \lambda(E)^{-1/2}\|\phi\| \quad \forall\, \phi \in V, \quad P(E)\phi = 0, \tag{14.6}$$

$$\|\psi\| \leqq \mu(E)^{1/2}|\psi| \quad \forall\, \psi \in E. \tag{14.7}$$

When it is not necessary to mention the dependance on E, we will write simply P, Q, ρ, λ, μ instead of $P(E)$, and so forth.

We will now prove (14.5).

LEMMA 14.1. *Under the above assumptions, there exists $\rho = \rho(E)$ such that* (14.5) *holds.*

Proof. If this were not true, we could find two sequences ϕ_j, ψ_j, $j \geqq 1$, $\phi_j \in E$, $\psi_j \in V$, $P\psi_j = 0$ such that

$$\|\phi_j\|\,\|\psi_j\| \geqq |((\phi_j, \psi_j))| \geqq \left(1 - \frac{1}{j}\right)\|\phi_j\|\,\|\psi_j\|.$$

Setting $\phi_j' = \phi_j/\|\phi_j\|$, $\psi_j' = \psi_j/\|\psi_j\|$, we find

$$1 \geqq |((\phi_j', \psi_j'))| \geqq 1 - \frac{1}{j}. \tag{14.8}$$

We can extract a subsequence, still denoted j, such that ϕ_j' converges to some limit ϕ, $\|\phi\| = 1$, $\phi \in E$ (E has finite dimension), and ψ_j' converges weakly in V to ψ, $\psi \in V$, $\|\psi\| \leqq 1$, $P\psi = 0$. At the limit, (14.8) gives

$$|((\phi, \psi))| = 1, \quad \|\phi\| = 1, \quad \|\psi\| \leqq 1,$$

so that $\|\psi\| = 1$, $\psi = k\phi \neq 0$, in contradiction with $P\psi = 0$. $\square$

An inequality. We consider the two solutions u, v of (14.1)–(14.2), and we set

$$w = u - v, \quad p = Pw, \quad q = Qw, \quad e = f - g,$$

$(P = P(E),\, Q = Q(E) = I - P(E))$.

We apply the operator Q to the difference between (14.1) and (14.2). We obtain

$$\frac{dq}{dt} + \nu QAw + QB(v, w) + QB(w, u) = Qe.$$

We then take the scalar product in H with q,

$$\frac{1}{2}\frac{d}{dt}|q|^2 + \nu\|q\|^2$$
$$= -(B(v, p), q) - (B(p, u), q) - (B(q, u), q) + (Qe, q) - \nu((p, q)).$$

Using Lemma 2.1 (or (2.36)) and (14.5)–(14.7), we find that the right-hand side of this inequality is less than or equal to

$$\lambda_1|Qe|\,\|q\| + \nu\rho\mu^{1/2}|p|\,\|q\| + c_1(|Au| + |Av|)\,|p|\,\|q\| + c_1|Au|\,|q|\,\|q\|.$$

If $t \geqq \alpha$, $\alpha > 0$ arbitrary, then this expression is in turn less than or equal to

$$\frac{\varepsilon\nu}{3}\|q\|^2+\frac{3\lambda_1^2}{4\varepsilon\nu}|Qe|^2+\frac{\varepsilon\nu}{3}\|q\|^2+\frac{3\nu}{\varepsilon}\rho^2\mu\,|p|^2+\frac{\varepsilon\nu}{3}\|q\|^2$$
$$+\frac{3c_1^2}{4\varepsilon\nu}(|Au|+|Av|)^2\,|p|^2+c_1c''\lambda^{-1/2}\|q\|^2$$
$$\leqq \varepsilon\nu\,\|q\|^2+\frac{3\lambda_1^2}{4\varepsilon\nu}|Qe|^2+\frac{3}{\varepsilon}\left(\frac{(c_1c'')^2}{\nu}+\nu\rho^2\mu\right)|p|^2+c_1c''\lambda^{-1/2}\|q\|,$$

where $\varepsilon > 0$ is arbitrary.

If

$$\lambda(E)>\left(\frac{c_1c''}{\nu}\right)^2, \tag{14.9}$$

then we set

$$\nu' = \nu - c_1c''\lambda(E)^{1/2}>0, \qquad \varepsilon=\frac{1}{2}-\frac{c_1c''\lambda(E)^{1/2}}{2\nu}, \tag{14.10}$$

and we have established:

LEMMA 14.2. *If* (14.3)–(14.5)–(14.7) *and* (14.9) *hold then, for* $t \geqq \alpha > 0$,

$$\frac{d}{dt}|q|^2+\nu'\,\|q\|^2\leqq\frac{3\lambda_1^2}{2\varepsilon\nu}|Qe|^2+\frac{6}{\varepsilon}\left(\frac{(c_1c'')^2}{\nu}+\nu\rho^2\mu\right)|p|^2, \tag{14.11}$$

with ν', ε *as in* (14.10).

We now prove

THEOREM 14.1. *We assume that the space dimension is* $n = 2$ *or* 3 *and that* u *and* v *are solutions of* (14.1)–(14.2) *uniformly bounded in* V. *Let* E *be a finite dimensional subspace of* V *such that* (14.9) *is satisfied.*

Then, if

$$|P(E)(u(t)-v(t))|\to 0, \tag{14.12}$$

$$|(I-P(E))(f(t)-g(t))|\to 0 \quad \text{as } t\to\infty, \tag{14.13}$$

we have also

$$|(I-P(E))(u(t)-v(t))|\to 0 \quad \text{as } t\to\infty, \tag{14.14}$$

i.e.,

$$|u(t)-v(t)|\to 0 \quad \text{as } t\to\infty. \tag{14.15}$$

Remark 14.1. Theorem 14.1 shows that if the condition (14.9) is satisfied the behavior for $t\to\infty$ of $u(t)$ is *completely* determined by that of $P(E)u(t)$ (although we do not know how to recover $u(t)$, knowing $P(E)u(t)$). Lemma 14.3 below shows that (14.9) is satisfied if E is "sufficiently large".

Proof of Theorem 14.1. We infer from (14.6) and (14.11) that

$$\frac{d}{dt}|q|^2+\nu'\lambda\,|q|^2 \leqq c_1'\,|Qe|^2+c_2'\,|p|^2$$

where

$$c_1'=\frac{3\lambda_1^2}{2\varepsilon\nu}, \qquad c_2'=\frac{6}{\varepsilon}\left(\frac{(c_1c'')^2}{\nu}+\nu\rho^2\mu\right),$$

whence for $t \geqq t_0 \geqq \alpha$,

$$|q(t)|^2 \leqq |q(t_0)|^2 e^{-\nu'\lambda(t-t_0)}+\int_{t_0}^{t}[c_1'\,|e(\tau)|^2+c_2'\,|p(\tau)|^2]e^{-\nu'\lambda(t-\tau)}\,d\tau. \tag{14.16}$$

Given $\delta>0$, there exists M (which we can assume $\geqq\alpha$) such that for $t\geqq M$

$$|P(u(t)-v(t))|^2 \leqq \delta,$$
$$|(I-P)(f(t)-g(t))|^2 \leqq \delta.$$

Therefore, for $t \geqq t_0+M$, (14.11) implies

$$|q(t)|^2 \leqq c(u,v)e^{-\nu'\lambda(t-t_0)}+\delta(c_1'+c_2')\int_{t-M}^{t} e^{-\nu'\lambda(t-\tau)}\,d\tau$$
$$+[c_1'c(f,g)+c_2'c(u,v)]\int_{t_0}^{t-M} e^{-\nu'\lambda(t-\tau)}\,d\tau,$$

where

$$c(u,v)=\sup_{t\geqq 0}|u(t)-v(t)|, \qquad c(f,g)=\sup_{t\geqq 0}|f(t)-g(t)|.$$

As $t\to\infty$,

$$\limsup_{t\to\infty}|q(t)|^2 \leqq \delta(c_1'+c_2')\left(\frac{1-e^{-\nu'\lambda M}}{\nu'\lambda}\right)$$
$$+[c_1'c(f,g)+c_2'c(f,g)]\frac{e^{-\nu'\lambda M}}{\nu'\lambda}.$$

We let $\delta\to 0$ and then $M\to\infty$ and we obtain (14.14). $\square$

As mentioned before, the following lemma shows that condition (14.9) is always satisfied if E is sufficiently large.

LEMMA 14.3. *If E_j, $j\geqq 1$, is an increasing sequence of subspaces of V such that $\bigcup_j E_j$ is dense in V, then $\lambda(E_j)\to+\infty$ as $j\to\infty$.*

Proof. In fact, we will prove that

(14.17) For every integer m there exists j_m and, for $j\geqq j_m$, we have $\lambda(E_{j_m})\geqq\lambda_m$,

where λ_m is the mth eigenvalue of A.

By assumption, for every k,

$$\operatorname*{Inf}_{\psi \in E_j} \|w_k - \psi\| \to 0 \quad \text{as } j \to \infty.$$

Hence, for given m and $\delta > 0$, there exists j_m such that

$$\operatorname*{Inf}_{\psi \in E_j} \|w_k - \psi\| \leqq \delta \quad \text{for } k = 1, \ldots, m \text{ and every } j \geqq j_m.$$

Thus, for every $j \geqq j_m$, there exist $\tilde{w}_1, \ldots, \tilde{w}_m$ in E_j, with $\|w_j - \tilde{w}_j\| \leqq \delta$. Therefore if $\phi \in V$, $(I - P(E_j))\phi = 0$, we have

$$\begin{aligned}
\|\phi\|^2 &= \|P_m\phi\|^2 + \|(I - P_m)\phi\|^2 \\
&\geqq \lambda_{m+1} |(I - P_m)\phi|^2 + \lambda_1 |P_m\phi|^2 \\
&\geqq \lambda_{m+1} |\phi|^2 - (\lambda_{m+1} - \lambda_1) |P_m\phi|^2 \\
&= \lambda_{m+1} |\phi|^2 - (\lambda_{m+1} - \lambda_1) \sum_{j=1}^{m} (\phi, w_j - \tilde{w}_j)^2 \\
&\geqq (\lambda_{m+1} - (\lambda_{m+1} - \lambda_1)\lambda_1 m\delta^2) |\phi|^2.
\end{aligned}$$

This implies

$$\lambda(E_j) \geqq \lambda_{m+1} - (\lambda_{m+1} - \lambda_1)\lambda_1 m\delta^2,$$

and the results follow by letting $\delta \to 0$. □

14.2. Galerkin approximation. We are going to establish a result similar to that in Theorem 14.1, for the Galerkin approximation of Navier–Stokes equations when the space dimension $n = 2$. For simplicity we restrict ourself to the Galerkin approximation using the eigenfunctions w_m of A (as in § 3.3), and we will show that if m is sufficiently large, the behavior as $t \to \infty$ of the Galerkin approximation u_m is completely determined by the behavior as $t \to \infty$ of a certain number m_* of its modes, i.e. the behavior of $P_{m_*}u_m$, $m_* < m$.

We start with some complementary remarks on Galerkin approximations.

For fixed m, the Galerkin approximation u_m of the solution u of (14.1) is defined (see § 3.3) by

$$\begin{aligned}
&\frac{du_m}{dt} + \nu A u_m + P_m B(u_m) = P_m f, \qquad t > 0, \\
&u_m(0) = P_m u_0,
\end{aligned} \tag{14.18}$$

where P_m is the projector in $H, V, V', D(A), \ldots$, onto the space spanned by the first m eigenfunctions of A, $w_1, \ldots, w_m$.

We have established in § 3.3 some a priori estimates which are valid on a finite interval of time $[0, T]$; we can obtain a priori estimates valid for all time as follows.

We infer from (3.46) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_m(t)|^2+\nu\,\|u_m(t)\|^2&=(f(t),u_m(t))\leqq|f(t)|\,|u_m(t)|\\&\leqq\frac{1}{\sqrt{\lambda_1}}|f(t)|\,\|u_m(t)\|\\&\leqq\frac{\nu}{2}\|u_m(t)\|^2+\frac{2}{\nu\sqrt{\lambda_1}}|f(t)|^2,\end{aligned}$$

$$\frac{d}{dt}|u_m(t)|^2+\nu\,\|u_m(t)\|^2\leqq\frac{4}{\nu\lambda_1}|f(t)|^2\leqq\frac{4}{\nu\lambda_1}N(f)^2, \tag{14.19}$$

where $N(f)=\sup_{t\geqq 0}|f(t)|$. Thus for every $t\geqq s\geqq 0$

$$|u_m(t)|^2+\nu\int_s^t\|u_m(\sigma)\|^2\,d\sigma\leqq|u_m(s)|^2+\frac{4(t-s)N(f)^2}{\nu\lambda_1}, \tag{14.20}$$

and $|u_m(t)|$ is bounded for all time:

$$|u_m(t)|^2\leqq|u_0|^2e^{-\nu\lambda_1 t}+\frac{1-e^{-\nu\lambda_1 t}}{\nu^2\lambda_1}N(f)^2. \tag{14.21}$$

The following a priori estimate is also verified by u_m.

LEMMA 14.4. *$\|u_m(t)\|$ is bounded independently of m and t.*

Proof. For $T=\alpha>0$, this is proved on $[0,\alpha]$ in (3.59). In order to obtain the result for $t\geqq\alpha$, we consider the analogue of (3.11) for u_m (obtained by taking the scalar product of (14.18) with Au_m; see § 3):

$$\frac{d}{dt}\|u_m(t)\|^2+\nu\,|Au_m(t)|^2\leqq\frac{2}{\nu}|f(t)|^2+c_2'\,|u_m(t)|^2\,\|u_m(t)\|^4.$$

Therefore with (14.21)

$$\begin{aligned}\frac{d}{dt}\|u_m(t)\|^2+\nu\lambda_1\,\|u_m(t)\|^2&\leqq\frac{2}{\nu}N(f)^2+c_2'\,\|u_m(t)\|^4\\&\leqq c_3'(1+\|u_m\|^2)^2,\end{aligned} \tag{14.22}$$

and for $0\leqq s\leqq t$, we can show by integration that

$$(1+\|u_m(t)\|^2)\leqq(1+\|u_m(s)\|^2)\exp\left(c_2'\int_s^t(1+\|u_m(\sigma)\|^2)\,d\sigma\right).$$

If $t\geqq\alpha>0$, we integrate in s from $t-\alpha$ to t and we find

$$c_2'\alpha(1+\|u_m(t)\|^2)\leqq\left[\exp\left(c_2'\int_{t-\alpha}^t(1+\|u_m(\sigma)\|^2)\,d\sigma\right)-1\right].$$

Using (14.20), we see that the right-hand side of this inequality is bounded by a constant depending on α and the data, but independent of t and m. The lemma follows. □

We can now state the result announced.

Let u_m and v_m be the Galerkin approximations to the solution u and v of (14.1)–(14.2),

$$\frac{du_m}{dt}+\nu A u_m+P_m B(u_m)=P_m f, \qquad u_m(0)=P_m u_0, \tag{14.23}$$

$$\frac{dv_m}{dt}+\nu A v_m+P_m B(v_m)=P_m g, \qquad v_m(0)=P_m v_0. \tag{14.24}$$

THEOREM 14.2. *We assume that $n=2$ and that $m\geqq m_*$, m_* sufficiently large so that the condition* (14.29) *below is verified.*

Then, if

$$|P_{m_*}(u_m(t)-v_m(t))|\to 0 \quad \text{as } t\to\infty, \tag{14.25}$$

$$|(I-P_{m_*})(f(t)-g(t))|\to 0 \quad \text{as } t\to\infty, \tag{14.26}$$

we have

$$|(I-P_{m_*})(u_m(t)-v_m(t))|\to 0 \quad \text{as } t\to\infty, \tag{14.27}$$

i.e.,

$$u_m(t)-v_m(t)\to 0 \quad \text{as } t\to\infty. \tag{14.28}$$

Proof. We have to obtain an analogue of (14.11); we then proceed as for Theorem 14.1.

We set for $m_*\leqq m$ (m_* to be determined):

$$w_m=u_m-v_m, \quad p_{m_*}=P_{m_*}w_m, \quad q_{m_*}=Q_{m_*}w_m, \quad e=f-g, \quad e_{m_*}=Q_{m_*}e_m$$

($Q_m=I-P_m$). Then by subtracting (14.21) from (14.23) and applying Q_{m_*} to the equality which we obtain, we arrive at

$$\frac{dq_{m_*}}{dt}+\nu A q_{m_*}+Q_{m_*}P_m B(v_m,w_m)+Q_{m_*}P_m B(w_m,u_m)=Q_{m_*}P_m e.$$

Consequently,,

$$\frac{1}{2}\frac{d}{dt}|q_{m_*}|^2+\nu\,\|q_{m_*}\|^2=-(B(v_m,w_m),P_m q_{m_*})-(B(w_m,u_m),P_m q_{m_*})+(Q_{m_*}e,P_m q_{m_*}).$$

The right-hand side is equal to

$$\begin{aligned}(Q_{m_*}e,P_m q_{m_*})&-(B(v_m,p_{m_*}),P_m q_{m_*})-(B(v_m,(I-P_m)q_{m_*}),P_m q_{m_*})\\ &-(B(p_{m_*},u_m),P_m q_{m_*})-(B(q_{m_*},u_m),P_m q_{m_*}).\end{aligned}$$

Because of (2.31) and Lemma 14.4, this quantity is bounded by

$$\begin{aligned}|Q_{m_*}e|\,|q_{m_*}|&+c'\,|v_m|^{1/2}\,\|v_m\|^{1/2}\,|p_{m_*}|^{1/2}\,\|p_{m_*}\|^{1/2}\,\|q_{m_*}\|\\ &+c'\,|v_m|^{1/2}\,\|v_m\|^{1/2}\,|q_{m_*}|^{1/2}\,\|q_{m_*}\|^{3/2}\\ &+c'\,|p_{m_*}|^{1/2}\,\|p_{m_*}\|^{1/2}\,\|u_m\|\,\|q_{m_*}\|+c'\,|q_{m_*}|^{1/2}\,\|q_{m_*}\|^{3/4}\,\|u_m\|\\ \leqq \lambda_1^{1/2}\,|Q_{m_*}e|\,\|q_{m_*}\|&+c''\lambda_{m_*+1}^{1/4}\,|p_{m_*}|\,\|q_{m_*}\|+c''\lambda_{m_*+1}^{-1/4}\,\|q_{m_*}\|^2.\end{aligned}$$

If

$$\lambda_{m_*+1} > \left(\frac{c''}{\nu}\right)^4, \tag{14.29}$$

we set

$$\nu' = 2(\nu - c''\lambda_{m_*+1}^{-1/4}) > 0, \qquad \varepsilon = \frac{1}{2} - \frac{c''\lambda_{m_*+1}^{-1/4}}{2\nu} > 0, \tag{14.30}$$

and we bound the last quantity by

$$\frac{\varepsilon\nu}{2}\|q_{m_*}\|^2 + \frac{\lambda_1}{2\varepsilon\nu}|Q_{m_*}e|^2 + \frac{\varepsilon\nu}{2}\|q_{m_*}\|^2 + \frac{(c'')^2\lambda_{m_*+1}^{1/2}}{2\varepsilon\nu}|p_{m_*}|^2$$
$$+ c''\lambda_{m_*+1}^{-1/4}\|q_{m_*}\|^2.$$

Finally we find

$$\frac{d}{dt}|q_{m_*}|^2 + \nu'\|q_{m_*}\|^2 \leqq \frac{\lambda_1}{\varepsilon\nu}|Q_{m_*}e|^2 + \frac{c''\lambda_{m_*+1}^{1/2}}{\varepsilon\nu}|p_{m_*}|. \tag{14.31}$$

This inequality is totally similar to (14.11), and starting from (14.31) the proof of Theorem 14.2 is the same as that of Theorem 14.1. □

Remark 14.2. We did not establish above any connection between the behavior for $t \to \infty$ of $u(t)$ and of its Galerkin approximation $u_m(t)$. This remains an open problem. See however P. Constantin–C. Foias–R. Temam [1].

APPENDIX

Inertial Manifolds and Navier–Stokes Equations

Inertial manifold is a new concept introduced in 1985 after the publication of the first edition of these notes. Although related concepts and results existed earlier, inertial manifolds were introduced under this name in 1985 and systematically studied for nonlinear dissipative evolution equations since that time. It would be beyond the scope of this appendix to make a thorough presentation of inertial manifolds; we refer the interested reader to more specialized books and articles (see references below). In this appendix we recall only the concept, the main definitions, and the main results and concentrate on some results specifically related to the Navier–Stokes equations as they appear in the articles of C. Foias–G. Sell–R. Temam [1], [2], M. Kwak [1], and R. Temam–S. Wang [1]. We restrict ourselves to space dimension two. Some of the results are not in their final form; a number of problems are still open but we believe they are of sufficient interest to be presented in their current form.

A.1. Inertial manifolds and inertial systems. It was shown in § 12 that attractors for Navier–Stokes equations, which are bounded in the H^1-norm, have finite dimension. As indicated in the introduction of § 12, this shows that the long-time behavior of the solutions to these equations is finite dimensional. The concept of inertial manifold is an attempt to go further in the reduction of the dynamics of infinite-dimensional equations such as the Navier–Stokes equations to that of a finite-dimensional differential equation.

Consider a semigroup of continuous operators $\{S(t)\}_{t\geqq 0}$ in a Hilbert space H (scalar product $(\cdot,\cdot)$, norm $|\cdot|$). This could be, for instance, the semigroup defined in § 12 for the Navier–Stokes equations in space dimension two when $f(t) = f \in H$ is independent of t; $S(t)$ is then the nonlinear mapping $u(0) \to u(t)$ in H, whose existence is guaranteed by Theorem 3.1 (in particular (3.34)).

DEFINITION A1. Given a semigroup of operators $\{S(t)\}_{t\geqq 0}$ in a Hilbert space H, an inertial manifold for this semigroup (or for the corresponding evolution equation) is a Lipschitz finite-dimensional manifold $\mathcal{M}$ *in* H such that

$$S(t)\mathcal{M} \subset \mathcal{M}, \qquad \forall\, t \geqq 0, \tag{A.1}$$

$$\begin{aligned}&\text{for every } u_0 \in H,\ S(t)u_0 \text{ converges to } \mathcal{M}\\ &\text{at an exponential rate, i.e.,}\\ &\operatorname{dist}(S(t)u_0, \mathcal{M}) \leqq c_1 \exp(-c_2 t),\\ &\text{where } c_1, c_2 \text{ depend boundedly on } |u_0|.\end{aligned} \tag{A.2}$$

When an inertial manifold exists, the restriction of $S(t)$ to $\mathcal{M}$ defines a finite-dimensional semigroup of operators $\{\Sigma(t)\}_{t\geqq 0}$,

$$\Sigma(t) = S(t)|_{\mathcal{M}},$$
$$\Sigma(t) : \mathcal{M} \to \mathcal{M},$$

which fully reproduces the dynamics of the initial equation (or semigroup). This system is called an *inertial system*. More generally an *inertial system* (or an inertial form) for a semigroup $\{S(t)\}_{t\geqq 0}$ is a finite-dimensional system (or semigroup) $\{\Sigma(t)\}_{t\geqq 0}$ which fully reproduces the dynamics of the initial system.[1]

Having defined inertial manifolds (and inertial systems), we now make some comments and comparisons with attractors.

Remark A.1. When an attractor $\mathcal{A}$ exists as well as the inertial manifold $\mathcal{M}$ then obviously $\mathcal{A} \subset \mathcal{M}$ and, if Theorem 12.1 applies, then $\mathcal{A}$ like $\mathcal{M}$ has finite dimension. However, there are differences between attractors and inertial manifolds:

i) Attractors may be complicated sets, even fractals, while inertial manifolds are required to be smooth, at least Lipschitz manifolds, usually $\mathscr{C}^1$ manifolds.

ii) The convergence of orbits to the attractors may be slow, like for instance a negative power of t or even slower. However the convergence of the orbits to an inertial manifold is required to be exponential. Hence, after a short transient period, the orbits essentially lie on the inertial manifold $\mathcal{M}$ and most of the dynamics takes place on $\mathcal{M}$.

Remark A.2. See Remark A.4 for some comments on the connection between inertial manifolds and turbulence.

A.2. Survey of the main results. In this section we survey some typical results on inertial manifolds for general equations and give bibliographical references.

We consider in the Hilbert space H an evolution equation

$$\frac{du}{dt} + Au = R(u), \tag{A.3}$$

$$u(0) = u_0. \tag{A.4}$$

This could be the Navier–Stokes equation written in the form (2.43), with A replaced by νA and $R(u) = f - Bu$. We assume here that A is an unbounded self-adjoint closed positive operator in H with domain $D(A) \subset H$, and that R is a $\mathscr{C}^1$ mapping from the domain $D(A^\gamma)$ in H of A^γ, for some γ $(0 \leqq \gamma < 1)$, into H. The function $u = u(t)$ is a mapping from $\mathbb{R}_+$ (or some interval of $\mathbb{R}$) into $D(A)$.

We assume that the initial value problem (A.3), (A.4) is well posed; i.e., we assume that for every $u_0 \in H$ there exists a function u continuous from $\mathbb{R}_+$ into

[1] We realize that this definition is not very precise but prefer to avoid technicalities; see the references given below for the precise statements.

H which satisfies (A.3), (A.4) in some sense. According to Theorem 3.1, this is true for the Navier–Stokes equations in space dimension two (space periodic case or flow in a bounded domain). In that case we denote obviously by $S(t)$ the operator

$$S(t)\ :\ u(0) \in H \to u(t) \in H, \qquad t \geqq 0.$$

Assuming also that A^{-1} is compact, we see that there exists an orthonormal basis of H, $\{w_j\}_{j\in\mathbb{N}}$ which consists of eigenvectors of A, i.e., exactly as in (2.17),

$$\text{(A.5)} \qquad \begin{aligned} &Aw_j = \lambda_j w_j, && w_j \in D(A), \\ &0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3, \ldots, && \lambda_j \to \infty \text{ for } j \to \infty. \end{aligned}$$

Then any $u \in H$ (or $u \in D(A^\beta)$ for some $\beta \geqq 0$) can be expanded in the form

$$\text{(A.6)} \qquad u = \sum_{j=1}^{\infty} \hat{u}_j w_j,$$

where the series (A.6) converges in H (or in $D(A^\beta)$).

For any $N \in \mathbb{N}$, let P_N denote the orthogonal projector in H onto the space spanned by $w_1, \ldots, w_N$, and let $Q_N = I - P_N$ be the orthogonal projector in H onto the space spanned by the w_j, $j \geqq N+1$. Then for $u \in H$, we write

$$\text{(A.7)} \qquad u = y_N + z_N, \quad y_N = P_N u, \quad z_N = Q_N u.$$

It is easy to see that equation (A.3) is equivalent to the system

$$\text{(A.8)} \qquad \frac{dy}{dt} + Ay = PR(y+z),$$

$$\text{(A.9)} \qquad \frac{dz}{dt} + Az = QR(y+z).$$

As in (A.8), (A.9) we will omit the indices N and write P, Q, y, z instead of P_N, Q_N, y_N, z_N, when N is fixed and no confusion can arise.

There are by now many ways to construct an inertial manifold for equation (A.3)(A.4) (or the semigroup $\{S(t)\}_{t\geqq 0}$); in general they produce the inertial manifold $\mathcal{M}$ as the graph of a function Φ from $P_N H$ into $Q_N H$, for some N:

$$\text{(A.10)} \qquad \begin{aligned} &z_N = \Phi(y_N), \\ &\Phi\ :\ P_N H \to Q_N H. \end{aligned}$$

The following result proved in C. Foias–G. Sell–R. Temam [1], [2], is a general and typical result of existence of inertial manifolds. To avoid technicalities we state it in a fuzzy way without all the assumptions; the interested reader is referred to the references above for the precise statements.

THEOREM A.1. *The assumptions are those given above in this Appendix. We also assume*

i) *some technical hypotheses on A and R,*

ii) $\lambda_{N+1} \geqq \kappa_1$, *for some $N \in \mathbb{N}$,*

iii) $\lambda_{N+1} - \lambda_N \geqq \kappa_2(\lambda_{N+1}^\alpha + \lambda_N^\alpha)$,

for some $\kappa_1, \kappa_2 > 0$, and some α, $0 \leqq \alpha < 1$, which depend on A and R.

Then there exists a $\mathscr{C}^1$ mapping Φ from $P_N H$ into $Q_N H$ such that the graph $\mathscr{M}$ of Φ is an inertial manifold for equation (A.3), (A.4) (*or the semigroup* $\{S(t)\}_{t \geqq 0}$).

The technical hypotheses mentioned in i) are satisfied by many equations including the Navier–Stokes equations in space dimension two. Hypothesis ii) is easily satisfied for N sufficiently large, but the most restrictive assumption is hypothesis iii), also called the spectral gap condition. For instance, for the two-dimensional Navier–Stokes equations, $\alpha = 1/2$ and since,

$$\lambda_N \sim cN, \quad \text{as } N \to \infty,$$

(see R. Courant–H. Hilbert [1], G. Métivier [1]) we do not know if iii) is satisfied. This hypothesis is satisfied and Theorem A.1 applies for reaction–diffusion equations for which $\alpha = 0$ (dimension one or two), for certain dissipative evolution equations of mathematical physics (such as the Kuramoto–Sivashinsky, the Cahn–Hilliard, or the complex Ginzburg–Landau equations). It applies more generally to "very dissipative" equations, including the Navier–Stokes equations with enhanced viscosity described below.

Before describing these equations of Navier–Stokes type and proceeding with recent results and open problems, we make a few comments.

Remark A.3 (asymptotic completeness). Another interesting property of inertial manifolds is the *asymptotic completeness* property proved in C. Foias–G. Sell– E. Titi [1], and P. Constantin–C. Foias–B. Nicolaenko–R. Temam [1]; namely,

$$\begin{aligned} &\text{for every } u_0 \in H, \text{ there exists } \bar{u}_0 \in \mathscr{M} \\ &\text{and } \tau \in \mathbb{R} \text{ such that} \\ &|S(t)u_0 - S(t+\tau)\bar{u}_0| \to 0 \text{ when } t \to +\infty. \end{aligned} \tag{A.11}$$

Remark A.4 (inertial manifolds and turbulence). Consider an orbit $u = u(t)$ lying in $\mathscr{M}$; then in view of (A.6), (A.7), and (A.10), we have, at all times,

$$u(t) = \sum_{j=1}^{\infty} \hat{u}_j(t) w_j,$$

$$\begin{gathered} u(t) = y(t) + z(t), \quad y(t) = \sum_{j=1}^{N} \hat{u}_j(t) w_j, \quad z(t) = \sum_{j=N+1}^{\infty} \hat{u}_j(t) w_j, \\ z(t) = \Phi(y(t)), \quad t \geqq 0. \end{gathered} \tag{A.12}$$

In the physical language of turbulence, (A.12) means that the high frequency component of the flow, z, is *slaved* by its low frequency component, y. Other comments on the relevance of inertial manifolds to turbulence appear in R. Temam [11], [12]. See also the concept of approximate inertial manifolds in C. Foias–O. Manley–R. Temam [2]–[5].

Remark A.5. Detailed study of inertial manifolds for dissipative evolution equations appears in the references quoted above; see also J. Mallet-Paret–G. Sell [1] for reaction–diffusion equations in higher dimension, A. Debussche–R. Temam [2] for a general result of existence of inertial manifolds, including the case where the operator A is not self-adjoint and an estimate of their dimensions. Many more results and references appear in the books by P. Constantin–C. Foias–B. Nicolaenko–R. Temam [1], R. Temam [13], and the books edited by C. Foias–B. Nicolaenko–R. Temam [1] and C. Foias–G. Sell–R. Temam [3]. See also the books quoted above and the references therein about approximation of inertial manifolds and attractors. The relations of inertial manifolds with the concept of *slow manifolds* in meteorology is addressed in A. Debussche–R. Temam [1], [2] and discussed from a nontechnical point of view in R. Temam [12].

Remark A.6. Relations like (A.12) open the way for the use of approximate inertial manifolds for the development of multilevel methods for the numerical approximation of the Navier–Stokes equations; see the references quoted above and R. Temam [14].

We conclude § A.2 with the Navier–Stokes-related example.

Example A.1: *Navier–Stokes equation with enhanced viscosity.* We consider the Navier–Stokes equations in space dimension n with a higher-order viscosity term (see J. L. Lions [1]):

$$\frac{\partial u}{\partial t} + \mu(-\Delta)^r u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \tag{A.13}$$

$$\nabla\cdot u = 0. \tag{A.14}$$

The functions $u = u(x,t)$ and $p = p(x,t)$ are defined on $\mathbb{R}^n \times \mathbb{R}_+$, taking values in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, $u = (u_1,\dots,u_n)$; $r \in \mathbb{N}$, and μ,ν are strictly positive numbers. For $\mu = 0$, equations (A.13), (A.14) reduce to the incompressible Navier–Stokes equations.

We consider the space periodic case; hence u and p are periodic in each direction $x_1,\dots,x_n$, with period $L > 0$, i.e., (1.11) holds. We also assume as in Remark 1.1 and in the preceding chapters that

$$\int_Q u(x,t)\,dx = 0,$$

where $Q = (0,L)^n$.

We consider the same space H as in § 2.1 and set

$$Au = \mu(-\Delta)^r u - \nu\Delta u, \tag{A.15}$$

with domain $D(A) = V \cap \mathbb{H}^{2r}_p(Q)$ where V and $\mathbb{H}^{2r}_p(Q)$ are defined as in § 2.1. Writing

$$R(u) = \Pi(f - (u\cdot\nabla)u), \tag{A.16}$$

where Π is the orthogonal projector in $\mathbb{L}^2(Q)$ onto H, equation (A.3) is the same as (A.13), (A.14).

We consider the same trilinear form b as in (2.27) and we infer from Lemma 2.1 that b is trilinear continuous on $\mathbb{H}^{m_1}(Q) \times \mathbb{H}^{m_2+1}(Q) \times \mathbb{L}^2(Q)$ provided $m_1 = m_2 + 1 > n/4$. In particular for $m_1 = m_2 + 1 = r$, b is trilinear continuous on $D(A^{1/2}) \times D(A^{1/2}) \times H$, A as in (A.15), provided

$$r > \frac{n}{4} - \frac{1}{2},$$

e.g., $r \geqq 1$ in space dimension $2 \leqq n \leqq 5$. Using the methods of § 2, it is very easy to see that (1.11), (A.13), (A.14) define a well-posed initial value problem and that all hypotheses of Theorem A.1 are satisfied except iii) (see the technical details in C. Foias–G. Sell–R. Temam [1], [2]).

Now we turn our attention to hypothesis iii) in Theorem A.1. By R. Courant–D. Hilbert [1]–G. Métivier [1],

$$\lambda_N \sim cN^{2r/n} \quad \text{as } N \to \infty,$$

where c is an appropriate constant. Assume momentarily that $\lambda_N = cN^{2r/n}$; then for $\alpha = 1/2$,

$$\lambda_{N+1} - \lambda_N \sim c' N^{(2r/n)-1},$$
$$\lambda_{N+1}^{1/2} + \lambda_N^{1/2} \sim c'' N^{r/n},$$

and the spectral gap condition is satisfied for N large provided

$$r > n. \tag{A.17}$$

It can be shown that this condition, established by assuming that $\lambda_N = cN^{2r/n}$, remains partly valid if we only assume that $\lambda_N \sim cN^{2r/n}$ as $N \to \infty$; namely we obtain that the spectral gap condition is satisfied for a sequence N_i of $N_s, N_i \to \infty$ as $i \to \infty$.

In conclusion, under assumption (A.17), *equation* (A.13), (A.14) *with space periodicity boundary condition* (1.11) *possesses an inertial manifold.* A better (smaller) value of r can be obtained by choosing α more carefully (α smaller).

A.3. Inertial system for the Navier–Stokes equations. In this section we present a tentative approach for proving the existence of an inertial form for the space periodic Navier–Stokes equations in space dimension two, as in M. Kwak [1]. Similar results for the flow around a sphere, as in R. Temam–S. Wang [1], are given in § A.4.[2]

[2] Despite the statements made in these references and in the first printing of the second edition of this book, the existence of an inertial form for the Navier–Stokes equations in space dimension two is not yet fully proved. The difficulty is explained below. Nevertheless we thought the method proposed by M. Kwak sufficiently interesting and promising to warrant presentation in its present form. The reduction of the Navier–Stokes equations to a system of reaction–diffusion equations is also interesting on its own.

As observed before, Theorem A.1 does not apply to the two-dimensional Navier–Stokes equations with space periodic boundary conditions. The program of M. Kwak consists of imbedding the Navier–Stokes equations, through what we will call the Kwak transform, into a reaction–diffusion system for which Theorem A.1 might apply and then deducing the existence of an inertial system for the Navier–Stokes equations themselves.

The following presentation, based on R. Temam–S. Wang [1], is much simpler than the original proof; other results contained in R. Temam–S. Wang [1] are presented in § A.4. Of course we will not be able to give all the details but we will emphasize the main points. All the details can be found in R. Temam–S. Wang [1], hereafter referred to as [TW].

It is convenient, in view of § A.4, to introduce the function

$$\zeta = \operatorname{curl} u = D_1 u_2 - D_2 u_1.$$

Taking the curl of (1.5), we obtain the curl equation

$$\frac{\partial \zeta}{\partial t} - \nu\Delta\zeta + \operatorname{div}(\zeta u) = F = \operatorname{curl} f. \tag{A.18}$$

Space periodicity is also required for ζ ($\zeta(\cdot, t) \in H^1_p(Q)$ at each time t)[3] and from ζ we recover u by the equations

$$\begin{aligned} u &= \nabla^\perp \psi = (D_2\psi, -D_1\psi), \\ -\Delta\psi &= \zeta, \qquad \psi \in \dot{H}^1_p(Q), \end{aligned} \tag{A.19}$$

where ψ is the stream function. It is clear that (A.18), (A.19) are equivalent to the space periodic Navier–Stokes equations in dimension two.

The appropriate Kwak transform in this case consists of considering the functions

$$\begin{gathered} \zeta, \ \tilde{v} = \operatorname{grad} \zeta, \qquad \tilde{w} = \zeta u, \\ J(\zeta) = (\zeta, \tilde{v}, \tilde{w}). \end{gathered} \tag{A.19'}$$

By straightforward calculation, one can check that $J(\zeta)$ is solution of the system ($\varphi' = d\varphi/dt$):

$$\begin{gathered} \zeta' - \nu\Delta\zeta + \operatorname{div}\tilde{w} = F, \\ \tilde{v}' - \nu\Delta\tilde{v} + \operatorname{grad}(\operatorname{div}\tilde{w}) = \operatorname{grad} F, \\ \tilde{w}' - \nu\Delta\tilde{w} + 2\nu\tilde{v}\cdot\nabla u + (\tilde{v}\cdot u)u + \zeta \operatorname{curl}[\Delta^{-1}(\tilde{v}\cdot u)] = Fu + \zeta f. \end{gathered} \tag{A.20}$$

Here Δ^{-1} is the inverse of the Laplace operator from $\dot{L}^2(Q)$ into $\dot{H}^2_p(Q)$; $\tilde{v}\cdot\nabla u$ is the contracted product

$$\sum_{i=1}^{2} \tilde{v}_i D_i u = \sum_{i=1}^{2} (D_i\zeta)(D_i u).$$

[3] For the function spaces and norms the notations are those of § 2.

The embedded system. The embedded system (reaction–diffusion type system) consists of looking for functions ζ, v, w which are space periodic with period Q, and which satisfy the following equations:

$$
\begin{aligned}
\zeta' - \nu\Delta\zeta + \operatorname{div} w &= F, \\
v' - \nu\Delta v + \operatorname{grad}(\operatorname{div} w) &= \operatorname{grad} F, \\
w' - \nu\Delta w + 2\nu v\cdot\nabla u + (v\cdot u)u & \\
+ \zeta \operatorname{curl}[\Delta^{-1}(v\cdot u)] - Fu - \zeta f & \\
\underline{+ r\bar{\nu}[1 + \bar{\nu}^{-4}|\zeta|_{1/2}^{4}](w - \tilde{w})} &= 0.
\end{aligned}
\tag{A.21}
$$

Here $\bar{\nu} = \nu/L^2$, and $u, \tilde{v}, \tilde{w}$ are still defined by (A.19) and (A.19′) while v and w are now functions independent of ζ; $r > 0$ sufficiently large will be choosen hereafter. Except for the underlined terms, equations (A.21) are the same as (A.20) if we replace v and w by $\tilde{v}$ and $\tilde{w}$. We call (A.21) the *embedded system.*

For simplicity we write $U = (\zeta, v, w)$. The linear operator in (A.21) is the nonself-adjoint operator $U \to \mathbf{A}U$,

$$
\mathbf{A} = \begin{pmatrix} -\nu\Delta & 0 & \operatorname{div} \\ 0 & -\nu\Delta & \operatorname{grad}\operatorname{div} \\ 0 & 0 & -\nu\Delta \end{pmatrix}.
$$

We consider it an unbounded operator from $\mathbf{W}_2$ into $\mathbf{W}_0$, where for $s \geqq 0$, $\mathbf{W}_s = \dot{H}_p^s(Q) \times \dot{H}_p^s(Q)^2 \times \dot{H}_p^s(Q)$. Then, the notation being obvious, we write (A.21) like (A.3) in the form

$$
\frac{dU}{dt} + \mathbf{A}U = \mathbf{R}(U).
\tag{A.22}
$$

One can derive a priori estimates for the solutions of (A.21) and prove theorems of existence and uniqueness of solutions, similar to Theorem 3.1. As in § 3 we will establish a priori estimates on the solutions U of (A.21) assuming that they are sufficiently regular, skipping the details of the proof of existence and uniqueness; this is done in two steps:

i) The first step, a key one, is to show that (A.21) converges in some sense to (A.20) as $t \to \infty$; this is shown by proving that $v(t) - \tilde{v}(t)$ and $w(t) - \tilde{w}(t)$ converge to 0 as $t \to \infty$. Note that if ζ, v, w are solutions of (A.21) then

$$
\tilde{v}' - \nu\Delta\tilde{v} + \operatorname{grad}\operatorname{div} w = \operatorname{grad} F.
\tag{A.23}
$$

Subtracting this equation from the second equation (A.21) we find

$$
(v - \tilde{v})' - \nu\Delta(v - \tilde{v}) = 0.
$$

Hence

$$
\frac{d}{dt}|v - \tilde{v}|^2 + 2\nu\,\|v - \tilde{v}\|^2 = 0,
\tag{A.24}
$$

and if $\mu_1 = 4\pi^2/L^2$ is the first eigenvalue of the operator $-\Delta$ in $\dot{H}_p^2(Q)^2$,

$$\frac{d}{dt}\left|(v-\tilde{v})\right|^2 + 2\nu\mu_1\left|(v-\tilde{v})\right|^2 \leqq 0,$$

$$\left|(v-\tilde{v})(t)\right|^2 \leqq \left|(v-\tilde{v})(0)\right|^2 \exp(-2\nu\mu_1 t),$$

$$\left|(v-\tilde{v})(t)\right| \to 0 \text{ as } t \to \infty. \tag{A.25}$$

The proof is more involved for $w-\tilde{w}$. As for the derivation of the third equation (A.20), we first observe (using div $\tilde{w} = \tilde{v}\cdot u$) that $\tilde{w}$ satisfies

$$\begin{aligned}&\tilde{w}' - \nu\Delta\tilde{w} + 2\nu\,\tilde{v}\cdot\nabla u + (\operatorname{div} w)u\\&\quad + \zeta\operatorname{curl}[\Delta^{-1}(\operatorname{div} w)] = Fu + \zeta f.\end{aligned} \tag{A.26}$$

Subtracting this equation from the third equation (A.21) we obtain

$$\begin{aligned}&(w-\tilde{w})' - \nu\Delta(w-\tilde{w}) + 2\nu(v-\tilde{v})\cdot\nabla u\\&\quad + (v\cdot u - \operatorname{div} w)u + \zeta\operatorname{curl}[\Delta^{-1}(v\cdot u - \operatorname{div} w)]\\&\quad + r\bar{\nu}[1+\bar{\nu}^{-4}|\zeta|_{1/2}^4](w-\tilde{w}) = 0.\end{aligned} \tag{A.27}$$

Upon taking the scalar product in $L^2(Q)$ of each side of (A.27) with $w-\tilde{w}$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left|w-\tilde{w}\right|^2 + \nu\left\|w-\tilde{w}\right\|^2 + r\bar{\nu}\left[1+\bar{\nu}^{-4}|\zeta|_{1/2}^4\right]\left|w-\tilde{w}\right|^2\\&\quad = -2\nu\left((v-\tilde{v})\cdot\nabla u, w-\tilde{w}\right) - \left((v\cdot u - \operatorname{div} w)u, w-\tilde{w}\right)\\&\qquad - \left(\zeta\operatorname{curl}\left[\Delta^{-1}(v\cdot u - \operatorname{div} w)\right], w-\tilde{w}\right).\end{aligned} \tag{A.28}$$

Using appropriate estimates (see [TW]) we infer from (A.28) that

$$\begin{aligned}&\frac{d}{dt}\left|w-\tilde{w}\right|^2 + \nu\left\|w-\tilde{w}\right\|^2 + (2r-c_1')\bar{\nu}\left[1+\bar{\nu}^{-4}|\zeta|_{1/2}^4\right]\left|w-\tilde{w}\right|^2\\&\quad \leqq \nu^3\left\|w-\tilde{w}\right\|^2,\end{aligned} \tag{A.29}$$

where c_1' is a suitable constant independent of r. We now choose $r = c_1'$. Dividing then (A.29) by ν^2 and adding this relation to (A.24) we obtain

$$\begin{aligned}&\frac{d}{dt}\left\{\left|v-\tilde{v}\right|^2 + \nu^{-2}\left|w-\tilde{w}\right|^2\right\} + \nu\left\{\left\|v-\tilde{v}\right\|^2 + \nu^{-2}\left\|w-\tilde{w}\right\|^2\right\}\\&\qquad + \frac{r}{\nu L^2}\left[1+\bar{\nu}|\zeta|_{1/2}^4\right]\left|w-\tilde{w}\right|^2 \leqq 0.\end{aligned} \tag{A.30}$$

In particular

$$\frac{d}{dt}\left\{|v-\tilde{v}|^2+\nu^{-2}|w-\tilde{w}|^2\right\}+\nu\mu_1\left\{|v-\tilde{v}|^2+\nu^{-2}|w-\tilde{w}|^2\right\}\leqq 0,$$

$$\left\{|(v-\tilde{v})(t)|^2+\nu^{-2}|(w-\tilde{w})(t)|^2\right\}$$
$$\leqq\left\{|(v-\tilde{v})(0)|^2+\nu^{-2}|(w-\tilde{w})(0)|^2\right\}\ \exp(-2\nu\mu_1 t),$$

and this shows that

$$|(v-\tilde{v})(t)|^2+\nu^{-2}|(w-\tilde{w})(t)|^2\to 0 \text{ as } t\to\infty. \tag{A.31}$$

This implies that *the embedded system "converges" to the original one* (i.e., (A.18) *or* (A.20)) *when* $t\to\infty$.

ii) In a second step we derive further a priori estimates.

Estimates for ζ, $\tilde{v}=\operatorname{grad}\zeta$, and $\tilde{w}=\zeta u$ are derived as follows. We take the scalar product in $L^2(Q)$ of the first equation (A.21) with ζ; we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\zeta|^2+\nu\|\zeta\|^2&=(F,\zeta)-(\operatorname{div} w,\zeta)\\&=(F,\zeta)-(\operatorname{div}(w-\tilde{w}),\zeta)\qquad(\text{since } (\operatorname{div}\tilde{w},\zeta)=0)\\&=(\operatorname{curl} f,\zeta)+(w-\tilde{w},\operatorname{grad}\zeta)\\&=(f,\nabla^{\perp}\zeta)+(w-\tilde{w},\operatorname{grad}\zeta)\\&\leqq\frac{\nu}{4}\left|\nabla^{\perp}\zeta\right|^2+\frac{1}{\nu}|f|^2+\frac{\nu}{4}|\nabla\zeta|^2+\frac{1}{\nu}|w-\tilde{w}|^2\\&\leqq\frac{\nu}{2}\|\zeta\|^2+\frac{1}{\nu}|f|^2+\frac{1}{\nu}|w-\tilde{w}|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}|\zeta|^2+\nu\|\zeta\|^2&\leqq\frac{2}{\nu}|f|^2+\frac{2}{\nu}|w-\tilde{w}|^2\\&\leqq\frac{2}{\nu}|f|^2+\frac{2}{\nu\mu_1}\|w-\tilde{w}\|^2.\end{aligned}\tag{A.32}$$

We multiply (A.32) by $\mu_1/4$ ($\mu_1=4\pi^2/L^2$) and add this relation to (A.30):

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{\mu_1}{4}|\zeta|^2+|v-\tilde{v}|^2+\nu^{-2}|w-\tilde{w}|^2\right\}\\&+\frac{\nu}{2}\left\{\frac{\mu_1}{4}\|\zeta\|^2+\|v-\tilde{v}\|^2+\nu^{-2}\|w-\tilde{w}\|^2\right\}\\&+\frac{r}{\nu L^2}\left[1+\frac{\nu}{L^2}|\zeta|_{1/2}^4\right]|w-\tilde{w}|^2\leqq\frac{\mu_1}{2\nu}|f|^2.\end{aligned}\tag{A.33}$$

From (A.33) we infer some estimates similar to (3.4)–(3.7). In particular

$$\frac{d}{dt}\left\{\frac{\mu_1}{4}|\zeta|^2+|v-\tilde{v}|^2+\nu^{-2}|w-\tilde{w}|^2\right\}$$
$$+\frac{\nu\mu_1}{2}\left\{\frac{\mu_1}{4}|\zeta|^2+|v-\tilde{v}|^2+\nu^{-2}|w-\tilde{w}|^2\right\}\leqq\frac{\mu_1}{2\nu}|f|^2,$$

which yields

$$|\zeta(t)|\leqq K_1 \quad \forall\ t\geqq 0, \tag{A.34}$$

where K_1 depends on $U_0, |f|, \nu$ and L.

Further estimates are derived as in (3.9)–(3.17) by considering higher norms.

Recall that, for any $s \geqq 0$, $\mathbf{W}_s = \dot{H}^s_p(Q)\times\dot{H}^s_p(Q)^2\times\dot{H}^s_p(Q)$ and that $D(\mathbf{A}) = \mathbf{W}_2$. Then, based on these estimates one proves (see [TW] and compare to Theorem 3.2) the following theorem.

THEOREM A.2. *For f given in $\dot{\mathbf{H}}^1_p(Q)^2 = \dot{H}^1_p(Q)^2$ and U_0 given in $D(\mathbf{A})$, there exists a unique solution $U = U(t)$ of* (A.21) *such that $U(0) = U_0$, and*

$$U \in \mathscr{C}(\mathbb{R}_+;\mathbf{W}_1),$$
$$U \in L^2(0,T;\mathbf{W}_3),\quad U' \in L^2(0,T;\mathbf{W}_1)\quad \forall\ T>0.$$

Furthermore

i) *If $U_0 = J(\zeta_0)$, then $U(t) = J(\zeta(t))\ \forall\ t \geqq 0$.*

ii) *If $U_0 \neq J(\zeta_0)$, then*

$$|U(t)-J(\zeta(t))|\to 0 \quad \text{as } t\to\infty.$$

Application of Theorem A.1. We would now like to apply Theorem A.1 or, more precisely, a version of Theorem A.1 adapted to the nonself-adjoint case (see A. Debussche–R. Temam [2], [3], hereafter referred to as [DT2,3]). Remember that $\mathbf{A}$ is not self-adjoint; however its eigenvalues are the numbers $(4\pi^2\nu/L^2)(k_1^2+k_2^2)$, $k_1, k_2 \in \mathbb{N}$ and its eigenfunctions are proper combinations of sines and cosines.

One of the main points in applying Theorem A.1 is to check the spectral gap condition, i.e., hypothesis iii) of Theorem A.1. Here the λ_N are the numbers $(4\pi^2\nu/L^2)(k_1^2+k_2^2)$, $k_1, k_2 \in \mathbb{N}$ numbered in increasing order and according to their multiplicity.

Considered in $D(\mathbf{A})$, equation (A.21) is of reaction–diffusion type, i.e., its nonlinear terms are *continuous functions* of U in $D(\mathbf{A})$; for this reason $\alpha = 0$ in hypothesis iii). Thus the spectral gap condition reduces to a well-known problem in number theory; namely that there are arbitrarily large gaps among the integers which are sum of two squares (of integers). Such gaps indeed exist.

Because we need the nonself-adjoint version of Theorem A.1 appearing, for instance, in [DT2,3], a new difficulty arises which is not yet resolved. Indeed some of the requirements in [DT2,3] involve bounding, independently of N, the norms of the operators $e^{\mathbf{A}t}P$ and $e^{-\mathbf{A}t}Q$ where $t \geqq 0$ and $P = P_N$ and $Q = Q_N$ are the spectral projectors as above. Although $\mathbf{A}$ is not self-adjoint, its generalized eigenvectors (root vectors) are orthogonal and the projectors P and Q are orthogonal. However, the Jordan blocks of $\mathbf{A}$ produce some contributions to $e^{\mathbf{A}t}, e^{-\mathbf{A}t}$, which we are not able to control. The verification of this hypothesis (hypothesis (H1) in [DT3]) has been overlooked in the references using the Kwak transform.

If an inertial manifold and an inertial system for the embedded equation exist then, due to statement i) in Theorem A.2, the inertial form of the embedded system is also an inertial form of the two dimensional Navier–Stokes equation. Write $U = Y + Z$ and write

$$Z = \Phi(Y),$$

the equation of the inertial manifold for the embedded system as in (A.7) and (A.10). Let $\mathbf{P}$ be the corresponding operator $P = P_N$; then from (A.8)–(A.10) and (A.22) we infer the inertial system

$$\frac{dY}{dt} + \mathbf{A}Y = \mathbf{PR}(Y + \Phi(Y)). \tag{A.35}$$

That is, we would have proved the following result:

(A.35′) There exists a finite-dimensional differential system, namely (A.35), which produces the same dynamics[4] as the two-dimensional space periodic Navier–Stokes equations.

Remark A.7. i) We assumed that the flow has the same period L in each direction x_1, x_2. If the periods L_1, L_2 are different, then the existence of an inertial manifold hinges on the fact that there are arbitrarily large gaps in the set of numbers $((L_2^2/L_1^2)k_1^2 + k_2^2)$, $k_1, k_2 \in \mathbb{N}$. This remains true if L_2/L_1 is rational and Theorem A.3 is valid in this case as well (see C. Foias–G. Sell–R. Temam [1], [2]).

ii) For the flow in a bounded domain (see § 2.5), the embedded system is not of reaction–diffusion type and the previous method no longer applies.

iii) If $\mathscr{A}_0$ is the global attractor for the Navier–Stokes equations and $\mathscr{A}$ the attractor for the embedded system then $\mathscr{A} = J(\mathscr{A}_0)$ and, as far as we know, (A.21) is the simplest imbedded system for which this property is valid.

iv) We could have considered the Navier–Stokes equations in vector form instead of the curl equation, using a suitable embedded system. However, for some geometric reason this is not straightforward for flows around a sphere considered in § A.4 and, for the sake of simplicity, it was better to start from the curl equation in both cases.

A.4. Flow around a sphere. In this section we consider the flow of an incompressible fluid around a two-dimensional sphere; we follow [TW].

[4] That is, the same equilibrium points, same time-periodic or time quasi-periodic orbits, same attractors, and so on.

There are two reasons for considering such flows:

• The first reason is the obvious interest of such flows in meteorology and climate type problem.

• The second reason is related to the spectral gap condition arising in Theorem A.2. The Laplace operator is now replaced by the Laplace–Beltrami operator on the sphere which we still denote by Δ. The eigenvalues and eigenfunctions of Δ are known; the eigenfunctions are spherical harmonics and the eigenvalues are the numbers $n(n+1)$, $n \in \mathbb{N}$, repeated according to their multiplicity. Then the λ_N are numbers of the form $\nu n(n+1)$, and *there are indeed large gaps in the spectrum.*

The corresponding Navier–Stokes equations for the flow around the unit sphere S^2 are written

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla_u u + \operatorname{grad} p = f, \tag{A.36}$$

$$\operatorname{div} u = 0. \tag{A.37}$$

Here Δ is the Laplace–Beltrami operator on the unit sphere S^2 (for vector functions), $\nabla_u v$ is the covariant derivative of v in the direction u, grad and div are the gradient and divergence operators on S^2. The analytic expressions using spherical coordinates can be found in [TW] and in many places in the literature (see, e.g., T. Aubin [1]). We have

$$\begin{aligned} \nabla_u u &= \operatorname{grad}\left(\frac{|u|^2}{2}\right) - u \times \operatorname{curl} u, \\ \Delta u &= \operatorname{grad}(\operatorname{div} u) - \operatorname{curl}\operatorname{curl} u. \end{aligned} \tag{A.38}$$

Here the curl of a scalar function ψ or a vector function u are defined by

$$\begin{aligned} \operatorname{curl} \psi &= \mathbf{n} \times \operatorname{grad} \psi, \\ \operatorname{curl} u &= -\operatorname{div}(\mathbf{n} \times u), \end{aligned} \tag{A.39}$$

where $\mathbf{n}$ is the unit outward normal on S^2.

Now set

$$\zeta = \operatorname{curl} u \tag{A.40}$$

and observe that u is completely determined by ζ using $\operatorname{div} u = 0$, which implies

$$u = \operatorname{curl} \psi, \qquad -\Delta\psi = \zeta. \tag{A.41}$$

Taking the curl of each side of (A.36) we obtain

$$\frac{\partial \zeta}{\partial t} - \nu\Delta\zeta + \operatorname{div}(\zeta u) = F = \operatorname{curl} f. \tag{A.42}$$

This equation is the same as (A.18) except that Δ is now the Laplace–Beltrami operator and div, curl have to be properly defined. In fact, except for some minor and obvious modifications, all that was said in § A.3 from (A.18) to (A.35′) applies here. The embedded system is the same as (A.21) (see [TW]). The only modification necessary occurs when we apply Theorem A.2 and check the spectral gap condition: here we do not need to invoke the result in number theory. Instead we observe that there are indeed arbitrarily large gaps in the spectrum which consists of the numbers $\nu n(n+1)$. If λ_N and λ_{N+1} are two different consecutive eigenvalues $\lambda_N = \nu n(n+1)$, $\lambda_{N+1} = \nu(n+1)(n+2)$, then

$$\lambda_{N+1} - \lambda_N = 2\nu(n+1). \tag{A.43}$$

Hence all hypotheses of Theorem A.1 are easily verified, in particular the spectral gap condition. In fact, (A.43) would give the value of N (expressed in terms of the data) for which the spectral gap condition is satisfied and hence an estimate of the dimension of the inertial manifolds of the embedded system and of the inertial form of the Navier–Stokes equations for the flow around a sphere. However, here we need again the nonself-adjoint version of Theorem A.1 of [DT3] and in that respect we meet the same difficulty as in § A.3. Instead of verifying hypothesis (H1) of [DT3] we might perhaps consider different constructions of the inertial manifold of the embedded system, using another function space. Also the reduction to the reaction–diffusion system (A.21) is not canonical; one might obtain a different reaction–diffusion system for which the operator **A** is self-adjoint. These problems are open.

Remark A.8. It would be very easy to incorporate the Coriolis force $k \times u$ in the model above.

Comments and Bibliography

Most of the results in §§ 2 and 3 are classical and, as mentioned, can be found in the books of O. A. Ladyzhenskaya [1], J. L. Lions [1] and R. Temam [6], and in J. Serrin [1]. The only difference is that we have emphasized a little more the space periodic case which is not treated in detail in the literature; also, at the end of § 3 (§ 3.4) we give a simple but interesting result of generic solvability in the large, due to A. V. Fursikov [1].

The new a priori estimates in § 4 are borrowed from C. Foias–C. Guillopé–R. Temam [1]. The results on the Hausdorff dimension of the singular set of a weak solution to the Navier–Stokes equations in § 5 are two typical results, if not the most recent. They rely on the ideas of B. Mandelbrot [1], and on the work of V. Scheffer [1], [4] and C. Foias–R. Temam [5]. The result in § 5.1 (time singularities) is just a restatement by V. Scheffer [1] of an old result of J. Leray [3], and also generalizes a result of S. Kaniel and M. Shinbrot [1]. The result in § 5.2, leading to a Hausdorff dimension of singularity $\leqq \frac{5}{2}$ in space and time ($n = 3$), is an extension due to C. Foias–R. Temam [5] of a result by V. Scheffer [3], [4]. The best result presently available along those lines is that of L. Caffarelli–R. Kohn–L. Nirenberg [1] (the one-dimensional Hausdorff measure of the singular set vanishes).

The set of compatibility conditions given in § 6 was derived in R. Temam [8]. The first compatibility condition is mentioned in J. G. Heywood [2]; for the compatibility conditions, cf. also O. A. Ladyzhenskaya–V. A. Solonnikov [3]. In § 7 we give a simple proof of time analyticity of the solutions to the Navier–Stokes equations (with values in $D(A)$), which is a "complexification" of the standard Galerkin approximation procedure; this presentation is slightly simpler than the original one given in C. Foias–R. Temam [2], [5]. For other results on time analyticity, cf. also C. Foias–C. Guillopé–R. Temam [1], and for a proof based on semigroup operators which apparently applies to N.S.E., cf. F. J. Massey III [1].

Finally, in the last section of Part I (§ 8), we use an a priori estimate of § 4, to determine the Lagrangian representation of the flow associated with a weak solution of the N.S.E.; for further developments on this new result, cf. C. Foias–C. Guillopé–R. Temam [2].

The properties of the set of stationary solutions to the N.S.E. based on the Sard–Smale theorem, and derived in § 10, are among the results proved in C. Foias–R. Temam [3], [4]. Results of genericity with respect to the boundary conditions are not presented here; they use somewhat more sophisticated tools in analysis and topology, and can be found in J. C. Saut–R. Temam [1], [2]. For other recent results on the set of stationary solutions, cf. also C. Foias–J. C. Saut [3] and Gh. Minéa [1].

The results in §§ 11 and 12 are taken from C. Foias–R. Temam [5]. The squeezing property of the trajectories is used in § 12, and has played a role in the proof of other results on the structure of a turbulent flow: cf. in particular C. Foias–R. Temam [7], [8]. The results mentioned in § 12.3 are proved in C. Guillopé [1]; a similar result is also announced in J. G. Heywood–R. Rannacher [1].

For the numerical approximation of the N.S.E. on a finite time interval (and for stationary solutions), see the references mentioned in § 13; see also R. Temam [6], and M. Fortin–R. Peyret–R. Temam [1], R. Peyret [1], R. Temam [1], [2], [4], [5], F. Thomasset [1], and the references therein. The compactness theorem used in the proof of convergence is new. The results in § 14 are part of a work in progress; cf. C. Foias–R. Temam [9], C. Foias–O. Manley–R. Temam–Y. Trève [1] and the references there.

Comments and Bibliography: Update for the Second Edition

A new result published since the first edition of these notes is the Gevrey class regularity of the space periodic solutions of the Navier–Stokes equations which appeared in C. Foias–R. Temam [13]. This implies a result of space analyticity of the solutions, providing an exponential decay rate of the Fourier coefficients when the force f itself is in the same class. The proof, which is short and rather elementary, is based on appropriate forms of energy estimates.

The results of Chapter 3 on the existence and uniqueness of solutions in space dimension three have been partly improved by G. Raugel–G. Sell [1], who consider three-dimensional flows in a thin domain (domain thin in the third direction). Namely, if the thickness ϵ and the data satisfy some set of inequalities, then the solution exists for large times and remains smooth.

The "new a priori estimates" in Chapter 4 have been extended in several directions by G. F. D. Duff [1]–[4] who addresses the case of a bounded domain and of nonzero (nonpotential) volume forces. See also in D. Chae [1], [2] results extending those of § 4 to Gevrey classes, combining the methods of C. Foias–C. Guillopé–R. Temam [1] and C. Foias–R. Temam [13]. The energy inequalities in Lemma 4.1 (derived here as a preliminary result for the "new a priori estimates" in § 4.3) have been improved by W. D. Henshaw–H. O. Kreiss–L. G. Reyna [1]. They make explicit the dependence of the bounds on ν and on associated nondimensional numbers and their proof leads to better bounds than those obtained if one makes explicit the bounds from Lemma 4.1.

New results partly related to those of Chapter 8 (Lagrangian Representation of the Flow) were proved by R. Coifman–P. L. Lions–Y. Meyer–S. Semmes [1].

The problems considered in Part II have been the object of extensive research during the past decade; a number of new results have appeared concerning the behavior for $t \to \infty$ of the solutions, the concept of attractor, and the idea of finite dimensionality of flows. For the attractors, the result of finite dimensionality of Navier–Stokes attractors presented in Chapter 12 (and based on C. Foias–R. Temam [5]) have been improved using the concept of Lyapunov exponents and ideas from dynamical systems theory. New bounds have been derived on the dimension of attractors which are physically relevant and are related to and consistent with the conventional theory of turbulence of Kraichnan (in dimension two) and Kolmogorov (in dimension three); see P. Constantin–C. Foias–O. Manley–R. Temam [1], P. Constantin–C. Foias [1], P. Constantin–C. Foias–R. Temam [2], [3]. A thorough description of the results and many more results on attractors for dissipative evolution equations can be found in the following books and in the references therein: A. V. Babin–M. I. Vishik [1], C. Foias–O. Manley–R. Temam

[5], J. Hale [1], O. A. Ladyzhenskaya [2], E. Lieb [1], D. Ruelle [4], [5], M. I. Vishik [2]; see also the books edited by C. Foias–B. Nicolaenko–R. Temam [1] and C. Foias–G. Sell–R. Temam [3].

Another result pertaining to finite dimensionality of flows is the concept of inertial manifolds addressed in the Appendix. The corresponding comments and bibliographical references are given in the Appendix itself. Finally results on finite dimensionality of flows also appear as part of the concepts of determining modes and determining points. Determining modes were introduced and studied in C. Foias–O. Manley–R. Temam–Y. Trève [1]; determining points (or nodes) were introduced and studied in C. Foias–R. Temam [11], [12]. New developments appear in D. A. Jones–E. S. Titi [1]–[3]; reference [3] contains the latest results.

In relation to Part III and in particular Chapter 14, we would like to mention the development of new multiresolution algorithms based on the use of approximate inertial manifolds. This includes the nonlinear Galerkin method and the incremental unknown method. See C. Foias–O. Manley–R. Temam [2], [3], M. Marion–R. Temam [1], [2], C. Foias–M. Jolly–I. F. Kevrekidis–G. R. Sell–E. S. Titi [1], T. Dubois–F. Jauberteau–R. Temam [1], [2], R. Temam [15], [16], J. G. Heywood–R. Rannacher [2], D. A. Jones–L. G. Margolin–E. S. Titi [1], and B. Garcia-Archilla–J. de Frutos [1], among others; see also R. Temam [16].

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