Front Motion in Viscous Conservation Laws with Stiff Source Terms

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Abstract

We study a scalar, bi-stable reaction-diffusion-convection equation in $\mathbb{R}^N$. With a hyperbolic scaling, we reduce it to a singularly perturbed equation. In the singular limit, solutions converge to functions which take on only two different values in bulk regions. We also derive an equation describing the motion of the interface between the two bulk regions. To the lowest order, the normal speed $s(\nu)$ of the interface depends only on the unit normal vector ν, where the wave speed $s(\nu)$ is determined by nonlinear reaction and convection terms.

When the convection term is even and the reaction term is odd in their argument, the wave speed $s(\nu)$ vanishes identically for all directions ν. In this situation, a parabolic scaling reduces the equation to another singularly perturbed equation for which we also establish convergence of solutions in the singular limit. The singular limit dynamics is governed by an anisotropic mean curvature flow, in which the anisotropy comes from the convection term.

Our method of proof for convergence consists of constructing approximate solutions of any order and the comparison principle.

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1 Introduction

1.1 Statement of the Problem

We consider the reaction-diffusion-convection equation

$$u_t + \text{div } f(u) = \Delta u + g(u), \qquad (x,t) \in \mathbb{R}^N \times (0,T) \tag{RDC}$$

in the N-dimensional space, where $u = u(x,t) \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}^N$ is a flux, and $g : \mathbb{R} \to \mathbb{R}$ a nonlinear reaction term of bistable type. By a hyperbolic scaling $(x,t) \to (x/\varepsilon, t/\varepsilon)$, the equation becomes

$$u_t + \text{div } f(u) = \varepsilon\Delta u + \varepsilon^{-1} g(u), \qquad (x,t) \in \mathbb{R}^N \times (0,T), \tag{1}$$

where $\varepsilon > 0$ is a positive parameter. When $\varepsilon > 0$ is small, this is a conservation law with small viscousity and stiff source term.

Our objective is to describe the dynamics in the singular limit for the scalar multi-dimensional equation (1). In terms of the original scale in (RDC), the singular limit $\varepsilon \to 0$ gives a representation of the large–time behavior of the spatial variation over large domains in the solutions of (RDC).

Let u^ε be the unique solution of the Cauchy problem for (1) with initial condition

$$u(x,0) = \phi(x). \tag{2}$$

It is our goal to determine the existence and structure of the limiting function

$$u^0(x,t) = \lim_{\varepsilon \to 0^+} u^\varepsilon(x,t)$$

for $(x,t) \in \mathbb{R} \times [0,T]$. Throughout the paper, we assume that the source term g and the convective term f satisfy the following conditions.

(H1) (i) The function g belongs to $C^\infty(\mathbb{R})$ and possesses exactly three zeros $u_- < 0 < u_+$ with

$$g'(u_-) < 0, \quad g'(0) > 0, \quad g'(u_+) < 0 \quad \text{(bi-stability of reaction)}.$$

(ii) The vector f belongs to $C^\infty(\mathbb{R}, \mathbb{R}^N)$.

1.2 Existence and uniqueness

Under assumption [**(H1)**, local existence and uniqueness of solutions to either (RDC) or (1) with fixed ε can be shown as follows (see [1] for details). The Cauchy problem

$$\left.\begin{array}{rcl} u_t &=& \Delta u + F(u, \nabla u) \\ u(x,0) &=& u_0(x) \end{array}\right\} \tag{3}$$

is considered as an abstract ordinary differential equation

$$\left.\begin{array}{rcl} \dot{u} & = & \Delta_{\mathrm{BC}} u + N_F(u) \\ u(0) & = & u_0, \end{array}\right\} \tag{4}$$

where $u(t) \in X = \mathrm{BC}_{\mathrm{unif}}(\mathbb{R}^N)$, Δ_{BC} is a realization of Δ on $\mathrm{BC}_{\mathrm{unif}}(\mathbb{R}^N)$, the space of bounded, uniformly continuous functions, and N_F is the Nemitskii operator associated with F.

One first shows that Δ_{BC} is the infinitesimal generator of an analytic semigroup. By standard arguments, one can then show that (4), and hence (3), has a unique local solution

$$u \in C^0([0,T], \mathrm{BC}_{\mathrm{unif}}(\mathbb{R}^N)) \cap C^1((0,T], \mathrm{BC}_{\mathrm{unif}}(\mathbb{R}^N))$$

provided that $F \in C^0(\mathbb{R} \times \mathbb{R}^N)$ has partial derivatives $\partial F/\partial u$ and $\partial F/\partial p_j$ which are bounded on bounded subsets of $\mathbb{R} \times \mathbb{R}^N$ and the initial condition u_0 is contained in some interpolation space. For our purposes it will suffice to assume $u_0 \in \mathrm{BC}^2_{\mathrm{unif}}(\mathbb{R}^N)$ which is contained in the domain $\mathcal{D}(\Delta_{\mathrm{BC}})$ and therefore also in the interpolation space.

In our situation $F(u, \nabla u) = g(u) - f'(u) \cdot \nabla u$ is continuous and the partial derivatives $\partial F/\partial u = g'(u) - f''(u) \cdot \nabla u$ and $\partial F/\partial p_j = -f_j'(u)$ are uniformly continuous on bounded sets if $f \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $g \in C^1(\mathbb{R})$.

If F grows at most linearly with respect to $|\nabla u|$, then any solution which remains bounded in the L^∞-norm is a global classical solution. In our situation, however, the L^∞-bound is an easy consequence of the dissipativity condition **(H1)**-(i) and the parabolic comparison principle. In particular, this shows that for initial data $u_0 \in \mathrm{BC}^2_{\mathrm{unif}}(\mathbb{R}^N)$ we get classical solutions on the infinite time interval $[0, \infty)$.

1.3 Planar Waves

Planar wave solutions to (1) of the form $u(x,t) = U(\frac{x \cdot \nu - st}{\varepsilon})$ satisfy a one-dimensional ordinary differential equation

$$U''(z) + (s - f'(U(z)) \cdot \nu)\, U'(z) + g(U(z)) = 0, \quad z \in \mathbb{R} \quad (' = \mathrm{d}/\mathrm{d}z) \tag{5}$$

which can be written as a first order system

$$\begin{array}{rcl} U' & = & V \\ V' & = & -\,(s - f'(U) \cdot \nu)\, V - g(U). \end{array} \tag{6}$$

This system possesses exactly three stationary points with $U \in \{u_-, 0, u_+\}$ and $V = 0$. The linearization at $(u^*, 0)$ ($u^* = u_-, 0, u_+$) is

$$\begin{array}{rcl} U' & = & V \\ V' & = & -g'(u^*)U - (s - f'(u^*) \cdot \nu)V. \end{array}$$

with eigenvalues

$$\mu_{\pm}(s) = \frac{-(s - f'(u^*)\cdot\nu) \pm \sqrt{(s - f'(u^*)\cdot\nu)^2 - 4g'(u^*)}}{2}. \tag{7}$$

Hence the two stable zeros u_- and u_+ always correspond to saddle equilibria of (6).

Using the fact that (6) defines a "rotated vector field" (see [3]), it can be shown that the unstable manifold of $(u_-, 0)$ and the stable manifold of $(u_+, 0)$ intersect for precisely one wave speed $s(\nu)$. In [5] existence and uniqueness of traveling waves have been established for a class of equations which contains (1). We state the corresponding result in our situation:

Proposition 1.1 ([5], Theorem 2.4).

(i) *There exists a unique wave speed $s(\nu)$ such that for $s = s(\nu)$ there is a unique heteroclinic orbit of (6) connecting $(u_-, 0)$ (at $z = -\infty$) to $(u_+, 0)$ (at $z = +\infty$).*

(ii) *The wave speed $s(\nu)$ depends on ν as smooth as $f'(u)$ and $g(u)$ do on u.*

(iii) *The corresponding travelling wave profile $Q(z;\nu)$, with $Q(0;\nu) = 0$, is smooth in (z,ν) and monotone increasing in z.*

Once the existence of planar waves is established, it is rather elementary to characterize the wave speeds.

Lemma 1.2. *Traveling wave solutions of (6) have the following properties.*

(1) *The wave speed $s(\nu)$ of the planar traveling wave satisfies the following two identities:*

(i) $s(\nu)(u_+ - u_-) = (f(u_+) - f(u_-))\cdot\nu + \int\limits_{-\infty}^{+\infty} g(Q(z;\nu))\,\mathrm{d}z$

(ii) $s(\nu) = \dfrac{G(u_-) - G(u_+) + \int_{-\infty}^{+\infty} Q_z^2(z;\nu) f'(Q(z;\nu))\cdot\nu\,\mathrm{d}z}{\int_{-\infty}^{+\infty} Q_z^2(z;\nu)\,\mathrm{d}z}$

where G is an anti-derivative of g.

(2) *If f is even and g is odd, then $Q(z)$ is odd, $Q(-z) = -Q(z)$, and $s(\nu) \equiv 0$ for $\nu \in \mathbb{R}^N$ (although we will always consider unit vectors $|\nu| = 1$ in the sequel).*

Proof: (1) (i) Since $Q(z)$ decays exponentially fast to u_- (resp. u_+) as $z \to -\infty$ (resp. $+\infty$), we may integrate the traveling wave equation (5) over $\mathbb{R}$. This yields exactly the given identity.

(1)(ii) Since the derivatives of the heteroclinic solution Q decay exponentially at $z = \pm\infty$, we may multiply equation (5) by Q_z and integrate from $z = -\infty$ to $z = +\infty$ to get

$$s(\nu) \int_{-\infty}^{+\infty} Q_z^2(z;\nu)\,\mathrm{d}z = \int_{-\infty}^{+\infty} g(Q(z;\nu))Q_z(z;\nu)\,\mathrm{d}z + \int_{-\infty}^{+\infty} f'(Q(z;\nu)) \cdot \nu Q_z(z;\nu)^2\,\mathrm{d}z.$$

(2) It is easy to verify that the function $\hat{Q}(z) := -Q(-z)$ satisfies (6) with $s = s(\nu)$ begin replaced by $s = -s(\nu)$ and the same boundary conditions as $Q(z)$ does. Therefore, the uniqueness of the solution pair $(Q(z), s(\nu))$ for (6) implies $s(\nu) = -s(\nu)$ and $\hat{Q}(z) \equiv Q(z)$, and hence $s(\nu) = 0$ and $Q(z) = -Q(-z)$.

■

Remark 1.3. *Characterization (1)(i) is analogous to the well-known Rankine–Hugoniot condition for viscous shocks of conservation laws. The presence of a stiff source results in an additional term which does depend not only on the asymptotic states but also on the whole "viscous profile" Q.*

We believe that the converse to the statemtent Lemma 1.2 (2) is valid. Namely, if the wave speed $s(\nu)$ vanishes identically in all directions ν, then f is even and g is odd. However, we have been unable to prove this statement.

Definition 1.4. *A* ***hypersurface*** *M of class C^2 is a subset of $\mathbb{R}^N$ which is locally a graph of a C^2-function:*

$$M = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N;\ x_N = F(x_1, x_2, \ldots, x_{N-1})\}.$$

By the **principal curvatures** *of M at a point x_0 in the direction ν ($\nu \perp_{x_0} T_{x_0}M$), we mean $N-1$ eigenvalues $\kappa_1, \kappa_2, \ldots, \kappa_{N-1}$ of the symmetric matrix $D^2F(x_0)$, where x_N stands for the coordinate function in ν-dirction, while $x_1, \ldots, x_{N-1}$ are coordinate functions on $T_{x_0}M$, the tangent plane to M at x_0.*

The **mean curvature** *of M at a point x_0 is the sum of principal curvatures*

$$H(x_0) = \kappa_1 + \kappa_2 + \ldots + \kappa_{N-1}.$$

1.4 Main results

We describe dynamics of (1) in singular limits as $\varepsilon \to 0$. Depending on whether or not the wave speed $s(\nu)$ identically vanishes, we have to distinguish two cases.

Theorem 1.1. *Assume that $s(\nu) \not\equiv 0$. Under the condition* **(H1)**, *we consider the Cauchy problem*

$$\left.\begin{aligned} u_t^\varepsilon &= \varepsilon\Delta u^\varepsilon - f'(u^\varepsilon)\cdot\nabla u^\varepsilon + \varepsilon^{-1}g(u^\varepsilon) \\ u^\varepsilon(x,0) &= \phi^\varepsilon(x). \end{aligned}\right\} \tag{8}$$

Then, there exist a pair of functions $\underline{u}_0^\varepsilon(x) < \overline{u}_0^\varepsilon(x)$ *on* $\mathbb{R}^N$ *and a time* $T > 0$ *such that if* $\phi^\varepsilon(x) \in \mathrm{BC}^2_{\mathrm{unif}}(\mathbb{R}^N)$ *with*

$$\underline{u}_0^\varepsilon(x) < \phi^\varepsilon(x) < \overline{u}_0^\varepsilon(x),$$

then the solution $u^\varepsilon(x,t)$ *converges to a limit function* $u^0(x,t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ *for almost all* $(x,t) \in \mathbb{R}^N \times [0,T]$. *The limit* $u^0(x,t)$ *is a piecewise constant function taking on only two values* u_- *and* u_+. *The bulk regions* $\Omega^\pm(t) := \{x \in \mathbb{R}^N;\ u^0(x,t) = u_\pm\}$ *are separated by a hypersurface* $\Gamma(t)$ *which evolves according to the equation*

$$V = s(\nu),$$

where V *stands for the normal speed of the interface* $\Gamma(t)$ *and* ν *is the unit normal vector on* $\Gamma(t)$ *pointing into the bulk region* $\Omega^+(t)$.

Moreover, if the ε*-dependent interface* $\Gamma^\varepsilon(t)$ *is defined by*

$$\Gamma^\varepsilon(t) = \{x \in \mathbb{R}^N \mid u^\varepsilon(x,t) = 0\},$$

then its motion is described by

$$V^\varepsilon = s(\nu^\varepsilon) + \varepsilon \left\{ H^\varepsilon(y,t) + \sum_{p,q=1}^N \mathbf{T}^\varepsilon_{pq} K^{pq}_\varepsilon(y,t) \right\} + \mathcal{O}(\varepsilon^2). \tag{9}$$

Here, ν^ε *is the unit normal vector to* $\Gamma^\varepsilon(t)$, $s(\nu^\varepsilon)$ *the speed of a planar travelling wave propagating in direction* ν^ε, $H^\varepsilon(y,t)$ *stands for the mean curvature of* $\Gamma^\varepsilon(t)$ *at* $y \in \Gamma^\varepsilon(t)$, $(\mathbf{T}^\varepsilon_{pq})$ *is a symmetric, positive semi-definite matrix depending on* (f, g, ν^ε), *and* K^{pq}_ε *is a symmetric tensor related to the second fundamental form of* $\Gamma^\varepsilon(t)$.

When f is even and g is odd, Lemma 1.2 implies that $s(\nu) \equiv 0$. In this case, a result analogous to Theorem 1.1 holds for a parabolically scaled version of (8), which is stated as follows.

Theorem 1.2. *Assume that* f *is even and* g *is odd. Under the condition* **(H1)**, *we consider the Cauchy problem*

$$\left.\begin{array}{rcl} u^\varepsilon_t &=& \Delta u^\varepsilon - \varepsilon^{-1} f'(u^\varepsilon)\cdot\nabla u^\varepsilon + \varepsilon^{-2} g(u^\varepsilon) \\ u^\varepsilon(x,0) &=& \phi^\varepsilon(x). \end{array}\right\} \tag{10}$$

There exist a class of initial functions ϕ^ε *and a* $T > 0$, *as in Theorem 1.1, such that the solution* u^ε *of (10) has a limit, i.e.,* $u^0(x,t) := \lim\limits_{\varepsilon\to 0} u^\varepsilon(x,t)$ *exists for a.e.* $(x,t) \in \mathbb{R}^N \times [0,T]$.

The limit $u^0(x,t)$ *is a piecewise constant function taking on only two values* u_- *and* u_+. *The regions* $\Omega^\pm(t) := \{x \in \mathbb{R}^N;\ u^0(x,t) = u_\pm\}$ *are separated by a hypersurface* $\Gamma(t)$ *which evolves according to the equation*

$$V = H + \sum_{p,q=1}^N \mathbf{T}_{pq} K^{pq} = \sum_{p,q=1}^N \left(\delta_{pq} + \mathbf{T}_{pq}\right) K^{pq}, \tag{11}$$

where V stands for the normal speed of the interface $\Gamma(t)$ and $H, \mathbf{T}$ and K are the same as in Theorem 1.1.

If we consider (1) with suitable initial functions $\phi(x)$, solutions develop internal layers after time of $\mathcal{O}(\varepsilon|\log \varepsilon|)$ near the zero level set Γ_0 of ϕ. Generically, Γ_0 is of class C^2, and the solution profile would satisfy

$$\underline{u}_0^\varepsilon(x) < u^\varepsilon(x, \mathcal{O}(\varepsilon|\log \varepsilon|) < \overline{u}_0^\varepsilon(x).$$

Therefore, from this moment on, our theorems 1.1 and 1.2 apply to this general situation.

Recently, Fan and Jin [4] have studied (1) in the case of a symmetric flux function f and an odd source term g. Since $s(\nu) \equiv 0$, in this situation, we are in the setting of Theorem 1.2. The front motion is then driven by some kind of generalized mean curvature, or, anisotropic curvature.

For $f \equiv 0$ and symmetric g, our equation (RDC) is the so called Allen-Cahn equation. The formation and motion of internal layers for

$$u_t = \varepsilon^2 \Delta u + g(u)$$

for $0 < \varepsilon \ll 1$ have been studied for a long time. For example, in [2] it is shown that interfaces which have formed move according to mean curvature flow. Their analysis uses different temporal and spatial scales without separating them completely. Our approach is therefore closer to the one in [7] for the spatially inhomogeneous Allen-Cahn equation, where the outer solution and the approximate internal layer are constructed separately up to any order in ε. This allows us to show how the convection term in (RDC) introduces a new type of geometric evolution in the interface equation. The additional term $\sum \mathbf{T}_{pq} K^{pq}$ in (11) has two parts where $\mathbf{T}$ is determined by the convective and reaction terms f and g from the equation and the unit normal vector ν of the interface, while K^{pq} depends on the geometry of the interface.

The paper is organized in the following way: Section 2 deals with the formal construction of the approximate interface equation. In Section 3, we prove our main theorems on the convergence. The last section is devoted to explicit computations in the case that f and g have some special form.

2 The approximate interface equation

In this section, we present an asymptotic expansion method to construct approximate solutions to (1). The procedure below may look rather formal. However, in §3.1, we will make it mathematically rigorous. A similar asymptotic expansion is valid for (10), and modifications needed are explained in §3.2.

2.1 The outer expansion

We rewrite (1) as

$$\varepsilon u_t + \varepsilon f'(u) \cdot \nabla u - \varepsilon^2 \Delta u = g(u)$$

and expand its solutions as follows.

$$u_{\text{out}}(x,t) = u^0_{\text{out}}(x,t) + \varepsilon u^1_{\text{out}}(x,t) + \dots .$$

Inserting this expansion, (1) leads at zeroth order to the equation

$$g(u^0_{\text{out}}(x,t)) = 0.$$

As we are looking for a solution which takes values close to u_- in one subdomain of $\mathbb{R}^N$ and values close to u_+ on the complement of that set, we choose the following as our lowest order outer expansion:

$$u^0_{\text{out}}(x,t) = \begin{cases} u_- & \text{in } \Omega^-(t), \\ u_+ & \text{in } \Omega^+(t), \end{cases}$$

in which $\Omega^\pm(t)$ are to be determined. In fact, we will later derive a motion law for the common boundary $\Gamma(t)$ of $\Omega^\pm(t)$.

Since g does neither depend on x nor on t, we get at order ε the equation

$$g'(u^0_{\text{out}}(x,t))u^1_{\text{out}}(x,t) = 0$$

which implies $u^1_{\text{out}}(x,t) = 0$ from our assumption $g'(u_\pm) < 0$. For the same reason, all higher orders vanish as well:

$$u^2_{\text{out}}(x,t) = u^3_{\text{out}}(x,t) = \dots = 0.$$

Therefore, we arrive at the simple outer "expansion"

$$u_{\text{out}}(x,t) = \begin{cases} u_- & \text{for } x \in \Omega^-(t), \\ u_+ & \text{for } x \in \Omega^+(t). \end{cases}$$

We will assume that $\Omega^-(t)$ and $\Omega^+(t)$ are separated by a smooth hypersurface $\Gamma(t)$. Close to this interface we expect to have a sharp layer of width $\mathcal{O}(\varepsilon)$. To resolve this layer we will use a "stretched" variable near $\Gamma(t)$.

2.2 Interface coordinates

To derive the inner expansion, we will use coordinates adapted to the moving interface. We want to describe $\Gamma(t)$ as the image set of some mapping γ_0. To this end, we set

$$\Gamma(t) := \{x \in \mathbb{R}^N \mid x = \gamma_0(y,t), y \in \Gamma_0\}$$

where Γ_0 is an $(N-1)$-dimensional smooth reference manifold. We assume that the parametrization γ_0 is smooth and chosen in such a way that

$$\frac{\partial \gamma_0}{\partial t} \perp \frac{\partial \gamma_0}{\partial y_i} \quad \text{for } i = 1, \ldots, N-1. \tag{12}$$

We can now parameterize a neighborhood of the surface $\Gamma(t)$ by coordinates (r, y) such that

$$x = \gamma_0(y,t) + r\nu(y,t) =: \gamma(y,t,r)$$

where $\nu(y,t)$ is a unit normal vector to $\Gamma(t)$ at y, or more precisely, at $\gamma_0(y,t)$. For definiteness, we assume throughout that ν points into $\Omega^+(t)$. We also define an r-shifted interface $\Gamma(t,r)$ by

$$\Gamma(t,r) := \{x = \gamma(y,t,r) \mid y \in \Gamma_0\}.$$

Consequences of this definition are the identities

$$\nu \cdot \nu = 1, \quad \nu \cdot \frac{\partial \nu}{\partial y_i} = 0 \;\text{ for } i = 1, \ldots, N-1, \quad \nu \cdot \frac{\partial \nu}{\partial t} = 0,$$

where "$\cdot$" stands for the Euclidean inner product.

We now introduce a space-time coordinate transformation:

$$x = \gamma(y, \bar{t}, r), \quad t = \bar{t}.$$

It is straightforward, but tedious, to write equation (1) in the new coordinates. The Jacobi matrix for the coordinate transformation is

$$J = \left(\; \nu(y,t) \;\middle|\; \tfrac{\partial \gamma_0}{\partial y}(y,t) + r \tfrac{\partial \nu}{\partial y}(y,t) \; \right).$$

This implies that

$$\begin{aligned}
\begin{pmatrix} \frac{\partial}{\partial r} \\[2ex] \frac{\partial}{\partial y} \end{pmatrix} &= \begin{pmatrix} \nu(y,t)^{\mathrm{T}} \\ \left(\frac{\partial \gamma_0}{\partial y} + r\frac{\partial \nu}{\partial y}\right)^{\mathrm{T}} \end{pmatrix} \frac{\partial}{\partial x} \\
&= \underbrace{\begin{pmatrix} \nu(y,t)^{\mathrm{T}} \\ \left(\frac{\partial \gamma_0}{\partial y} + r\frac{\partial \nu}{\partial y}\right)^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \nu(y,t) & \left(\frac{\partial \gamma_0}{\partial y} + r\frac{\partial \nu}{\partial y}\right) \end{pmatrix}}_{=\begin{pmatrix} 1 & 0 \\ 0 & g(y,r,t) \end{pmatrix}} \begin{pmatrix} \nu(y,t) & \left(\frac{\partial \gamma_0}{\partial y} + r\frac{\partial \nu}{\partial y}\right) \end{pmatrix}^{-1} \frac{\partial}{\partial x}
\end{aligned}$$

where the components of the symmetric metric tensor g are given by

$$g_{ij}(y,t,r)=\sum_{k=1}^{N}\frac{\partial\gamma^k}{\partial y^i}\frac{\partial\gamma^k}{\partial y^j}=\sum_{k=1}^{N}\left(\frac{\partial(\gamma_0)^k}{\partial y_i}+r\frac{\partial\nu^k}{\partial y^i}\right)\left(\frac{\partial(\gamma_0)^k}{\partial y^j}+r\frac{\partial\nu^k}{\partial y^j}\right)$$

which is the symmetric metric tensor on the r-shifted interface $\Gamma(t,r)$. Consequently, we have

$$\frac{\partial}{\partial x}=\left(\begin{array}{cc}\nu(y,t) & \left(\frac{\partial\gamma_0}{\partial y}+r\frac{\partial\nu}{\partial y}\right)\end{array}\right)\left(\begin{array}{cc}1 & 0\\ 0 & g^{-1}(y,r,t)\end{array}\right)\left(\begin{array}{c}\frac{\partial}{\partial r}\\ \\ \frac{\partial}{\partial y}\end{array}\right)$$

or, equivalently,

$$\frac{\partial}{\partial x_i}=\nu^i\frac{\partial}{\partial r}+\sum_{j,k=1}^{N-1}\frac{\partial\gamma^i}{\partial y^j}g^{jk}\frac{\partial}{\partial y^k},$$

where the g^{jk} are the entries of g^{-1}, the inverse matrix of (g_{jk}). The gradient operator, therefore, transforms as follows:

$$\nabla_{(x)}=\nu\frac{\partial}{\partial r}+\sum_{j,k=1}^{N-1}\frac{\partial\gamma}{\partial y^j}g^{jk}\frac{\partial}{\partial y^k}.$$

The Laplace operator then becomes

$$\Delta=\frac{\partial^2}{\partial r^2}-H(y,t,r)\frac{\partial}{\partial r}+\Delta_\Gamma(t,r),$$

where $H(y,t,r)$ is the mean curvature of the r-shifted interface $\Gamma(t,r)$ and $\Delta_\Gamma(t,r)$ is the Laplace-Beltrami operator on $\Gamma(t,r)$ acting on functions of y.

Similarly, one can verify that the time derivative transforms as follows:

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\bar{t}}-(\gamma_0)_{\bar{t}}\cdot\nu)\frac{\partial}{\partial r}-(r\nu_{\bar{t}}\cdot(\gamma_0)_y+r^2\nu_{\bar{t}}\cdot\nu_y)g^{-1}\frac{\partial}{\partial y}$$

Dropping the "bar" from $\bar{t}$, equation (1) in the new coordinates becomes

$$\begin{aligned}\varepsilon\frac{\partial u}{\partial t}=&\varepsilon\left\{((\gamma_0)_t\cdot\nu)\frac{\partial u}{\partial r}+\left(r\nu_t\cdot(\gamma_0)_y+r^2\nu_t\cdot\nu_y\right)g^{-1}\frac{\partial u}{\partial y}\right\}\\&+\varepsilon^2\left\{\frac{\partial^2u}{\partial r^2}-H(y,t,r)\frac{\partial u}{\partial r}+\Delta_\Gamma(r,t)u\right\}\\&-\varepsilon\left\{(f'(u)\cdot\nu)\frac{\partial u}{\partial r}+f'(u)\frac{\partial\gamma}{\partial y}g^{-1}\frac{\partial u}{\partial y}\right\}+g(u).\end{aligned}\tag{1^ε}$$

2.3 The inner expansion

Setting $r = \varepsilon z$, equation (1^ε) becomes

$$\begin{aligned} 0 \quad = \quad & \frac{\partial^2 u}{\partial z^2} + [(\gamma_0)_t \cdot \nu - f'(u) \cdot \nu] \frac{\partial u}{\partial z} + g(u) \\ & + \varepsilon \left\{ -\frac{\partial u}{\partial t} - H^{(0)} \frac{\partial u}{\partial z} - f'(u) \cdot \nabla_{\Gamma(t)} u \right\} \\ & + \varepsilon^2 \left\{ \Delta^{\Gamma(t)} u - z H^{(1)} \frac{\partial u}{\partial z} + z \nu_t \cdot \nabla_{\Gamma(t)} u - z \left(f'(u) \cdot \nabla^{(1)}_{\Gamma(t)} u \right) \right\} \\ & + \sum_{j \geq 3} \varepsilon^j \mathcal{P}_j(y, t, z) u. \end{aligned} \tag{13}$$

Here, with $g^{jk} = g^{jk}(y, t, 0)$, tangential gradients are defined by

$$\nabla_{\Gamma(t)} = \frac{\partial \gamma_0}{\partial y} g^{-1}(y, t, 0) \frac{\partial}{\partial y} = \sum_{j,k=1}^{N-1} \frac{\partial \gamma_0}{\partial y^j} g^{jk} \frac{\partial}{\partial y^k}, \tag{14}$$

$$\nabla^{(1)}_{\Gamma(t)} = \frac{\partial}{\partial r} \left[\frac{\partial \gamma}{\partial y}(y, t, r) g^{-1}(y, t, r) \right]_{r=0} \frac{\partial}{\partial y} = \sum_{j,s,l,k=1}^{N-1} \frac{\partial \gamma_0}{\partial y^j} g^{js} h_{sl} g^{lk} \frac{\partial}{\partial y^k}, \tag{15}$$

where (h_{sl}) is the second fundamental form of $\Gamma(t)$ with respect to ν. Moreover, the curvature terms are defined by $H^{(0)} = H(y, t, 0)$ and

$$H^{(1)} = \frac{\partial}{\partial r}|_{r=0} H(y, t, r) = \sum_{j=1}^{N-1} (\kappa_j)^2, \quad \text{(sum of squared principal curvatures)},$$

while $\Delta^{\Gamma(t)} = \Delta_\Gamma(t, 0)$ is the Laplace-Beltrami operator on $\Gamma(t)$ and $\mathcal{P}_j$ are differential operators acting on functions of (y, t, z) which we need not use explicitly.

We seek a solution of the form

$$u_{\text{in}}(z, y, t) = u^0_{\text{in}}(z, y, t) + \varepsilon u^1_{\text{in}}(z, y, t) + \dots .$$

Substituting this expression into (13), we will determine u^j_{in} ($j \geq 0$) with appropriate conditions.

At order zero we have to satisfy the equation

$$\frac{\partial^2 u^0_{\text{in}}}{\partial z^2} + \left((\gamma_0)_t \cdot \nu - f'(u^0_{\text{in}}) \cdot \nu \right) \frac{\partial u^0_{\text{in}}}{\partial z} + g(u^0_{\text{in}}) = 0 \tag{16}$$

with boundary condition

$$u^0_{\text{in}}(-\infty) = u_-, \qquad u^0_{\text{in}}(+\infty) = u_+.$$

¿From Proposition 1.1 we know that for any direction ν there is a unique wave speed $s = s(\nu)$ such that (16) possesses a monotone heteroclinic orbit from u_- to u_+, if and only if $(\gamma_0)_t \cdot \nu = s(\nu)$. Therefore, the lowest order interface equation is given by

$$V := \gamma_t^0 \cdot \nu = s(\nu). \tag{17}$$

We denote this heteroclinic orbit by $Q(z;\nu)$ and assume for definiteness that the phase shift in z is adjusted such that $Q(0;\nu) = 0$.

We will assume that the first term of the inner solution is just the heteroclinic wave profile in the corresponding normal direction with a "phase shift" $a_0(y,t)$:

$$u_{\text{in}}^0(z,y,t) = Q(z + a_0(y,t); \nu(y,t)).$$

The term $a_0(y,t)$ accounts for the fact that we expect the interface $\Gamma(t)$ to be determined only up to order $\mathcal{O}(\varepsilon)$ corresponding to the width of the transition layer. At this stage, the shift $a_0(y,t)$ is not known. We will derive the equation that governs a_0 in the next stage of approximation.

At order $\mathcal{O}(\varepsilon)$ we have to satisfy the equation

$$\frac{\partial u_{\text{in}}^0}{\partial t} = \frac{\partial^2 u_{\text{in}}^1}{\partial z^2} + \alpha(z)\frac{\partial u_{\text{in}}^1}{\partial z} + \beta(z) u_{\text{in}}^1 - H(y,t,0)\frac{\partial u_{\text{in}}^0}{\partial z} - f'(u_{\text{in}}^0)\cdot \nabla_{\Gamma(t)} u_{\text{in}}^0 \tag{18}$$

together with the boundary condition

$$u_{\text{in}}^1(-\infty) = u_{\text{in}}^1(+\infty) = 0,$$

where

$$\begin{aligned} \alpha(z) &= s(\nu) - f'(u_{\text{in}}^0)\cdot \nu, \\ \beta(z) &= g'(u_{\text{in}}^0) - f''(u_{\text{in}}^0)\cdot \nu \frac{\partial u_{\text{in}}^0}{\partial z}. \end{aligned}$$

Note that this equation has to be considered as an equation for the unknown u_{in}^1 as a function of z with y and t playing the role of parameters.

We will now state a condition which guarantees the solvability of this inhomogeneous linear second order equation. From this condition we will get an equation for the evolution of a_0.

Rewrite (18) as an inhomogeneous linear second order equation

$$\phi_{zz} + \alpha(z)\phi_z + \beta(z)\phi = h(z) \tag{19}$$

where

$$h(z) = \frac{\partial u_{\text{in}}^0}{\partial t} + H^{(0)}\frac{\partial u_{\text{in}}^0}{\partial z} + f'(u_{\text{in}}^0)\cdot \nabla_{\Gamma(t)} u_{\text{in}}^0.$$

The solvability for (19) is summarized as follows.

Proposition 2.1. *For a given bounded function $h(z)$, the problem (19) has a bounded solution if and only if the solvability condition*

$$\int_{-\infty}^{\infty} Q_z(z+a_0)e^{A(z)}h(z)\,\mathrm{d}z = 0 \tag{20}$$

is satisfied, where

$$\begin{aligned} A(z) := \int_{-a_0}^{z} \alpha(\tau)\,\mathrm{d}\tau &= \int_{-a_0}^{z} (s(\nu) - f'(Q(\tau+a_0;\nu))\cdot\nu)\,\mathrm{d}\tau \\ &= \int_{0}^{z+a_0} (s(\nu) - f'(Q(\tau;\nu))\cdot\nu)\,\mathrm{d}\tau =: \tilde{A}(z+a_0). \end{aligned}$$

Moreover, if $h(z)$ decays to zero at an exponential rate, then the corresponding solution of (19) does have the same property with the following explicit formula.

$$\phi(z) = a(y,t)Q_z(z+a_0) + Q_z(z+a_0)\int_{0}^{z+a_0} \frac{\mathrm{e}^{-\tilde{A}(z')}}{Q_z(z')^2}\int_{-\infty}^{z'} Q_z(z'')\mathrm{e}^{\tilde{A}(z'')}h(z''-a_0)\,\mathrm{d}z''\mathrm{d}z',$$

where $a(y,t)$ is any function of the "parameters" (y,t) and the second term is normalized so that it takes the value 0 for $z=-a_0$.

The proof of this proposition is elementary, and hence omitted.

To translate the condition (20) stated in Proposition 2.1 into information about the motion of the transition layer, we use the relations

$$\begin{aligned} \frac{\partial u_{\mathrm{in}}^0}{\partial t} &= Q_z(z+a_0;\nu)\frac{\partial a_0}{\partial t} + Q_\nu(z+a_0;\nu)\nu_t \\ \frac{\partial u_{\mathrm{in}}^0}{\partial z} &= Q_z(z+a_0;\nu) \\ \nabla_{\Gamma(t)}u_{\mathrm{in}}^0 &= Q_z(z+a_0;\nu)\nabla_{\Gamma(t)}a_0 + Q_\nu(z+a_0;\nu)\nabla_{\Gamma(t)}\nu \end{aligned}$$

which follow immediately from $u_{\mathrm{in}}^0(z,y,t) = Q(z+a_0(y,t);\nu(y,t))$. They in turn imply that the inhomogeneous term for (18) is given by

$$h(z) = Q_z\left[\frac{\partial a_0}{\partial t} + H^{(0)} + f'(Q)\cdot\nabla_{\Gamma(t)}a_0\right] + Q_\nu\cdot\nu_t + f'(Q)\cdot Q_\nu\nabla_{\Gamma(t)}\nu$$

where, here and below, Q is evaluated at $z+a_0(y,t)$ unless explicitly stated otherwise.

Inserting this into the first order solvability condition, we obtain the first order interface equation

$$\frac{\partial a_0}{\partial t} = -\frac{\partial s}{\partial \nu}\nabla_{\Gamma(t)} a_0 + \hat{h}_0(y,t), \tag{21}$$

where

$$\begin{aligned} \frac{\partial s}{\partial \nu} &= M_0^{-1}\int_{-\infty}^{\infty} Q_z e^{A(z)}(f(Q))_z \,\mathrm{d}z, \quad M_0 := \int_{-\infty}^{\infty} Q_z^2 e^{A(z)}\,\mathrm{d}z, \\ \hat{h}_0(y,t) &= -H(y,t,0) - M_0^{-1}\int_{-\infty}^{\infty} Q_z e^{A(z)}\left\{Q_\nu \cdot \nu_t + f'(Q)\cdot Q_\nu \nabla_{\Gamma(t)}\nu\right\}\,\mathrm{d}z. \end{aligned}$$

We remark that neither M_0 nor the last integral depends on a_0, since the integration variable z is shifted by a_0, such as

$$M_0 := \int_{-\infty}^{\infty} Q_z^2 e^{A(z)}\,\mathrm{d}z = \int_{-\infty}^{\infty} \tilde{Q}_z^2 e^{\tilde{A}(z)}\,\mathrm{d}z,$$

where $\tilde{Q} = Q(z-a_0)$, $\tilde{A}(z) = A(z-a_0)$ which are independent of a_0.

CHECK !!! $\tilde{Q} = Q(z-a_0)$ or $\tilde{Q} = Q(z)$?

Therefore, $\hat{h}_0(y,t)$ does not depend on a_0. Notice that (21) is an inhomogeneous version of the linearization of the lowest order interface equation (17).

When the solvability condition is satisfied for (18), it has the family of bounded solutions

$$u_{\mathrm{in}}^1(y,t,z) = a_1(y,t)Q_z + \overline{u}_{\mathrm{in}}^1(y,t,z), \tag{22}$$

where $\overline{u}_{\mathrm{in}}^1$ is a bounded solution of (18) with $\overline{u}_{\mathrm{in}}^1(y,t,0) = 0$. Since $h_1(y,t,z)$ decays exponentially to zero as $z \to \pm\infty$, so does u_{in}^1. The coefficient function a_1 is to be determined so that the next order equation be solvable.

Higher order approximations are obtained in a similar way. At order $\mathcal{O}(\varepsilon^2)$ the equation is of the same form as (19):

$$\frac{\partial^2 u_{\mathrm{in}}^2}{\partial z^2} + \alpha(z)\frac{\partial u_{\mathrm{in}}^2}{\partial z} + \beta(z)u_{\mathrm{in}}^2 = h_2(y,t,z). \tag{23}$$

Applying Proposition 2.1, we find that (23) has a bounded solution u_{in}^2, if and only if

$$\int_{-\infty}^{+\infty} Q_z e^{A(z)} h_2(y,t,z)\,\mathrm{d}z = 0 \tag{24}$$

where

$$\begin{aligned} h_2(y,t,z) &= (f''(u_{\rm in}^0)\cdot\nu)u_{\rm in}^1\frac{\partial u_{\rm in}^1}{\partial z} + \frac{1}{2}(f'''(u_{\rm in}^0)\cdot\nu)\left(u_{\rm in}^1\right)^2\frac{\partial u_{\rm in}^0}{\partial z} - \frac{1}{2}g''(u_{\rm in}^0)\left(u_{\rm in}^1\right)^2 \\ &\quad + \frac{\partial u_{\rm in}^1}{\partial t} + H^{(0)}\frac{\partial u_{\rm in}^1}{\partial z} + f'(u_{\rm in}^0)\cdot\nabla_{\Gamma(t)}u_{\rm in}^1 + u_{\rm in}^1 f''(u_{\rm in}^0)\cdot\nabla_{\Gamma(t)}u_{\rm in}^0 \\ &\quad - \Delta^{\Gamma(t)}u_{\rm in}^0 + zH^{(1)}\frac{\partial u_{\rm in}^0}{\partial z} - z\nu_t\cdot\nabla_{\Gamma(t)}u_{\rm in}^0 + zf'(u_{\rm in}^0)\cdot\nabla_{\Gamma(t)}^{(1)}u_{\rm in}^0. \end{aligned}$$

Since $h_2(y,t)$ decays to zero at an exponential rate as $z\to\pm\infty$, so does the solution $u_{\rm in}^2$. Using (22) for $u_{\rm in}^1$ in this expression of h_2, we have, with $u_{\rm in}^0(y,t,z) = Q(z+a_0(y,t))$ being utilized,

$$h_2(y,t,z) = \frac{1}{2}I_2^2(z)(a_1)^2 + I_2^1(z)a_1 + Q_z\frac{\partial a_1}{\partial t} + Q_z f'(Q)\cdot\nabla_{\Gamma(t)}a_1 + I_2^0(z), \tag{25}$$

where

$$I_2^2(z) = \left\{Q_z^2\left(f''(Q)\cdot\nu\right)\right\}_z - \left\{g'(Q)\right\}_z Q_z, \tag{26}$$

$$\begin{aligned} I_2^1(z) &= (f''(Q)\cdot\nu)\left\{Q_z\overline{u}_{\rm in}^1\right\}_z + \left\{f''(Q)\cdot\nu\right\}_z(Q_z\overline{u}_{\rm in}^1) - \left\{g'(Q)\right\}_z\overline{u}_{\rm in}^1 \quad (:= I_2^{1,1}) \\ &\quad + \{Q_z\}_t + H^{(0)}Q_{zz} + f'(Q)\cdot\nabla_{\Gamma(t)}Q_z + Q_z f''(Q)\cdot\nabla_{\Gamma(t)}Q \quad (:= I_2^{1,0}), \end{aligned} \tag{27}$$

and $I_0(z)$ represents terms independent of a_1. Substituting (25) into (24), we find that the solvability condition is reduced to

$$\frac{\partial a_1}{\partial t} = -\frac{\partial s}{\partial \nu}\nabla_{\Gamma(t)}a_1 + \int_{-\infty}^{\infty} Q_z e^{A(z)}I_2^0(z)\ \mathrm{d}z. \tag{28}$$

In fact, integrating by parts and using the fact that $P(z) := Q_z e^{A(z)}$ satisfies the adjoint equation of the homogeneous version of (19);

$$P_{zz} - \alpha(z)P_z + g'(Q)P = 0,$$

one can verify $\int I_2^2(z)P(z)\ \mathrm{d}z = 0$. To this end we calculate

$$\begin{aligned} \int_{-\infty}^{\infty} I_2^2(z)P(z)\ \mathrm{d}z &= \int_{-\infty}^{\infty}(Q_z^2(f''(Q)\cdot\nu))_z P\ \mathrm{d}z - \int_{-\infty}^{\infty}(g'(Q))_z Q_z P\ \mathrm{d}z \\ &= \int_{-\infty}^{\infty} -Q_z^2 f''(Q)\cdot\nu P_z + g'(Q)Q_zP_z + g'(Q)Q_{zz}P\ \mathrm{d}z \\ &= \int_{-\infty}^{\infty}\beta(z)Q_zP_z + g'(Q)Q_{zz}P\ \mathrm{d}z \\ &= \int_{-\infty}^{\infty} -Q_{zzz}P_z - \alpha(z)Q_{zz}P_z + g'(Q)Q_{zz}P\ \mathrm{d}z \\ &= \int_{-\infty}^{\infty} Q_{zz}(\underbrace{P_{zz} - \alpha P_z + g'(Q)P}_{=0})\ \mathrm{d}z = 0. \end{aligned}$$

Integrating by parts, using the equation for P and the equation (18) for $\overline{u}^1_{\rm in}$, one can also find that $\int I^1_2(z)P(z)\,\mathrm{d}z = 0$. To prove this claim we evaluate

$$
\begin{aligned}
\int_{-\infty}^{\infty} I_2^{1,1} P\,\mathrm{d}z &= \int_{-\infty}^{\infty} (f''(Q)\cdot\nu)\left\{Q_z\overline{u}^1_{\rm in}\right\}_z P + \{f''(Q)\cdot\nu\}_z\,(Q_z\overline{u}^1_{\rm in})P - \{g'(Q)\}_z\,\overline{u}^1_{\rm in}P\,\mathrm{d}z\\
&= \int_{-\infty}^{\infty} -(f''(Q)\cdot\nu)\,Q_z\overline{u}^1_{\rm in}P_z + g'(Q)\overline{u}^1_{{\rm in},z}P + g'(Q)\overline{u}^1_{\rm in}P_z\,\mathrm{d}z\\
&= \int_{-\infty}^{\infty} \beta\overline{u}^1_{\rm in}P_z + (\alpha P_z - P_{zz})\overline{u}^1_{{\rm in},z}\,\mathrm{d}z\\
&= \int_{-\infty}^{\infty} \beta\overline{u}^1_{\rm in}P_z + \alpha\overline{u}^1_{{\rm in},z}P_z + \overline{u}^1_{{\rm in},zz}P_z\,\mathrm{d}z\\
&= \int_{-\infty}^{\infty} \left(\frac{\partial u^0_{\rm in}}{\partial t} + H^{(0)}\frac{\partial u^0_{\rm in}}{\partial z} + f'(u^0_{\rm in})\cdot\nabla_{\Gamma(t)}u^0_{\rm in}\right) P_z\,\mathrm{d}z\\
&= -\int_{-\infty}^{\infty} \left(\frac{\partial u^0_{\rm in}}{\partial t} + H^{(0)}\frac{\partial u^0_{\rm in}}{\partial z} + f'(u^0_{\rm in})\cdot\nabla_{\Gamma(t)}u^0_{\rm in}\right)_z P\,\mathrm{d}z\\
&= -\int_{-\infty}^{\infty} I_2^{1,0}P\,\mathrm{d}z.
\end{aligned}
$$

Therefore, (28) follows from (24).

At order $\mathcal{O}(\varepsilon^k)$ ($k\in\mathbb{N},\ k\geq 3$), the equation is again of the same form as (19):

$$
\frac{\partial^2 u^k_{\rm in}}{\partial z^2} + \alpha(z)\frac{\partial u^k_{\rm in}}{\partial z} + \beta(z)u^k_{\rm in} = h_k(z). \tag{29}
$$

Here, h_k depends only on $u^0_{\rm in}, u^1_{\rm in}, \ldots, u^{k-1}_{\rm in}$ and is given by

$$
h_k = I^1_k(z)a_{k-1} + Q_z\frac{\partial a_1}{\partial t} + Q_z f'(Q)\cdot\nabla_{\Gamma(t)}a_{k-1} + I^0_k(z),
$$

where $a_{k-1}(y,t)$ is the coefficient of Q_z in the $(k-1)$-th approximation

$$
u^{k-1}_{\rm in}(y,t,z) = a_{k-1}Q_z + \overline{u}^{k-1}_{\rm in} \tag{30}
$$

$$
\begin{aligned}
I^1_k =&(f''(Q)\cdot\nu)\left\{Q_z u^1_{\rm in}\right\}_z + \{f''(Q)\cdot\nu\}_z\,(Q_z u^1_{\rm in}) - \{g'(Q)\}_z\,u^1_{\rm in} \quad (:= I^{1,1}_k)\\
&+ \{Q_z\}_t + H^{(0)}Q_{zz} + f'(Q)\cdot\nabla_{\Gamma(t)}Q_z + Q_z f''(Q)\cdot\nabla_{\Gamma(t)}Q \quad (:= I^{1,0}_k),
\end{aligned}
$$

and I^0_k depends only on $a_0,\ldots,a_{k-2}$. We notice that no term depending on $(a_{k-1})^2$ is present, so the analysis for $k\geq 3$ is slightly simpler than it is for $k=2$. Moreover, $I^{1,1}_k$ is the same as $I^{1,1}_2$ with $\overline{u}^1_{\rm in}$ being replaced by $u^1_{\rm in}$ and $I^{1,0}_k = I^{1,0}_2$. Therefore, by the same computation as above, $\int I^1_k P\,\mathrm{d}z = 0$ follows. Applying the solvability condition to (29), we obtain the k-th order interface equation

$$
\frac{\partial a_{k-1}}{\partial t} = -\frac{\partial s}{\partial \nu}\nabla_{\Gamma(t)}a_{k-1} + \hat{h}_{k-1}(y,t). \tag{31}
$$

where $\hat{h}_{k-1}$ represents the terms that depend only on $a_0, \dots, a_{k-2}$, but not on a_{k-1}.

For an integer $m \geq 1$, we now define the m-th order inner approximation $u_{\mathrm{in}}^{\varepsilon,m}$ by

$$u_{\mathrm{in}}^{\varepsilon,m}(y,t,z) = \sum_{k=0}^{m} \varepsilon^k u_{\mathrm{in}}^k(y,t,z) \tag{32}$$

where we set $a_m(y,t) \equiv 0$ in the expression of the last term $u_{\mathrm{in}}^m = a_m Q_z + \overline{u}_{\mathrm{in}}^m$.

3 Proof of main theorems

3.1 Proof of Theorem 1.1

When the initial interface is smooth, the initial value problem for (17)

$$\gamma_t^0 \cdot \nu = s(\nu), \qquad \gamma_0(y,0) = \overline{\gamma}^0(y)$$

has a smooth solution on $t \in [0,T]$ for some $T = T(\overline{\gamma}^0) > 0$. We then solve the initial value problem for (21) with $a_0(y,0) \equiv 0$ on the time interval $[0,T]$. Since this is a linear inhomogeneous first order equation with a non-characteristic initial surface, the problem has a unique solution on $[0,T]$, which we denote by $a_0(y,t)$. Similarly, the initial value problems for (28) and (31) with $a_1(y,0) \equiv 0$ and $a_{k-1}(y,0)$ have a unique solutions, denoted by $a_1(y,t)$ ane $a_{k-1}(y,t)$. In this way, we can determine all functions in (32) up to any order $m \in \mathbb{N}$.

There also exists a $\delta > 0$ so that the correspondence

$$\Gamma_0 \times (-2\delta, 2\delta) \ni (y,r) \mapsto x = \gamma_0(y,t) + r\nu(y,t)$$

smoothly parameterizes the 2δ-neighborhood of $\Gamma(t) = \{x = \gamma_0(y,t) \mid y \in \Gamma_0\} \subset \mathbb{R}^N$ for $t \in [0,T]$. For x in the 2δ-neighborhood of $\Gamma(t)$, the inverse of the parametrization

$$x \mapsto (y,r)$$

is expressed by

$$y = \hat{y}(x,t), \qquad r = \hat{r}(x,t).$$

We denote by $\Omega^-(t)$ (resp. $\Omega^+(t)$) the component of $\mathbb{R}^N \backslash \Gamma(t)$ where $r < 0$ (resp. $r > 0$). For $d > 0$, we define

$$\Omega_d^{\pm}(t) = \{x \in \Omega^{\pm}(t) \mid \mathrm{dist}(x, \Gamma(t)) > d\}.$$

Let us denote by $\Theta^-(r), \Theta^0(r)$ and $\Theta^+(r)$ smooth cut-off functions satisfying

$$0 \leq \Theta^-(r), \Theta^0(r), \Theta^+(r) \leq 1$$

and

$$\begin{array}{llll} \Theta^-(r) \equiv 1 & (r \le -2\delta), & \Theta^-(r) \equiv 0 & (r \ge -\delta) \\ \Theta^0(r) \equiv 1 & (|r| \le \delta), & \Theta^0(r) \equiv 0 & (|r| \ge 2\delta) \\ \Theta^+(r) \equiv 1 & (r \ge 2\delta), & \Theta^+(r) \equiv 0 & (r \le \delta) \end{array} \tag{33}$$

We now fix an integer $m \ge 1$ and define m-th order approximation $u^{\varepsilon,m}(x,t)$ by

$$\begin{cases} u^{\varepsilon,m}(x,t) = u_{\text{out}}(x,t) \quad \text{for } x \in \Omega^{\pm}_{2\delta}(t), \\ \\ u^{\varepsilon,m}(x,t) = u_{\text{out}}(x,t)\Theta^-(\hat{r}) + u_{\text{out}}(x,t)\Theta^+(\hat{r}) \\ \qquad\qquad +\Theta^0(\hat{r})u^{\varepsilon,m}_{\text{in}}(\hat{y},t,\frac{\hat{r}}{\varepsilon}) \quad \text{for } x \in \mathbb{R}^N \backslash (\Omega^-_{2\delta}(t) \cup \Omega^+_{2\delta}(t)), \end{cases} \tag{34}$$

for $t \in [0,T]$, where $\hat{r} = \hat{r}(x,t)$, $\hat{y} = \hat{y}(x,t)$.

¿From our construction of the outer and inner expansions in §2, we obtain the following.

Proposition 3.1. *Let $m \ge 1$ be an integer. Then, $u^{\varepsilon,m}(x,t)$ defined in (34) satisfies*

$$\|\varepsilon u^{\varepsilon,m}_t + \varepsilon f'(u^{\varepsilon,m}) \cdot \nabla u^{\varepsilon,m} - \varepsilon^2 \Delta u^{\varepsilon,m} - g(u^{\varepsilon,m})\|_{L^\infty(\mathbb{R}^N \times [0,T])} = \mathcal{O}(\varepsilon^{m+1})$$

as $\varepsilon \to 0$.

To prove Theorem 1.1, we use the comparison principle. We consider the auxiliary problems

$$u_t + f'(u) \cdot \nabla u = \varepsilon \Delta u + \varepsilon^{-1} \{g(u) \pm \varepsilon^m\} \tag{35}$$

for a fixed integer $m \ge 1$. We apply to the problem (35) the same procedure as in §2.1 to obtain *sub-* and *super-* outer solutions $u = \underline{u}^\varepsilon_{\text{out}}(x)$ and $\overline{u}^\varepsilon_{\text{out}}(x)$ defined by

$$\underline{u}^\varepsilon_{\text{out}}(x) = \begin{cases} u_- + g'(u_-)\varepsilon^m, & \text{for } x \in \Omega^-(t), \\ u_+ + g'(u_+)\varepsilon^m, & \text{for } x \in \Omega^+(t), \end{cases}$$

$$\overline{u}^\varepsilon_{\text{out}}(x) = \begin{cases} u_- - g'(u_-)\varepsilon^m & \text{for } x \in \Omega^-(t), \\ u_+ - g'(u_+)\varepsilon^m, & \text{for } x \in \Omega^+(t). \end{cases}$$

Note that $\underline{u}^\varepsilon_{\text{out}}(x) < \overline{u}^\varepsilon_{\text{out}}(x)$ for $x \in \Omega$ because of $g'(u_\pm) < 0$.

Applying the procedure of §2.3 to (35), we obtain m-th order *sub-* and *super-* inner solutions $\underline{u}^m_{\text{in}}(y,t,z)$ and $\overline{u}^m_{\text{in}}(y,t,z)$. These sub- and super- inner solutions satisfy

$$\frac{\partial^2 \underline{u}^m_{\text{in}}}{\partial z^2} + \alpha(z)\frac{\partial \underline{u}^m_{\text{in}}}{\partial z} + \beta(z)\underline{u}^m_{\text{in}} = h_m(z) - 1$$

and

$$\frac{\partial^2 \overline{u}^m_{\text{in}}}{\partial z^2} + \alpha(z)\frac{\partial \overline{u}^m_{\text{in}}}{\partial z} + \beta(z)\overline{u}^m_{\text{in}} = h_m(z) + 1,$$

respectively. Since the difference between (1) and (35) appears only in the $\mathcal{O}(\varepsilon^m)$-term, inner solutions u_{in}^k with lower indices ($0 \le k < m$) remain the same for both (1) and (35). By using the variation of constants formula presented in Proposition 2.1, it is easy to verify that

$$\underline{u}_{\text{in}}^m(y,t,z) \le u_{\text{in}}^m(y,t,z) \le \overline{u}_{\text{in}}^m(y,t,z).$$

Then m-th order sub and super inner approximations $\underline{u}_{\text{in}}^{\varepsilon,m}$ and $\overline{u}_{\text{in}}^{\varepsilon,m}$ are defined by the same formula as (32) with u_{in}^m being replaced by $\underline{u}_{\text{in}}^m$ and $\overline{u}_{\text{in}}^m$. We also define $\underline{u}^{\varepsilon,m}(x,t)$ and $\overline{u}^{\varepsilon,m}(x,t)$ by the same formula as (34), except that u_{out} is replaced by $\underline{u}_{\text{out}}^\varepsilon$ and $\overline{u}_{\text{out}}^\varepsilon$, and $u_{\text{in}}^{\varepsilon,m}(y,t,z)$ is replace by $\underline{u}_{\text{in}}^{\varepsilon,m}(y,t,z)$ and $\overline{u}_{\text{in}}^{\varepsilon,m}(y,t,z)$, respectively. The order relations for the outer and inner solutions above now inherit to the approximations

$$\underline{u}^{\varepsilon,m}(x,t) \le u^{\varepsilon,m}(x,t) \le \overline{u}^{\varepsilon,m}(x,t) \quad x \in \mathbb{R}^N,\ t \in [0,T].$$

Moreover, by using Proposition 3.1, we have, for small $\varepsilon > 0$,

$$\begin{aligned}\varepsilon\underline{u}_t^{\varepsilon,m} &+ \varepsilon f'(\underline{u}^{\varepsilon,m}) \cdot \nabla\underline{u}^{\varepsilon,m} - \varepsilon^2\Delta\underline{u}^{\varepsilon,m} - g(\underline{u}^{\varepsilon,m}) \\ &= -\varepsilon^m + \mathcal{O}(\varepsilon^{m+1}) < 0, \quad x \in \mathbb{R}^N,\ t \in [0,T],\end{aligned}$$

and

$$\begin{aligned}\varepsilon\overline{u}_t^{\varepsilon,m} &+ \varepsilon f'(\overline{u}^{\varepsilon,m}) \cdot \nabla\overline{u}^{\varepsilon,m} - \varepsilon^2\Delta\overline{u}^{\varepsilon,m} - g(\overline{u}^{\varepsilon,m}) \\ &= +\varepsilon^m + \mathcal{O}(\varepsilon^{m+1}) > 0, \quad x \in \mathbb{R}^N,\ t \in [0,T].\end{aligned}$$

Therefore, $\underline{u}^{\varepsilon,m}$ (resp. $\overline{u}^{\varepsilon,m}$) is a subsolution (resp. supersolution) of (1). The comparison principle yields that (1) has a solution $u^\varepsilon(x,t)$ satisfying

$$\underline{u}^{\varepsilon,m}(x,t) \le u^\varepsilon(x,t) \le \overline{u}^{\varepsilon,m}(x,t) \quad x \in \mathbb{R}^N,\ t \in [0,T].$$

Since $\|\overline{u}^{\varepsilon,m} - \underline{u}^{\varepsilon,m}\|_{L^\infty} = \mathcal{O}(\varepsilon^m)$, we conclude that

$$|u^\varepsilon(x,t) - u^{\varepsilon,m}(x,t)| = \mathcal{O}(\varepsilon^m) \qquad x \in \mathbb{R}^N,\ t \in [0,T]. \tag{36}$$

By our construction of $u^{\varepsilon,m}$ in §2, we have

$$\lim_{\varepsilon\to0} u^{\varepsilon,m}(x,t) = \begin{cases} u_- & \text{for } x \in \Omega^-(t) \\ u_+ & \text{for } x \in \Omega^+(t) \end{cases}, \quad t \in [0,T].$$

This, together with (36), completes the proof of convergence part.

We now prove the validity of (9) in Theorem 1.1. Notice that the wave profile at the lowest order approximation u_{in}^0 was shifted as $Q(z + a_0)$ by the amount $-a_0$ in the stretched coordinate. Therefore, the $\mathcal{O}(\varepsilon)$-approximation to the interface $\gamma^\varepsilon(y,t)$ is given by

$$\gamma^\varepsilon(y,t) = \gamma_0(y,t) - \varepsilon a_0(y,t)\nu(y,t) + \mathcal{O}(\varepsilon^2).$$

Then the unit normal vector $\nu^\varepsilon(y,t)$ to the interface γ^ε is represented as

$$\nu^\varepsilon(y,t) = \nu(y,t) + \varepsilon\nabla_{\Gamma(t)}a_0 + \mathcal{O}(\varepsilon^2).$$

Therefore, by using (21) and $s(\nu^\varepsilon) = s(\nu) + \varepsilon\partial s/\partial\nu\nabla_\Gamma a_0 + \mathcal{O}(\varepsilon^2)$, we have

$$\begin{aligned}
\gamma_t^\varepsilon\cdot\nu^\varepsilon =&(\gamma_0)_t\cdot\nu - \varepsilon\frac{\partial a_0}{\partial t} + \mathcal{O}(\varepsilon^2) \\
=&s(\nu) + \varepsilon\left\{\frac{\partial s}{\partial\nu}\nabla_{\Gamma(t)}a_0 - \hat{h}_0(y,t)\right\} + \mathcal{O}(\varepsilon^2) \\
=&s(\nu^\varepsilon) + \varepsilon H^\varepsilon(y,t) \\
&+ \varepsilon M_0^{-1}\int_{-\infty}^{\infty} Q_z \mathrm{e}^{A(z)}\left\{Q_\nu\cdot\nu_t + f'(Q)\cdot Q_\nu\nabla_{\Gamma(t)}\nu\right\}\,\mathrm{d}z + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{37}$$

where $H^\varepsilon(y,t)$ is the mean curvature of the interface represented by γ^ε. Let us now examine the integrand in the last term.

First of all, when γ_0 evolves according to (17), the t-derivative of ν is given by $\nu_t = -\nabla_{\Gamma(t)}s(\nu)$. For simplicity, we let s_p and Q_p denote, respectively, $\partial s/\partial\nu_p$ and $\partial Q/\partial\nu^p$ ($p = 1,\ldots,N$). According to the definition of the tangential gradient $\nabla_{\Gamma(t)}$ in (14), we have[1]

$$\nabla_{\Gamma(t)}s(\nu) = \frac{\partial\gamma_0}{\partial y^j}g^{jk}(y,t)\frac{\partial s(\nu)}{\partial y^k} = s_p\frac{\partial\gamma_0}{\partial y^j}g^{jk}(y,t)\frac{\partial\nu^p}{\partial y^k} = s_p\nabla_{\Gamma(t)}\nu^p.$$

Therefore, the q-th component of the tangential derivative is given by

$$\left[\nabla_{\Gamma(t)}s(\nu)\right]^q = s_p\left[\nabla_{\Gamma(t)}\nu^p\right]^q = s_p\frac{\partial(\gamma_0)^q}{\partial y^j}g^{jk}\frac{\partial\nu^p}{\partial y^k},$$

which gives rise to

$$Q_\nu\cdot\nu_t = Q_q\nu_t^q = -Q_q\left[\nabla_{\Gamma(t)}s(\nu)\right]^q = -Q_q s_p\frac{\partial(\gamma_0)^q}{\partial y^j}g^{jk}\frac{\partial\nu^p}{\partial y^k}.$$

Here and in the sequel, g_{jk} and g^{jk} are all evaluated at $(y,t,r=0)$. Similarly, we have

$$Q_\nu\nabla_{\Gamma(t)}\nu = Q_q\nabla_{\Gamma(t)}\nu^q = Q_q\frac{\partial\gamma_0}{\partial y^j}g^{jk}\frac{\partial\nu^q}{\partial y^k}$$

and hence

$$f'(Q)\cdot Q_\nu\nabla_{\Gamma(t)}\nu = f_p'(Q)\left[Q_q\nabla_{\Gamma(t)}\nu^q\right]^p = Q_q f_p'\frac{\partial(\gamma_0)^p}{\partial y^j}g^{jk}\frac{\partial\nu^q}{\partial y^k},$$

[1] In the sequel, summations over repeated indices are applied for $p,q = 1,\ldots,N$ and for $j,k,l,s = 1,\ldots,N-1$.

where f'_p means the p-th component of the vector f'. The definition of the second fundamental form $(h_{ks})_{k,s=1}^{N-1}$ of $\Gamma(t)$ yields

$$\frac{\partial \nu^p}{\partial y^k} = -h_{ks} g^{sl} \frac{\partial (\gamma_0)^p}{\partial y^l}, \tag{38}$$

which in turn shows the symmetry

$$\frac{\partial (\gamma_0)^p}{\partial y^j} g^{jk} \frac{\partial \nu^q}{\partial y^k} = \frac{\partial (\gamma_0)^q}{\partial y^j} g^{jk} \frac{\partial \nu^p}{\partial y^k}.$$

Using this symmetry and (38), we get

$$\begin{aligned} Q_\nu \nu_t + f'(Q) \cdot Q_\nu \nabla_{\Gamma(t)} \nu &= -\left\{ Q_q \left[\nabla_{\Gamma(t)} s(\nu) \right]^q - Q_p f'_q \left[\nabla_{\Gamma(t)} \nu^p \right]^q \right\} \\ &= Q_q (s_p - f'_p) \frac{\partial (\gamma_0)^p}{\partial y^j} \frac{\partial (\gamma_0)^q}{\partial y^l} g^{jk} h_{ks} g^{sl} \end{aligned} \tag{39}$$

On the other hand, Proposition 2.1 implies that Q_q has the following representation

$$Q_q(z) = -Q_z \int_{-a_0}^{z} \frac{1}{PQ_z} \left(\int_{-\infty}^{z'} PQ_z (s_q - f'_q) \, \mathrm{d}z'' \right) \, \mathrm{d}z'.$$

Therefore, (37) becomes

$$\gamma_t^\varepsilon \cdot \nu^\varepsilon = s(\nu^\varepsilon) + \varepsilon \left\{ H^\varepsilon + \mathbf{T}_{pq} K^{pq} \right\} + \mathcal{O}(\varepsilon^2), \tag{40}$$

where $\mathbf{T}_{pq}$ is given by

$$\mathbf{T}_{pq} = -M_0^{-1} \int_{-\infty}^{\infty} PQ_z (s_p - f'_p) \left\{ \int_{-a_0}^{z} \frac{1}{PQ_z} \left(\int_{-\infty}^{z'} PQ_z (s_q - f'_q) \, \mathrm{d}z'' \right) \, \mathrm{d}z' \right\} \, \mathrm{d}z \tag{41}$$

and K^{pq} is defined by

$$K^{pq} = \frac{\partial (\gamma_0)^p}{\partial y^j} \frac{\partial (\gamma_0)^q}{\partial y^l} g^{jk} h_{ks} g^{sl}, \tag{42}$$

which is symmetric. Note that we have evaluated $\mathbf{T}_{pq}$ and K^{pq} for $\Gamma(t)$. However, the difference between $\Gamma(t)$ and $\Gamma^\varepsilon(t)$ is of order $\mathcal{O}(\varepsilon)$. Therefore, the differences between $\mathbf{T}_{pq}^\varepsilon, K_\varepsilon^{pq}$ and $\mathbf{T}_{pq}, K^{pq}$ are contained in $\mathcal{O}(\varepsilon^2)$-terms in (9). This proves the validity of the interface equation (9) in Theorem 1.1. ■

By shifting the integration variables z'', z', z to $z'' + a_0, z' + a_0, z + a_0$, we see that $\mathbf{T}_{pq}$ do not depend on a_0.

Lemma 3.2. *The matrix* $\mathbf{T}$ *is independent of* a_0*, symmetric and positive semi-definite.*

Proof: We use the representation (41) and integrate by parts to obtain

$$\mathbf{T}_{pq} = M_0^{-1} \int_{-\infty}^{\infty} \frac{1}{PQ_z} L_p(z) L_q(z)\, \mathrm{d}z,$$

where

$$L_p(z) := \int_{-\infty}^{z} PQ_z (s_p - f_p')\, \mathrm{d}z'.$$

It is obvious that $\mathbf{T}_{pq} = \mathbf{T}_{qp}$. Moreover, we can write

$$\mathbf{T} = M_0^{-1} \int_{-\infty}^{\infty} \frac{1}{PQ_z} L(z) \otimes L(z)\, \mathrm{d}z, \tag{43}$$

where the vector-valued function L is defined by

$$L(z) = {}^t(L_1(z), \ldots, L_N(z)).$$

Evidently, (43) shows the positive semi-definiteness of $\mathbf{T}$.

∎

Lemma 3.3. *The sum* $\mathbf{T}_{pq} K^{pq}$ *is a weighted sum of principal curvatures;*

$$\mathbf{T}_{pq} K^{pq} = \sum_{i=1}^{N-1} w^i \kappa_i,$$

where κ_i *(*$i = 1, \ldots, N-1$*) are principal curvatures of* $\Gamma(t)$*.*

Proof: Let us first recall from (42) that the $N \times N$-matrix $K = (K^{pq})$ is expressed as

$$K = \frac{\partial \gamma_0}{\partial y} g^{-1} h g^{-1} \left(\frac{\partial \gamma_0}{\partial y} \right)^t, \tag{44}$$

where $g = (g_{ij})$ and $h = (h_{ij})$ are the first and second fundamental forms of $\Gamma(t)$. By definition, the principal curvatures κ_i ($i = 1, \ldots, N-1$) are roots of the characteristic polyonmial $\det(h - \kappa g)$. In other words, the principal curvatures are eigenvalues of the symmetric $(N-1) \times (N-1)$-matrix hg^{-1}. Let us diagonalize hg^{-1} at each $\gamma_0(y, t) \in \Gamma(t)$

by choosing an appropriate parametrization $y \mapsto \gamma_0(y,t)$. Then, from (42) and (44), we have

$$\mathbf{T}_{pq}K^{pq} = \sum_{i=1}^{N-1} w^i \kappa_i,$$

where

$$w^i = \sum_{p,q=1}^{N} \sum_{j=1}^{N-1} \mathbf{T}_{pq} \frac{\partial(\gamma_0)^p}{\partial y^j} \frac{\partial(\gamma_0)^q}{\partial y^i} g^{ji} \qquad \text{(no summation over } i \text{ is applied).}$$

■

3.2 Proof of Theorem 1.2.

The proof consists of the same steps as the proof of theorem 1.1, so we just indicate the differences.

The outer solution is the same as before in §2.1. However, when we compute the inner expansion, the first line of (1^ε) is multiplied by ε^2, instead of ε. Setting $r = \varepsilon z$ in this modified version of equation (1^ε), we obtain

$$\begin{aligned} 0 \;=\; & \frac{\partial^2 u}{\partial z^2} - (f'(u)\cdot\nu)\frac{\partial u}{\partial z} + g(u) \\ & +\varepsilon\left\{\left((\gamma_0)_t - H^{(0)}\right)\frac{\partial u}{\partial z} - f'(u)\cdot\nabla_{\Gamma(t)}u\right\} \\ & +\varepsilon^2\left\{-\frac{\partial u}{\partial t} + \Delta^{\Gamma(t)}u - zH^{(1)}\frac{\partial u}{\partial z} - z\left(f'(u)\cdot\nabla^{(1)}_{\Gamma(t)}u\right)\right\} + \sum_{j\geq 3}\varepsilon^j\hat{\mathcal{P}}_j(y,t,z)u, \end{aligned} \tag{45}$$

where $\hat{\mathcal{P}}_j$ are differential operators acting on functions of (y,t,z), which we need not use explicitly.

We seek a solution of the form

$$u_{\text{in}}(z,y,t) = u^0_{\text{in}}(z,y,t) + \varepsilon u^1_{\text{in}}(z,y,t) + \ldots.$$

Substituting this expression into (45), we will determine u^j_{in} ($j \geq 0$) successively, starting from $j=0$ to $j=m$ for any $m \in \mathbb{N}$.

At order zero, (45) gives rise to the equation

$$\frac{\partial^2 u^0_{\text{in}}}{\partial z^2} - f'(u^0_{\text{in}})\cdot\nu\frac{\partial u^0_{\text{in}}}{\partial z} + g(u^0_{\text{in}}) = 0$$

with boundary condition

$$u^0_{\text{in}}(-\infty) = u_-, \qquad u^0_{\text{in}}(+\infty) = u_+.$$

By our assumption $s(\nu) \equiv 0$ and Proposition 1.1 this problem always possesses a unique solution $u_{\text{in}}^0(z) = Q(z + a_0; \nu)$. As before we leave some freedom by introducing a shift $a_0(y,t)$ which will be determined later.

At order $\mathcal{O}(\varepsilon)$, (45) implies

$$\frac{\partial^2 u_{\text{in}}^1}{\partial z^2} + \alpha(z) \frac{\partial u_{\text{in}}^1}{\partial z} + \beta(z) u_{\text{in}}^1 = h_1(z), \tag{46}$$

where

$$h_1 = \left(-(\gamma_0)_t \cdot \nu + H(y,t)\right) Q_z + f'(Q) \cdot \nabla_{\Gamma(t)} Q.$$

Here and in the sequel, Q and Q_z, etc. always mean $Q(z + a_0; \nu)$ and $Q_z(z + a_0; \nu)$, etc. Applying to (46) the solvability theory (cf. Proposition 2.1), we obtain

$$\begin{aligned}(\gamma_0)_t \cdot \nu =& H(y,t) + M_0^{-1} \int_{-\infty}^{\infty} P f'(Q) \cdot \nabla_{\Gamma(t)} Q \, \mathrm{d}z \\ =& H(y,t) + M_0^{-1} \int_{-\infty}^{\infty} \tilde{P} f'(\tilde{Q}) \cdot \tilde{Q}_\nu \nabla_{\Gamma(t)} \nu \, \mathrm{d}z + M_0^{-1} \int_{-\infty}^{\infty} \tilde{Q}_z \tilde{P} f'(\tilde{Q}) \, \mathrm{d}z \cdot \nabla_{\Gamma(t)} a_0,\end{aligned}$$

where $\tilde{Q} = Q(z)$ and $\tilde{P} = P(z - a_0) = Q_z(z) \mathrm{e}^{A(z - a_0)}$ which are independent of a_0. Since $Q(z)$ and $f'(Q(z))$ are odd functions (cf. Lemma 1.2 (2) and f even), and hence $P(z - a_0)$ is even in z, the last integral vanishes; $\int_{-\infty}^{\infty} \tilde{Q}_z \tilde{P} f'(\tilde{Q}) \, \mathrm{d}z = 0$. Therefore, arguing as above, we find

$$(\gamma_0)_t \cdot \nu = H(y,t) + \mathbf{T}_{pq} K^{pq},$$

in which $\mathbf{T}_{pq}$ is the same as before, defined by

$$\mathbf{T}_{pq} = M_0^{-1} \int_{-\infty}^{\infty} \frac{1}{P Q_z} L_p(z) L_q(z) \, \mathrm{d}z \quad \text{with} \quad L_p(z) := -\int_{-\infty}^{z} P Q_z f_p' \, \mathrm{d}z',$$

where $s(\nu) \equiv 0$ is used and $f_p' = f_p'(Q)$, the p-th component of $f'(Q)$.

Notice that a_0 is still to be determined. In fact, an evolution equation for a_0 will be derived from the solvability condition for u_{in}^2 below. Once the solvability condition for (46) is fulfilled, it has a unique family of solutions

$$u_{\text{in}}^1 = a_1(y,t) Q_z + \overline{u}_{\text{in}}^1,$$

where a_1 is to be determined and $\overline{u}_{\text{in}}^1$, normalized as $\overline{u}_{\text{in}}^1(z = -a_0) = 0$, depends only on a_0 (and on $\nu(y,t)$). More precisely, we have

$$\overline{u}_{\text{in}}^1(z) = \overline{u}_{\text{even}}^1(z) + \overline{u}_{\text{odd}}^1(z) \tag{47}$$

$$\overline{u}^1_{\mathrm{even}}(z) = Q_z(z+a_0)\int_0^{z+a_0}\frac{1}{\tilde{P}(z')Q_z(z')}\left(\int_{-\infty}^{z'}\tilde{P}(z'')\tilde{h}_1^{\mathrm{even}}(z'')\,\mathrm{d}z''\right)\,\mathrm{d}z', \tag{48}$$

$$\overline{u}^1_{\mathrm{odd}}(z) = Q_z(z+a_0)\int_0^{z+a_0}\frac{1}{\tilde{P}(z')Q_z(z')}\left(\int_{-\infty}^{z'}\tilde{P}(z'')\tilde{h}_1^{\mathrm{odd}}(z'')\,\mathrm{d}z''\right)\,\mathrm{d}z', \tag{49}$$

where h_1^{even} and h_1^{odd} stand for the even and odd part of h_1 with $\tilde{h}_1^{\mathrm{even}}(z) = h_1^{\mathrm{even}}(z-a_0)$ and $\tilde{h}_1^{\mathrm{odd}}(z) = h_1^{\mathrm{odd}}(z-a_0)$. For reference, we give them explicitly.

$$h_1^{\mathrm{even}}(z) = (-\mathbf{T}_{pq}K^{pq})Q_z + Q_p f'(Q)\cdot\nabla_{\Gamma(t)}\nu^p, \quad h_1^{\mathrm{odd}}(z) = Q_z f'(Q)\cdot\nabla_{\Gamma(t)}a_0.$$

Recall again that $Q(z)$ is an odd function. This yields that $\tilde{h}_1^{\mathrm{even}}(z)$ and $\tilde{h}_1^{\mathrm{odd}}(z)$ are even and odd functions, respectively, and satisfy

$$\int_{-\infty}^{\infty} Ph_1^{\mathrm{even}}\,\mathrm{d}z = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} Ph_1^{\mathrm{odd}}\,\mathrm{d}z = 0.$$

Therefore, we find that $\overline{u}^1_{\mathrm{even}}(z-a_0)$ and $\overline{u}^1_{\mathrm{odd}}(z-a_0)$ defined in (48) and (49) are, respectively, even and odd functions. Moreover, we emphasize the following

$$\overline{u}^1_{\mathrm{odd}}(z) = Q_\nu(z+a_0)\cdot\nabla_{\Gamma(t)}a_0 = Q_p[\nabla_{\Gamma(t)}a_0]^p.$$

These facts will play a rôle later.

Now, (45) yields the following equation for u^2_{in};

$$\frac{\partial^2 u^2_{\mathrm{in}}}{\partial z^2} + \alpha(z)\frac{\partial u^2_{\mathrm{in}}}{\partial z} + \beta(z)u^2_{\mathrm{in}} = h_2(z), \tag{50}$$

where h_2 is given below. We will derive an evolution equation for a_0 by applying Proposition 2.1 to (50). In fact, we show that the condition

$$\int_{-\infty}^{\infty} Ph_2\,\mathrm{d}z = 0 \tag{51}$$

leads to a linear parabolic equation for a_0, where h_2 is given by

$$\begin{aligned} &h_2(y,t,z) \\ =&\frac{1}{2}I_2^2(z)(a_1)^2 + I_2^{1,1}(z)a_1 + \overline{I}_2^{1,0}(z)a_1 + Q_z f'(Q)\cdot\nabla_{\Gamma(t)}a_1 + \overline{I}_2^0(z) \\ &+\left\{\frac{\partial u^0_{\mathrm{in}}}{\partial t} - \Delta^\Gamma u^0_{\mathrm{in}} + zH^{(1)}\frac{\partial u^0_{\mathrm{in}}}{\partial z} + z\left(f'(u^0_{\mathrm{in}})\cdot\nabla^{(1)}_{\Gamma(t)}u^0_{\mathrm{in}}\right)\right\}, \end{aligned} \tag{52}$$

which is similar to (25). Indeed, I_2^2 and $I_2^{1,1}$ have the same expression as in (26) and (27), while $\overline{I}_2^{1,0}$ and $\overline{I}_2^0$ are given by

$$\overline{I}_2^{1,0} = (-(\gamma_0)_t \cdot \nu + H)\, Q_{zz} + \big(f''(Q) \cdot \nabla_{\Gamma(t)} Q\big)\, Q_z + f'(Q) \cdot \nabla_{\Gamma(t)} Q_z, \tag{53}$$

$$\begin{aligned}\overline{I}_2^0 =& f''(Q) \cdot \nu \overline{u}_{\mathrm{in}}^1 \frac{\partial \overline{u}_{\mathrm{in}}^1}{\partial z} + \frac{1}{2} f'''(Q) \cdot \nu (\overline{u}_{\mathrm{in}}^1)^2 Q_z - \frac{1}{2} g''(Q)(\overline{u}_{\mathrm{in}}^1)^2 \\ &+ (-(\gamma_0)_t \cdot \nu + H)\, \frac{\partial \overline{u}_{\mathrm{in}}^1}{\partial z} + \big(f''(Q) \cdot \nabla_{\Gamma(t)} Q\big)\, \overline{u}_{\mathrm{in}}^1 + f'(Q) \cdot \nabla_{\Gamma(t)} \overline{u}_{\mathrm{in}}^1.\end{aligned} \tag{54}$$

Let us explicitly analyze the condition (51). By the same computation as in §2.3, one can show that

$$\int_{-\infty}^{\infty} P I_2^2 \,\mathrm{d}z = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} P I_2^{1,1} \,\mathrm{d}z = -\int_{-\infty}^{\infty} P \overline{I}_2^{1,0} \,\mathrm{d}z.$$

The anti-symmetry of the integrand implies $\int P(f'(Q) \cdot \nabla_{\Gamma(t)} a_1) Q_z \,\mathrm{d}z = 0$, eliminating a_1 completely from (51).

When we compute the integral $\int P \overline{I}_2^0 \,\mathrm{d}z$ in (51), we encounter terms which are linear and quadratic in $\nabla_{\Gamma(t)} a_0$. These terms come from $\overline{u}_{\mathrm{odd}}^1 = Q_p [\nabla_{\Gamma(t)} a_0]^p$ (here, and below, summations over repeated indices $p, q, p' = 1, \dots, N$ are applied). By using anti-symmetry of integrands, we find that integrals multiplying quadratic terms in $\nabla_{\Gamma(t)} a_0$ vanish, in which we use the facts that f is even and g is odd, and hence $f'(Q(z)), g''(Q(z)), Q_p(z)$ are odd, and $f''(Q(z)), Q_z(z)$ are even in z. More explicitly, we have

$$\begin{aligned}\int_{-\infty}^{\infty} P \overline{I}_2^0 \,\mathrm{d}z &= \int_{-\infty}^{\infty} P \hat{F}_p \,\mathrm{d}z [\nabla_{\Gamma(t)} a_0]^p + \int_{-\infty}^{\infty} P Q_p f_q'(Q) \,\mathrm{d}z [\nabla_{\Gamma(t)} \{[\nabla_{\Gamma(t)} a_0]^p\}]^q + \dots \\ &= \int_{-\infty}^{\infty} P \hat{F}_p \,\mathrm{d}z [\nabla_{\Gamma(t)} a_0]^p - M_0 \mathbf{T}_{pq} [\nabla_{\Gamma(t)} \{[\nabla_{\Gamma(t)} a_0]^p\}]^q + \dots,\end{aligned} \tag{55}$$

where "…" stand for terms which are all together independent of a_0 as well as $\nabla_{\Gamma(t)} a_0$, while $\hat{F}_p$ is defined by

$$\begin{aligned}\hat{F}_p =& (f''(Q) \cdot \nu Q_p \overline{u}_{\mathrm{even}}^1)_z - g''(Q) Q_p \overline{u}_{\mathrm{even}}^1 - (-(\gamma_0)_t \cdot \nu + H^{(0)})(Q_p)_z \\ &+ (f_p'(Q) \overline{u}_{\mathrm{even}}^1)_z + \frac{\partial}{\partial \nu^q} \{f_{p'}'(Q) Q_p\} [\nabla_{\Gamma(t)} \nu^q] p'.\end{aligned} \tag{56}$$

We now deal with the third line of (52).

For the first three terms in the third line of (52), we have

$$\begin{aligned}\frac{\partial \overline{u}_{\rm in}^0}{\partial t} - \Delta^\Gamma \overline{u}_{\rm in}^0 + zQ_z H^{(1)} =& Q_z \frac{\partial a_0}{\partial t} - Q_z \Delta^\Gamma a_0 + zQ_z H^{(1)} - 2(Q_p)_z (\nabla_{\Gamma(t)} \nu^p \cdot \nabla_{\Gamma(t)} a_0) \\ & - Q_{zz} |\nabla_{\Gamma(t)} a_0|^2 - Q_{pq} \nabla_{\Gamma(t)} \nu^p \cdot \nabla_{\Gamma(t)} \nu^q - Q_p \Delta^\Gamma \nu^p + Q_p \nu_t^p.\end{aligned}$$

By using anti-symmetry of the integrand, one can show that $\int PQ_{zz}\ \mathrm{d}z = 0$, and hence the coefficient of $|\nabla_{\Gamma(t)} a_0|^2$ in (52) vanishes when substituted into (52). Moreover, we have

$$\int_{-\infty}^{\infty} zPQ_z\ \mathrm{d}z = \int_{-\infty}^{\infty} (z - a_0)\tilde{P}\tilde{Q}_z\ \mathrm{d}z = -M_0 a_0.$$

For the last term in the third line of (52), we have

$$\begin{aligned}\int_{-\infty}^{\infty} zPf'(Q) \cdot \nabla_{\Gamma(t)}^{(1)} u_{\rm in}^0\ \mathrm{d}z =& \int_{-\infty}^{\infty} zPQ_z f_p'(Q)\ \mathrm{d}z [\nabla_{\Gamma(t)}^{(1)} a_0]^p + \int_{-\infty}^{\infty} zPQ_q f_p'(Q)\ \mathrm{d}z [\nabla_{\Gamma(t)}^{(1)} \nu^q]^p \\ =& \int_{-\infty}^{\infty} z\tilde{P}\tilde{Q}_z f_p'(\tilde{Q})\ \mathrm{d}z [\nabla_{\Gamma(t)}^{(1)} a_0]^p + a_0 \mathbf{T}_{pq} [\nabla_{\Gamma(t)}^{(1)} \nu^q]^p.\end{aligned}$$

We finally conclude that the solvability condition (51) for the problem (50) is equivalent to the following linear parabolic equation

$$\begin{aligned}\frac{\partial a_0}{\partial t} =& \Delta^\Gamma a_0 + H^{(1)} a_0 + \mathbf{T}_{pq} [\nabla_{\Gamma(t)} \{[\nabla_{\Gamma(t)} a_0]^p\}]^q - a_0 \mathbf{T}_{pq} [\nabla_{\Gamma(t)} \nu^q]^p \\ & + \mathbf{F}(y,t) \cdot \nabla_{\Gamma(t)} a_0 + \mathbf{G}(y,t) \cdot \nabla_{\Gamma(t)}^{(1)} a_0 + \hat{h}_2(y,t),\end{aligned} \tag{57}$$

where the principal part $\Delta^\Gamma a_0 + \mathbf{T}_{pq} [\nabla_{\Gamma(t)} \{[\nabla_{\Gamma(t)} a_0]^p\}]^q$ on the right hand side is uniformly elliptic because of the positive semi-definiteness of $\mathbf{T}$. In (57), $\mathbf{F}(y,t)$ and $\mathbf{G}(y,t)$ are vector fields on $\Gamma(t)$ defined by

$$\begin{aligned}\mathbf{F}(y,t) =& 2M_0^{-1} \int_{-\infty}^{\infty} P(Q_p)_z\ \mathrm{d}z (\nabla_{\Gamma(t)} \nu^p \cdot\ ) - M_0^{-1} \int_{-\infty}^{\infty} P\hat{F}\ \mathrm{d}z \\ \mathbf{G}(y,t) =& - M_0^{-1} \int_{-\infty}^{\infty} z\tilde{P}\tilde{Q}_z f'(\tilde{Q})\ \mathrm{d}z\end{aligned}$$

which are independent of a_0. We note that the homogeneous part of (57) is the linearization of th interface equation (11), although we do not bother proving this fact.

Similarly, at order $\mathcal{O}(\varepsilon^k)$, the solvability condition gives rise to

$$\begin{aligned}\frac{\partial a_{k-2}}{\partial t} =&\Delta^\Gamma a_{k-2} + H^{(1)} a_{k-2} + \mathbf{T}_{pq}[\nabla_{\Gamma(t)}\{[\nabla_{\Gamma(t)} a_{k-2}]^p\}]^q - a_{k-2}\mathbf{T}_{pq}[\nabla_{\Gamma(t)}\nu^q]^p \\ &+ \mathbf{F}(y,t)\cdot\nabla_{\Gamma(t)} a_{k-2} + \mathbf{G}(y,t)\cdot\nabla^{(1)}_{\Gamma(t)} a_{k-2} + \hat{h}_k(y,t),\end{aligned} \tag{58}$$

where $\hat{h}_k(y,t)$ depends only on $a_0,\ldots,a_{k-3}$, but not on a_{k-2}.

In this way, we can construct approximate solutions as high order as we wish. Then, the remaining part of proof is the same as the proof of Theorem 1.1. This completes the proof of Theorem 1.2. ∎

4 Examples and Discussion

Following an approach described in [6] for $f \equiv 0$, we can explicitly determine the wave speed and the heteroclinic orbits when g is cubic and f is quadratic in u.

Lemma 4.1. *Consider the nonlinear eigenvalue problem (5) with*

$$\begin{aligned} g(u) &= -R(u-u_-)u(u-u_+), \quad u_- < 0 < u_+, \quad R > 0, \\ f(u) &= \frac{1}{2}u^2\mathbf{a} + u\mathbf{b}, \quad \mathbf{a},\ \mathbf{b} \in \mathbb{R}^N. \end{aligned}$$

The wave speed $s(\nu)$ and travelling wave profile $Q(z)$ are then explicitly given by

$$\begin{aligned} s(\nu) &= \frac{u_- + u_+}{4}\left\{\mathbf{a}\cdot\nu - \sqrt{(\mathbf{a}\cdot\nu)^2 + 8R}\right\} + \mathbf{b}\cdot\nu \\ Q(z) &= \frac{-u_+u_- + u_+u_-\mathrm{e}^{-D(u_+-u_-)z}}{-u_- + u_+\mathrm{e}^{-D(u_+-u_-)z}} \end{aligned}$$

where

$$D = \frac{\sqrt{(\mathbf{a}\cdot\nu)^2 + 8R} - \mathbf{a}\cdot\nu}{4}.$$

Proof: The statement is verified directly by following the method described in §11.5 of [6] for scalar bi-stable reaction-diffusion equations. ∎

4.1 Symmetric case

Let us first consider a completely symmetric nonlinearities

$$f(u) = \frac{1}{2}u^2\mathbf{a}, \quad g(u) = -u(u^2-1)$$

with the vector $\mathbf{a}$ remaining as the only parameter.

The travelling wave profile is then $Q(z) = \tanh(Dz)$ which depends on the direction ν only through $D = \frac{1}{4}(\sqrt{(\mathbf{a}\cdot\nu)^2+8} - \mathbf{a}\cdot\nu)$. We have

$$Q_q := \frac{\partial Q}{\partial \nu^q} = -\frac{D\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}}\frac{z}{\cosh^2(Dz)}, \quad P = Q_z \mathrm{e}^A = D\left(\cosh(Dz)\right)^{-(\frac{\mathbf{a}\cdot\nu}{D}+2)} \tag{59}$$

This allows us to compute

$$M_0 = \int_{-\infty}^{\infty} \mathrm{e}^{A(z)} Q_z^2(z)\,\mathrm{d}z = D^2 \int_{-\infty}^{\infty} \left(\cosh(Dz)\right)^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z. \tag{60}$$

Similarly, we get from (41) with $s(\nu)\equiv 0$ and (59)

$$\begin{aligned} M_0\mathbf{T}_{pq} &= -\int_{-\infty}^{\infty} \mathrm{e}^{A(z)} Q_z Q_q f_p'(Q)\,dz \\ &= \frac{D^2\mathbf{a}_p\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}} \int_{-\infty}^{\infty} z\tanh(Dz)\left(\cosh(Dz)\right)^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z \\ &= \frac{D^2\mathbf{a}_p\mathbf{a}_q}{\sqrt{(\mathbf{a}\cdot\nu)^2+8}}\left(\frac{1}{\mathbf{a}\cdot\nu+4D}\right) \int_{-\infty}^{\infty} \left(\cosh(Dz)\right)^{-(\frac{\mathbf{a}\cdot\nu}{D}+4)}\,\mathrm{d}z \end{aligned}$$

Comparing this expression with (60) gives

$$\mathbf{T}_{pq} = \frac{\mathbf{a}_p\mathbf{a}_q}{(\mathbf{a}\cdot\nu)^2+8}.$$

In particular, we can see directly that the matrix $\mathbf{T}$ is positive semi-definite. The definition (42) of K^{pq} and elementary computations lead to

$$\mathbf{T}_{pq}K^{pq} = \frac{1}{2D^2+1}\mathbf{a}\cdot\nabla_{\Gamma(t)}D.$$

Therefore, the interface equation (11) in our special situation is written as

$$V = H(y,t) + \mathbf{a}\cdot\nabla_{\Gamma(t)}\left(\frac{\mathrm{Arctan}(\sqrt{2}D)}{\sqrt{2}}\right) \tag{61}$$

Note that $D > 0$ in the function $Q(z) = \tanh(Dz)$ signifies the "steepness" of the wave profile. Therefore, (61) means that the tangential variation of the "steepness"

$$\frac{\mathrm{Arctan}\left(\sqrt{2}D\right)}{\sqrt{2}}$$

of the wave profile $Q(z;\nu)$ is converted in the singular limit to the normal velocity of the interface. Moreover, this effect does not operate when $\mathbf{a}$ is normal to the interface, where the interface is driven by the mean curvature alone.

Another viewpoint is possible for (11) in the present situation. Applying Lemma 3.3, the interface equation has the following expression

$$V = \sum_{i=1}^{N-1} (1 + w^i)\kappa_i, \tag{62}$$

where

$$w^i = \frac{1}{(\mathbf{a}\cdot\nu)^2 + 8} \sum_{j=1}^{N-1} \left(\frac{\partial\gamma_0}{\partial y^j}\cdot\mathbf{a}\right)\left(\frac{\partial\gamma_0}{\partial y^i}\cdot\mathbf{a}\right) g^{ji} \quad \text{(no summation over } i\text{)}.$$

Therefore, (62) is considered as a motion driven by a weighted mean curvature. Note that w^i in (62) vanishes when $\mathbf{a}$ is parallel to ν. This means that the curvature effects on the normal velocity is enhanced when $\mathbf{a}$ is parallel to the interface.

It is of interest to note that the matrix $\mathbf{T}$ which stems from the first order differential operator $f'\cdot\nabla$ contributes in the singular limit to the curvature which is a second order operator.

4.2 Slightly asymmetric case

Let us consider ε-dependent flux and reaction terms

$$f(u) = \varepsilon u\mathbf{b}, \quad g(u) = -a^2 u(u - (1-\varepsilon))(u + (1+\varepsilon)) \quad \text{(with } a > 0\text{)}.$$

In this situation, Theorem 1.2 applies. The second term in (61) vanishes because of $D \equiv /\sqrt{2}$. However, ε-term in the wave speed $s(\nu)$ comes in and the interface equation is given by

$$V = H + \sqrt{2}a + \mathbf{b}\cdot\nu. \tag{63}$$

This is similar to a curvature flow driven by a constant force (cf. [8])

$$V = H + k \tag{64}$$

which comes as a singular limit of the Allen-Cahn equation with slightly asymmetric source term. In (64) k represents a constant driving force. In our equation (63), the driving force depends on the orientation of the interface. However, (63) is expressed as follows.

$$(\tilde{\gamma}_0)_t = H + \sqrt{2}a,$$

where $\tilde{\gamma}_0 = \gamma_0 - t(\mathbf{b}\cdot\nu)\nu$. Therefore, the motion of interface driven by (63) is a motion by mean curvature with a driving force plus constant translation with velocity $\mathbf{b}$.

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