Operational Method for Solving Generalized Abel Integral Equations of Second Kind

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Abstract

A modification of the Mikusinski operational calculus is used to obtain explicit solutions of some classes of integral equations of the second kind of Abel type. In particular, the solution of the generalized Abel equation

$$y(x) - \sum_{i=1}^{m} \lambda_i (I_{0+}^{\alpha_i \mu} y)(x) = f(x), \ \alpha_i > 0, \ 1 \le i \le m, \ \mu > 0, \ x > 0$$

with the Riemann-Liouville fractional integral operator I_{0+}^{μ} of order μ is expressed in terms of the Mittag-Leffler's function of several variables.

Keywords: operational calculus, generalized Abel integral equation, Mittag-Leffler's function, generalized Mittag-Leffler's function.

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1 Introduction

In recent years integral equations of Abel type and differential equations of fractional order have found applications in modelling many physical and chemical processes (see, for example, Babenko (1994), Gorenflo and Rutman (1994), Gorenflo and Vessella (1991), Mainardi (1994), Michalski (1991)). The main method, which has been applied for the solution of these equations, used the Laplace transform technique (Gorenflo and Rutman (1994), Mainardi (1994), Podlubny (1994)). This technique is very popular in applied mathematics. However, it has some drawbacks also, in particular, we can't use it for equations with increasing too fast right-hand sides. It is well known that these difficulties may be overcome by using the Mikusinski type operational calculi, which were used for the solution of ordinary differential equations both with constant and variable coefficients and for the solution of some partial differential equations (see, for example, Ditkin and Prudnikov (1963), Luchko (1993), Luchko and Yakubovich (1993), (1994), Meller (1963), Mikusinski (1959), Rodrigues (1989), Yakubovich and Luchko (1994), Yosida (1984)). It is worth mentioning that the operational method may be used for the solution of integral equations as well

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(Buschman (1972), Erdelyi (1962)). In this report, we present an application of the operational calculus for the Riemann-Liouville fractional differential operator to the solution of some classes of integral equations of the second kind. In particular, we consider the generalized Abel equation of the second kind

$$y(x) - \sum_{i=1}^{m} \lambda_i (I_{0+}^{\alpha_i \mu} y)(x) = f(x),\ \alpha_i > 0,\ 1 \le i \le m,\ \mu > 0,\ x > 0,$$

where

$$(I_{0+}^{\mu} y)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} y(t)dt,\ \mu > 0,\ x > 0, \tag{1.1}$$

$$(I_{0+}^{0} y)(x) = y(x),\ x > 0$$

defines the Riemann-Liouville fractional integral operator I_{0+}^{μ} of order μ. The used operational calculus is, as a matter of fact, a modification of the Mikusinski operational calculus, a modification that is more suitable for application to the solution of differential and integral equations of fractional order. Firstly, we consider the space C_α of functions which can have an integrable singularity in the neighborhood of zero, instead of the space $\mathrm{C}[0,\infty)$ of functions continuous on the half-axis used in Mikusinski's operational calculus. Since the kernel of the Riemann-Liouville integral operator I_{0+}^{μ} belongs to space C_α if $\mu - 1 > \alpha$ and $\alpha \ge -1$, this function space seems to be more suitable for the development of the operational calculus for the Riemann-Liouville fractional differential operator. Secondly, the abovementioned property allows us to consider the Riemann-Liouville integral operator I_{0+}^{μ} as the convolution product with the power function $x^{\mu-1}/\Gamma(\mu)$ in the space C_α if $\mu - 1 > \alpha$, and this power function plays in our considerations the same role, as the function $\{1\}$ in Mikusinski's operational calculus.

2 Operational calculus for the Riemann-Liouville fractional differential operator

We shall develop an operational calculus for the operator I_{0+}^{μ}, thereby considering μ as a fixed positive parameter. We begin by defining the function space $C_\alpha,\ \alpha \in \mathbf{R}$, which will be used in our investigation.

Definition 1 *A function $f(x),\ x > 0$ is said to be in the space $C_\alpha,\ \alpha \in \mathbf{R}$ if there exists a real number p, $p > \alpha$, such that*

$$f(x) = x^p f_1(x)$$

with a function $f_1(x)$ in $C[0,\infty)$.

For example, the function $x^{-\frac{1}{2}}|\log x|$ is in C_α if and only if $\alpha < -\frac{1}{2}$.

Clearly, C_α is a vector space and the set of spaces C_α is ordered by inclusion according to

$$C_\alpha \subset C_\beta \Leftrightarrow \alpha \ge \beta. \tag{2.1}$$

Theorem 1 *The Riemann-Liouville fractional integral operator I_{0+}^{μ} is a linear map of the space C_α, $\alpha \geq -1$, into itself, that is,*

$$I_{0+}^{\mu} : C_\alpha \to C_{\mu+\alpha} \subset C_\alpha.$$

Proof. The case $\mu = 0$ is evident. In the case $\mu > 0$, setting $t = x\tau$ in (1.1), and using Definition 1, we readily obtain

$$(I_{0+}^{\mu} y)(x) = \frac{x^{p+\mu}}{\Gamma(\mu)} \int_0^1 \tau^p (1-\tau)^{\mu-1} y_1(x\tau) d\tau = x^{p+\mu} y_2(x),$$

where $p > \alpha$ and $y_1 \in C[0, \infty)$.

The last integral is uniformly convergent with respect to x in every closed interval $[0, X]$, $X > 0$; consequently, we have $y_2 \in C[0, \infty)$ and $I_{0+}^{\mu} y \in \mathrm{C}_{\mu+\alpha}$. $\square$

Remark 1 *The operator I_{0+}^{μ}, $\mu > 0$ has the following convolutional representation in the space C_α, $\alpha \geq -1$:*

$$(I_{0+}^{\mu} f)(x) = (h_\mu \circ f)(x), \; h_\mu(x) := x^{\mu-1}/\Gamma(\mu), \; f \in C_\alpha, \tag{2.2}$$

where

$$(g \circ f)(x) = \int_0^x g(x-t) f(t) dt, \; x > 0$$

is the Laplace convolution.

Remark 2 *It is well known (Dimovski* (1982)*, Mikusinski* (1959)*, Yosida* (1984)*) that the Laplace convolution is commutative and associative. This fact and Remark* 1 *lead to the following property of the Riemann-Liouville fractional integral operator:*

$$(I_{0+}^{\delta} I_{0+}^{\eta} f)(x) = (I_{0+}^{\eta} I_{0+}^{\delta} f)(x), \; f \in C_\alpha, \; \alpha \geq -1, \; \delta \geq 0, \; \eta \geq 0. \tag{2.3}$$

Next, using the same reasoning and the Euler integral of the first kind for the evaluation of the convolution $(h_\delta \circ h_\eta)(x)$, we obtain

$$(I_{0+}^{\delta} I_{0+}^{\eta} f)(x) = (I_{0+}^{\delta+\eta} f)(x), \; f \in C_\alpha, \; \alpha \geq -1, \; \delta \geq 0, \; \eta \geq 0, \tag{2.4}$$

that is also well known. In particular, it follows from (2.3) *and* (2.4) *that*

$$(\underbrace{I_{0+}^{\mu} \dots I_{0+}^{\mu}}_{n} f)(x) = (I_{0+}^{n\mu} f)(x), \; f \in C_\alpha, \; \alpha \geq -1, \; \mu \geq 0, \; n \in \mathbf{N}. \tag{2.5}$$

As in the case of Mikusinski's operational calculus, we obtain

Theorem 2 *The space C_α with the operations $\circ$ and ordinary addition becomes a commutative ring $(C_\alpha, \circ, +)$ without divisors of zero.*

In analogy to Mikusinski's reasoning the ring C_α can be extended to the quotient field M_α by factorizing the set $C_\alpha \times (C_\alpha - \{0\})$ with respect to the equivalence relation

$$(f, g) \sim (f_1, g_1) \Leftrightarrow (f \circ g_1)(x) = (g \circ f_1)(x). \tag{2.6}$$

One can consider the elements of the field M_α as convolution quotients f/g and define the operations in M_α, as usual, by

$$\frac{f}{g} + \frac{f_1}{g_1} = \frac{f \circ g_1 + g \circ f_1}{g \circ g_1}, \tag{2.7}$$

$$\frac{f}{g} \cdot \frac{f_1}{g_1} = \frac{f \circ f_1}{g \circ g_1}. \tag{2.8}$$

The proof of the fact that the set M_α is a commutative field with respect to operations "+" and "$\cdot$" is based on Theorem 2.

It is easily seen that the ring C_α can be embedded in the field M_α by the map:

$$f \mapsto \frac{h_\mu \circ f}{h_\mu}, \tag{2.9}$$

with, by (2.2), $h_\mu(x) = x^{\mu-1}/\Gamma(\mu)$.

Moreover, the field $\mathbf{C}$ of complex numbers can also be embedded in M_α by the map

$$z \mapsto \frac{z h_\mu}{h_\mu}. \tag{2.10}$$

Definition 2 *The algebraic inverse of the Riemann-Liouville fractional integral operator I^μ_{0+} is said to be the element S of the field M_α, which is reciprocal to the element h_μ in the field M_α, that is,*

$$S = \frac{I}{h_\mu} \equiv \frac{h_\mu}{h_\mu \circ h_\mu} \equiv \frac{h_\mu}{h_{2\mu}}, \tag{2.11}$$

where $I = \frac{h_\mu}{h_\mu}$ denotes the identity element of the field M_α with respect to the operation of multiplication.

The main idea of applications of an operational calculus to solving differential, integral and other equations resides in the reduction of these to algebraic or differential equations of simple type in the corresponding convolution quotient field. Hence, it is important to reduce complicated operations in the initial ring of functions to simpler ones in the convolution quotient field. The relationship between the Riemann-Liouville fractional integral operator and the elements of the field M_α is given by

Theorem 3 *The Riemann-Liouville fractional integral operator I^μ_{0+} may be represented in the field M_α in the form*

$$(I^\mu_{0+} f)(x) = \frac{I}{S} \cdot f. \tag{2.12}$$

Proof. From Definition 2 we take that the element S of the field M_α is reciprocal to the element h_μ, conversely, the element h_μ is reciprocal to the element S, that is,

$$h_\mu = \frac{I}{S}.$$

Using the embedding (2.9) of the ring C_α in the field M_α, the fact that the convolution of two functions from the ring C_α corresponds to their multiplication in the field M_α, and Remark 1, we obtain the statement of this theorem. $\Box$

Remark 3 *Let $f \in C_\alpha$, $D^\mu_{0+} f \in C_\alpha$, $\alpha \geq -1$, where*

$$(D^\mu_{0+} f)(x) = \left(\frac{d}{dx}\right)^\eta (I^{\eta-\mu}_{0+} f)(x),\ \eta = \begin{cases} [\mu]+1, & \mu \notin \mathbf{N}, \\ \mu, & \mu \in \mathbf{N} \end{cases}$$

is the Riemann-Liouville fractional differential operator. Then we have the following relation in the convolution quotient field M_α (see Luchko and Yakubovich (1993)):

$$D^\mu_{0+} f = S \cdot f - S \cdot Ff,\ (Ff)(x) = \sum_{k=1}^{\eta} \frac{x^{\mu-k}}{\Gamma(\mu-k+1)} \lim_{z\to 0} (D^{\mu-k}_{0+} f)(z).$$

This operational relation was used for the solution of differential equations of fractional order in Luchko (1993), Luchko and Yakubovich (1993), Yakubovich and Luchko (1994).

For the sake of simplicity we will consider in our further discussions the case $\alpha = -1$, which turns out to be the most interesting one for applications. We already know (see Remark 2), that

$$h^n_\mu(x) := \underbrace{h_\mu \circ \ldots \circ h_\mu}_{n} = h_{n\mu}(x).$$

Let us extend this relation to the fractional power of $h_\mu(x)$:

$$h^\lambda_\mu(x) := h_{\lambda\mu}(x),\ \Re(\lambda) > 0. \tag{2.13}$$

We have for any $\Re(\lambda) > 0$ the inclusion $h^\lambda_\mu \in C_{-1}$, and the following relation can be easily obtained:

$$h^\alpha_\mu \circ h^\beta_\mu = h_{\alpha\mu} \circ h_{\beta\mu} = h_{(\alpha+\beta)\mu} = h^{\alpha+\beta}_\mu,\ \Re(\alpha) > 0,\ \Re(\beta) > 0.$$

We define also the fractional power of the element S by relation

$$S^\lambda = \begin{cases} h^{-\lambda}_\mu(x), & \lambda < 0, \\ I, & \lambda = 0, \\ \frac{I}{h^\lambda_\mu}, & \lambda > 0. \end{cases} \tag{2.14}$$

Using the same technique as in the case of Mikusinski's operational calculus (Mikusinski (1959), Yosida (1984)) we obtain the relation

$$S^\alpha \cdot S^\beta = S^{\alpha+\beta},\ \alpha, \beta \in \mathrm{C}. \tag{2.15}$$

Remark 4 *It follows from Theorem* 3 *and relations* (2.13) *and* (2.14)*, that the Riemann-Liouville fractional integral operator* $I_{0+}^{\alpha\mu}$*,* $\alpha, \mu > 0$ *may be represented in the field* M_{-1} *in the form*

$$(I_{0+}^{\alpha\mu} f)(x) = \frac{I}{S^\alpha} \cdot f. \tag{2.16}$$

The next important step in applications of an operational calculus is to represent solutions of some equations in the convolution quotient field by the elements of the initial ring of functions. It is known, that there exist long tables of corresponding representations in the case of Mikusinski's operational calculus. We can obtain a wide class of such functions for our operational calculus using

Theorem 4 *Let the power series of several complex variables* $z = (z_1, \ldots, z_m)$ *with complex coefficients be convergent at the point* $z_0 = (z_{10}, \ldots, z_{m0}) \neq 0$*, that is,*

$$\sum_{i_1,\ldots,i_m=0}^{\infty} b_{i_1,\ldots,i_m} z_{10}^{i_1} \times \cdots \times z_{m0}^{i_m} = A \in \mathbf{C}$$

and $\beta > 0$*,* $\alpha_i > 0$*,* $1 \leq i \leq m$*. Then the power series*

$$S^{-\beta} \sum_{i_1,\ldots,i_m=0}^{\infty} b_{i_1,\ldots,i_m} (S^{-\alpha_1})^{i_1} \times \cdots \times (S^{-\alpha_m})^{i_m} = \sum_{i_1,\ldots,i_m=0}^{\infty} b_{i_1,\ldots,i_m} h_{(\beta+\alpha_1 i_1+\cdots+\alpha_m i_m)\mu}(x),$$

where $h_\mu(x)$ *is given by* (2.2)*, defines an element of the ring* C_{-1}*.*

Proof. By using the definition of $h_\mu(x)$, we have

$$f(x) := \sum_{i_1,\ldots,i_m=0}^{\infty} b_{i_1,\ldots,i_m} h_{(\beta+\alpha_1 i_1+\cdots+\alpha_m i_m)\mu}(x) = x^{\beta\mu-1} f_1(x),$$

$$f_1(x) = \sum_{i_1,\ldots,i_m=0}^{\infty} \frac{b_{i_1,\ldots,i_m} (x^{\alpha_1\mu})^{i_1} \times \cdots \times (x^{\alpha_m\mu})^{i_m}}{\Gamma(\beta\mu + \alpha_1 i_1 \mu + \cdots + \alpha_m i_m \mu)}.$$

We must show now that the function $f_1(x)$ is continuous in the interval $[0, \infty)$. First of all, we obtain the estimate

$$\frac{1}{\Gamma(\beta\mu + \alpha_1 i_1 \mu + \cdots + \alpha_m i_m \mu)} \leq \frac{C}{\Gamma(\beta\mu/m + \alpha_1 i_1 \mu) \times \cdots \times \Gamma(\beta\mu/m + \alpha_m i_m \mu)},$$

$$C \in \mathbf{R},\ i_j \geq 0,\ 1 \leq j \leq m,$$

which is based on the inequality for Euler's Beta function

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \leq 1,\ s \geq 1,\ t \geq 1.$$

This inequality is a consequence of the logarithmic convexity of the Gamma function. Using the asymptotic behaviour of the Gamma function with large argument (Samko et al. (1993)) we obtain the estimate

$$\frac{1}{\Gamma(\beta\mu/m + \alpha_j i_j \mu)} \leq C_j \frac{A_j^{i_j} i_j^{\cdot a_j}}{i_j^{i_j \alpha_j \mu}},\ 1 \leq j \leq m,\ i_j = 0, 1, \ldots,$$

where $C_j,\ A_j,\ a_j$ are suitable constants. Furthermore, since the series

$$\sum_{i_1,\dots,i_m=0}^{\infty} b_{i_1,\dots,i_m} z_{10}^{i_1} \times \cdots \times z_{m0}^{i_m},\ z_0 \neq 0,$$

is convergent, we have

$$\sup_{i_1,\dots,i_m \geq 0} \mid b_{i_1,\dots,i_m} z_{10}^{i_1} \times \cdots \times z_{m0}^{i_m} \mid = M_0 < \infty.$$

Hence

$$\mid b_{i_1,\dots,i_m} \mid \leq \frac{M_0}{\mid z_{10} \mid^{i_1} \times \cdots \times \mid z_{m0} \mid^{i_m}},\ i_j = 0, 1, \dots,\ 1 \leq j \leq m.$$

Thus, for $x = X\ (0 < X < \infty)$, we obtain the estimate

$$\sum_{i_1,\dots,i_m=0}^{\infty} \left| \frac{b_{i_1,\dots,i_m} (X^{\alpha_1 \mu})^{i_1} \times \cdots \times (X^{\alpha_m \mu})^{i_m}}{\Gamma(\beta\mu + \alpha_1 i_1 \mu + \cdots + \alpha_m i_m \mu)} \right| \leq C M_0 \prod_{j=1}^{m} \sum_{i_j=0}^{\infty} \frac{C_j (X^{\alpha_j \mu})^{i_j} A_j^{i_j} i_j^{a_j}}{\mid z_{j0} \mid^{i_j} i_j^{i_j \alpha_j \mu}} < \infty,$$

which, by Abel's theorem, shows that the series representing $f_1(x)$ is uniformly convergent on every bounded closed interval $[0, X],\ 0 < X < \infty$. Consequently, $f_1 \in \mathrm{C}[0, \infty)$. $\square$

Applying Theorem 4, we may obtain various functions of S in the field M_{-1}, that can be represented by using the elements of the ring C_{-1}. In particular, we have the operational relations (2.17), (2.19), (2.21), (2.22).

$$\frac{I}{S-\rho} = S^{-1} \cdot \frac{I}{I - \rho S^{-1}} = S^{-1} \sum_{i=0}^{\infty} \rho^i S^{-i} = \tag{2.17}$$

$$\sum_{i=0}^{\infty} \rho^i h_{(i+1)\mu}(x) = \sum_{i=0}^{\infty} \frac{\rho^i x^{(i+1)\mu - 1}}{\Gamma(\mu i + \mu)} = x^{\mu-1} E_{\mu,\mu}(\rho x^\mu),$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta > 0,\ |z| < \infty. \tag{2.18}$$

$$\frac{I}{(S-\rho)^m} = S^{-m} \cdot \frac{I}{(I - \rho S^{-1})^m} = S^{-m} \sum_{i=0}^{\infty} \frac{(m)_i \rho^i}{i!} S^{-i} = \tag{2.19}$$

$$\sum_{i=0}^{\infty} \frac{(m)_i \rho^i}{i!} h_{(m+i)\mu}(x) = x^{\mu m - 1} \sum_{i=0}^{\infty} \frac{(m)_i (\rho x^\mu)^i}{i! \Gamma(\mu i + m\mu)} = x^{\mu m - 1} E_{\mu, m\mu}^m(\rho x^\mu),\ m \in \mathbf{N},$$

where

$$E_{\alpha,\beta}^m(z) := \sum_{i=0}^{\infty} \frac{(m)_i z^i}{i! \Gamma(\alpha i + \beta)},\ \alpha, \beta > 0,\ |z| < \infty, \tag{2.20}$$

$$(m)_i = \prod_{k=0}^{i-1} (m+k)$$

is Pochhammer's symbol. In particular,

$$E^1_{\alpha,\beta}(z) \equiv E_{\alpha,\beta}(z).$$

Let $\beta > 0,\ \gamma > 0$. We then have the relation

$$\frac{S^{-\beta}}{(S^{\gamma}-\rho)^m} = S^{-\beta-\gamma m}\cdot\frac{I}{(I-\rho S^{-\gamma})^m} = \tag{2.21}$$

$$S^{-\beta-\gamma m}\sum_{i=0}^{\infty}\frac{(m)_i\rho^i}{i!}(S^{-\gamma})^i = \sum_{i=0}^{\infty}\frac{(m)_i\rho^i}{i!}h_{(\beta+\gamma m+\gamma i)\mu}(x) =$$

$$x^{\gamma\mu m+\beta\mu-1}\sum_{i=0}^{\infty}\frac{(m)_i(\rho x^{\gamma\mu})^i}{i!\Gamma(\gamma\mu i+\gamma\mu m+\beta\mu)} = x^{\gamma\mu m+\beta\mu-1}E^m_{\gamma\mu,\gamma\mu m+\beta\mu}(\rho x^{\gamma\mu}),\ m\in\mathbf{N}.$$

Finally, we obtain an operational relation that will be used in solution of the generalized Abel integral equation. Let $\beta > 0,\ \alpha_j > 0,\ 1\le j\le m$. We then obtain the operational relation

$$\frac{S^{-\beta}}{I-\sum_{i=1}^m\lambda_i S^{-\alpha_i}} = S^{-\beta}\sum_{k=0}^{\infty}\left(\sum_{i=1}^m\lambda_i S^{-\alpha_i}\right)^k = \tag{2.22}$$

$$S^{-\beta}\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=k\\ l_1\ge 0,\ldots,l_m\ge 0}}(k;l_1,\ldots,l_m)\prod_{i=1}^m\left(\lambda_i S^{-\alpha_i}\right)^{l_i} =$$

$$\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=k\\ l_1\ge 0,\ldots,l_m\ge 0}}(k;l_1,\ldots,l_m)\prod_{i=1}^m\lambda_i^{l_i}h_{\beta\mu+\mu\sum_{i=1}^m\alpha_i l_i}(x) =$$

$$x^{\beta\mu-1}\sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=k\\ l_1\ge 0,\ldots,l_m\ge 0}}(k;l_1,\ldots,l_m)\frac{\prod_{i=1}^m(\lambda_i x^{\alpha_i\mu})^{l_i}}{\Gamma(\beta\mu+\mu\sum_{i=1}^m\alpha_i l_i)} =$$

$$x^{\beta\mu-1}E_{(\alpha_1\mu,\ldots,\alpha_m\mu),\beta\mu}(\lambda_1 x^{\alpha_1\mu},\ldots,\lambda_m x^{\alpha_m\mu}),$$

with the multivariate Mittag-Leffler function

$$E_{(a_1,\ldots,a_m),b}(z_1,\ldots,z_m) := \sum_{k=0}^{\infty}\sum_{\substack{l_1+\cdots+l_m=k\\ l_1\ge 0,\ldots,l_m\ge 0}}(k;l_1,\ldots,l_m)\frac{\prod_{i=1}^m z_i^{l_i}}{\Gamma(b+\sum_{i=1}^m a_i l_i)} \tag{2.23}$$

and the multinomial coefficients

$$(k;l_1,\ldots,l_m) := \frac{k!}{l_1!\times\cdot\times l_m!}.$$

Note here, that Podlubny (1994), using the Laplace transform method, has obtained similar formulas.

Theorem 4 permits us to obtain some other representations of functions of S in the field M_{-1} by elements of the ring C_{-1}, but we will restrict ourselves to those obtained above.

3 Applications Involving Generalized Abel Equations

In this part of our report we consider integral equations of the second kind of Abel type.

Theorem 5 *The unique solution of the integral equation of the second kind*

$$y(x) - \lambda \int_0^x K(x-t)y(t)dt = f(x), \ f \in C_{-1}, \ x > 0, \tag{3.1}$$

$$K(x) = \sum_{l=1}^{m} \sum_{j=1}^{m_l} c_{lj} x^{\mu j - 1} E^j_{\mu, j\mu}(z_l x^\mu),$$

in the space C_{-1} is represented in the form

$$y(x) = f(x) + \lambda \int_0^x R(x-t) f(t)dt, \tag{3.2}$$

where

$$R(x) = \sum_{l=1}^{m} \sum_{j=1}^{n_l} d_{lj} x^{\mu j - 1} E^j_{\mu, \mu j}(w_l x^\mu),$$

the constants w_l, $1 \le l \le m$ and d_{lj}, $1 \le l \le m$, $1 \le j \le n_l$ are determined by the representation of the rational function $\frac{P(z)}{Q(z) - \lambda P(z)}$ as the sum of partial fractions

$$\frac{P(z)}{Q(z) - \lambda P(z)} = \sum_{l=1}^{m} \sum_{j=1}^{n_l} d_{lj} \frac{1}{(z - w_l)^j}, \tag{3.3}$$

$$P(S) = \sum_{i=0}^{p} b_i S^i, \ Q(S) = \sum_{i=0}^{q} d_i S^i, \ p < q,$$

$$\frac{P(S)}{Q(S)} = \sum_{l=1}^{m} \sum_{j=1}^{m_l} c_{lj} \frac{I}{(S - z_l)^j}. \tag{3.4}$$

Proof. Using the relations (3.4), (2.19) and the embedding (2.9) we can reduce (3.1) in the field M_{-1} of convolution quotients to the equation

$$y - \lambda \frac{P(S)}{Q(S)} \cdot y = f$$

whose unique solution in M_{-1} has the form

$$y = \frac{Q(S)}{Q(S) - \lambda P(S)} \cdot f = f + \lambda \frac{P(S)}{Q(S) - \lambda P(S)} \cdot f. \tag{3.5}$$

It turns out that $\frac{P(S)}{Q(S) - \lambda P(S)} \cdot f$, as an element of M_{-1}, belongs to the ring C_{-1}. Let us prove this fact. The function $f(x)$ belongs to the space C_{-1} and $\frac{P(S)}{Q(S) - \lambda P(S)} \cdot f$ can be represented as an element of C_{-1} by using the operational relation (2.19). It follows now from Theorem 2 that the solution (3.5) of equation (3.1) can be represented as an element of the ring C_{-1} and this solution is given by (3.2). □

We consider some interesting examples of integral equations of form (3.1).

Example 1 Let us consider the simple Abel equation of second kind (see Gorenflo and Vessella (1991), Samko et al. (1993)):

$$y(x) - \lambda(I^{\mu}_{0+}y)(x) = f(x), \tag{3.6}$$

where the Riemann-Liouville fractional integration operator I^{μ}_{0+} is given by (1.1), the known function $f(x)$ belongs to the space C_{-1} and the unknown function $y(x)$ is required to lie in the same space.

Using relation (2.12) we reduce this integral equation in the field M_{-1} to the algebraic equation

$$y - \lambda\frac{I}{S} \cdot y = f, \tag{3.7}$$

whose unique solution in the field M_{-1} is given by

$$y = \frac{S}{S-\lambda} \cdot f = f + \lambda\frac{I}{S-\lambda} \cdot f. \tag{3.8}$$

In accordance with relation (2.17) and embedding (2.9) we have the following form of the solution of the Abel integral equation of the second kind (3.6) in the ring C_{-1}:

$$y(x) = f(x) + \lambda\int_0^x (x-t)^{\mu-1}E_{\mu,\mu}(\lambda(x-t)^{\mu})f(t)dt, \tag{3.9}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function (2.18).

The solution (3.9) of Abel integral equation of the second kind (3.6) may be easily reduced to the known form (see Samko et al. (1993)):

$$y(x) = \frac{d}{dx}\int_0^x E_{\mu}(\lambda(x-t)^{\mu})f(t)dt,$$

where

$$E_{\mu}(z) := E_{\mu,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+1)}$$

is the Mittag-Leffler function.

Example 2 Let us consider the special case of the generalized Abel equation of second kind, where $a_i = i,\ 1 \le i \le m$:

$$y(x) - \sum_{i=1}^{m}\lambda_i(I^{i\mu}_{0+}y)(x) = f(x). \tag{3.10}$$

This equation may be solved by using the same method as in Example 1. It follows from relation (2.16) that the integral equation (3.10) is reduced to the algebraic equation in the field M_{-1} of the form

$$y - \sum_{i=1}^{m}\lambda_i\frac{I}{S^i} \cdot y = f.$$

This algebraic equation has a unique solution in the field M_{-1}, namely,

$$y = \frac{S^m}{S^m - \lambda_1 S^{m-1} - \ldots - \lambda_m} \cdot f = f + \frac{\lambda_1 S^{m-1} + \ldots + \lambda_m}{S^m - \lambda_1 S^{m-1} - \ldots - \lambda_m} \cdot f.$$

Let

$$\frac{\lambda_1 S^{m-1} + \ldots + \lambda_m}{S^m - \lambda_1 S^{m-1} - \ldots - \lambda_m} = \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij} \frac{I}{(S - w_i)^j}.$$

Then, in accordance with relation (2.19) and the embedding (2.9) we have the following form of the solution of the integral equation (3.10) in the ring C_{-1}:

$$y(x) = f(x) + \int_0^x R(x-t) f(t) dt,$$

$$R(x) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} d_{ij} x^{\mu j - 1} E^j_{\mu j, \mu}(w_i x^\mu),$$

where the function $E^j_{\mu,\mu j}(w_l x^\mu)$ is defined by (2.20).

Example 3 Let us consider integral equation with the generalized Mittag-Leffler function as the kernel in the space C_{-1}:

$$y(x) - \lambda \int_0^x (x-t)^{\mu-1} E_{\mu,\mu}(\rho (x-t)^\mu) y(t) dt = f(x). \tag{3.11}$$

This integral equation is reduced to the algebraic equation in the field M_{-1} by using the relation (2.17) of the form

$$y - \lambda \frac{I}{S - \rho} \cdot y = f.$$

This equation has a unique solution in the field M_{-1}, which is given by

$$y = \frac{S - \rho}{S - \rho - \lambda} \cdot f = f + \lambda \frac{I}{S - \rho - \lambda} \cdot f.$$

Again using relation (2.17), we obtain the solution of equation (3.11) in the form

$$y(x) = f(x) + \lambda \int_0^x (x-t)^{\mu-1} E_{\mu,\mu}((\rho + \lambda)(x-t)^\mu) f(t) dt.$$

Example 4 Let us consider integral equation (3.11) in the case $\mu = \frac{1}{2}$. We have

$$E_{\frac{1}{2},\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{1}{2} + \frac{k}{2})} \tag{3.12}$$

$$= \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(\frac{1}{2} + k)} + \sum_{k=0}^{\infty} \frac{z^{2k+1}}{\Gamma(1+k)}$$

$$= \frac{1}{\sqrt{\pi}} {}_1F_1(1; \frac{1}{2}; z^2) + ze^{z^2} = \frac{1}{\sqrt{\pi}} + ze^{z^2}(1 + \mathrm{erf}(z)),$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

is the error function. Thus, using (3.12), we can rewrite the integral equation (3.11) in the form

$$y(x) - \lambda(I_{0+}^{\frac{1}{2}} y)(x) - \lambda\rho \int_0^x e^{\rho^2(x-t)} y(t)dt - \lambda\rho \int_0^x e^{\rho^2(x-t)} \mathrm{erf}(\rho\sqrt{x-t}) y(t)dt = f(x).$$

Using the result of Example 3 we obtain its solution as

$$y(x) = f(x) + \lambda(I_{0+}^{\frac{1}{2}} f)(x) + \lambda(\rho+\lambda) \int_0^x e^{(\rho+\lambda)^2(x-t)} f(t)dt$$

$$+\lambda(\rho+\lambda) \int_0^x e^{(\rho+\lambda)^2(x-t)} \mathrm{erf}(\rho\sqrt{x-t}) f(t)dt.$$

Finally, we obtain the exact solution of the generalized Abel integral equation of the second kind.

Theorem 6 *The unique solution of the generalized Abel integral equation of the second kind*

$$y(x) - \sum_{i=1}^{m} \lambda_i (I_{0+}^{\alpha_i \mu} y)(x) = f(x), \;\; \alpha_i > 0, \;\; 1 \le i \le m \tag{3.13}$$

in the space C_{-1} is represented in the form

$$y(x) = f(x) + \int_0^x Q(x-t) f(t)dt, \tag{3.14}$$

where

$$Q(x) = \sum_{j=0}^{m} \lambda_j x^{\mu\alpha_j - 1} E_{(\mu\alpha_1, \ldots, \mu\alpha_m), \mu\alpha_j}(\lambda_1 x^{\mu\alpha_1}, \ldots, \lambda_m x^{\mu\alpha_m}),$$

and the function

$$E_{(\mu\alpha_1, \ldots, \mu\alpha_m), \mu\alpha_j}(\lambda_1 x^{\mu\alpha_1}, \ldots, \lambda_m x^{\mu\alpha_m})$$

is the Mittag-Leffler's function of several variables defined by (2.23).

Proof. Using relation (2.16) and embedding (2.9) we reduce equation (3.13) to the equation in the field of convolutional quotients M_{-1}

$$y - \sum_{i=1}^{m} \frac{\lambda_i}{S^{\alpha_i}} \cdot y = f,$$

whose unique solution in the field M_{-1} is given by

$$y = \frac{S^\alpha}{S^\alpha - \sum_{i=1}^m \lambda_i S^{\alpha - \alpha_i}} \cdot f = \tag{3.15}$$

$$f+\frac{\sum_{i=1}^m \lambda_i S^{-\alpha_i}}{I-\sum_{i=1}^m \lambda_i S^{-\alpha_i}}\cdot f,\ \alpha=\sum_{j=1}^m \alpha_j.$$

We obtain now the solution of equation (3.13) in the ring C_{-1} in the form (3.14) by using the solution in the form (3.15), the operational relation (2.22) and the embedding of the ring C_{-1} in the field M_{-1}.

REFERENCES

M.A. Al-Bassam and Yu.F. Luchko, On generalized fractional calculus and its application to the solution of integro-differential equations, *J. of Frac. Calc.* **7**(1995) (to appear).

Yu.I. Babenko, Non integer differential equation: theory, Paris, 1994.

Yu.I. Babenko, Non integer differential equation in engineering: chemical engineering, Paris, 1994.

R.G. Buschman, Decomposition of an integral operator by use of Mikusinski calculus, *SIAM J. Math. Anal.* **3**(1972), 83-85.

I.H. Dimovski, Operational calculus for a class of differential operators, *C. R. Acad. Bulgare Sci.* **19**(1966), 1111-1114.

I.H. Dimovski, *Convolutional calculus*, Bulgarian Academy of Sciences, Sofia, 1982.

V.A. Ditkin and A.P. Prudnikov, The theory of operational calculus, which is generated by the Bessel equation, *Zh. Vychisl. Mat. i Mat. Fiz.* **3**(1963), 223-238; English translation in U.S.S.R. Comput. Math. and Math. Phys. **3**(1963), 296-315.

A. Erdelyi, *Operational calculus and generalized functions*, Holt, Rinehart and Winston, New York, 1962.

R. Gorenflo and S. Vessella, *Abel integral equations. Analysis and applications*, Springer-Verlag, Berlin, Heidelberg, New York, and Tokyo, 1991.

R. Gorenflo and R. Rutman, On ultraslow and intermediate processes, Proceedings of the Workshop for Transform Methods and Special Functions, Sofia, Bulgaria, 12-17 August 1994. Science Culture Technology Publishing (SCTP), Singapore, 1995, 61-81.

Yu.F. Luchko, *Some operational relations for the H-transforms and their applications*, Ph.D. thesis, Minsk, 1993. In Russian.

Yu.F. Luchko and H.M.Srivastava, The exact solution of certain differential equations of fractional order by using operational calculus, *Comput. Math. Appl.* (to appear).

Yu.F. Luchko and S.B. Yakubovich, Operational method of solution of some classes of integro-differential equations, *Differentzialnye uravneniya* **30**(1994), 269-280. In Russian.

Yu.F. Luchko and S.B. Yakubovich, Operational calculus for the generalized fractional differential operator and applications, *Math. Balkanika* **7**(1993), 119-130.

F. Mainardi, On the initial value problem for the fractional diffusion-wave equation, *Proceedinds of VII WASCOM*, Bologna, Italy, 4-9 October, 1993, World Scientific, Singapore, 1994 (to appear).

F. Mainardi, Fractional relaxation and fractional diffusion equations, mathematical aspects, *Proceedings of the 12-th IMACS World Congress*, Atalanta, USA, 11-15 July, 1994.
N.A. Meller, On some applications of operational calculus to problems of analysis, *Zh. Vychisl. Mat. i Mat. Fiz.* **3**(1963), 73-89. In Russian.
M.W. Michalski, On a certain differential equation of non-integer order, *Zeitschrift für Analysis und ihre Anwendungen* **10**(1991), 205-210.
M.W. Michalski, *Derivatives of noninteger order and their applications*, Dissertationes Mathematicae, CCCXXVIII, Polska Akademia Nauk, Institut Matematyczny, Warszawa, 1993.
J. Mikusinski, *Operational calculus*, Pergamon Press (International Series of Monographs on Pure and Applied Mathematics, Vol. **8**), New York, London, Paris, and Los Angeles, 1959.
I. Podlubny, *The Laplace transform method for linear differential equations of the fractional order*, Slovak Academy of Sciences, Institute of Experimental Physics, 1994.
J. Rodrigues, Operational calculus for the generalized Bessel operator, *Serdica* **15**(1989), 179-186.
S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach Science Publishers, New York and London, 1993. Translated from the Russian and revised.
S.B. Yakubovich and Yu.F. Luchko, *The hypergeometric approach to integral transforms and convolutions*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1994.
K. Yosida, *Operational calculus: a theory of hyperfunctions*, Springer-Verlag, Berlin, Heidelberg, New York, and Tokyo, 1984.

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