Lattice–based D–optimum design for Fourier regression

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Abstract

A theory of optimal orthogonal fractions is developed for Fourier regression models using integer lattice designs. These provide alternatives to simple grids (product designs) in the case when specified main effects and interaction terms are required to be analysed. The challenge is to obtain sample sizes which are polynomial in the dimension rather than exponential. This is achieved for certain models with special algorithms based on both special algebraic generation and more direct sequential search, which will be called linear.

1 Introduction

In the paper by Bates, Buck, Riccomagno and Wynn (1995) the simple fact was mentioned that certain integer lattice designs (see below for the definition) are D–optimal in the sense of Kiefer and Wolfowitz for Fourier regression models. It has been known for many years that integer equally spaced

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grids (product designs) have this property. The analogy is with polynomial regression in that if only a limited number of, or in fact no, interactions are required to be estimated then we can reduce the size of the experiment by using a fraction. Here the lattice will play the role of fractions in the polynomial theory. The alias theory turns out to be radically different with the cyclic group playing an important role via the harmonic nature of the theory. Briefly the lattice structure is able

to turn a model in high dimensions into a one–dimensional model.

2 Fourier models

2.1 Notation

We work with, what we call, *complete models* and describe them in the following format taken from the convention for the syntax of computer languages

$$ClassModel(Dimension; Order, < Order >; Number\, of\ Interactions)$$

where $Dimension \geq Number\, of\ Interactions \geq 1$. The first equality holds for a full interaction model and the second for an additive model.

2.2 One–dimensional model

For the one–dimensional Fourier regression model $F(1; m; 1)$

$$E(Y(x)) = \mu(x) = \theta_0 + \sqrt{2}\sum_{r=1}^{m} \sin(2\pi r x)\theta_r + \sqrt{2}\sum_{r=1}^{m} \cos(2\pi r x)\phi_r \qquad (1)$$

$x \in [0, 1)$, the equally spaced design points on an equidistant grid with at least $2m + 1$ supporting points is D–optimum in the sense of Kiefer and Wolfowitz (see Kiefer, 1959, and see also Pukelsheim, 1993, for a recent text). The uniform design with $2m + 1$ supporting points has minimal support, i.e. there are exactly as many supporting points in the design as parameters in the model. It is denoted by ξ^*. It is also A–, E– and integrated mean square error (IMSE) optimum (see Section 3.2).

2.3 Two–dimensional models

Hoel (1965) shows that *product designs* are D–optimal for product models. Thus, if ξ_1 is a D–optimum design measure for a linear model $E\left(Y(x_1)\right) = \sum_{i=0}^{m_1} \theta_i f_i(x_1)$ on a design space X_1 and similarly ξ_2 is D–optimum for a model $E\left(Y(x_2)\right) = \sum_{i=0}^{m_2} \phi_i g_i(x_2)$ on a space X_2, then the product measure $\xi_1 \times \xi_2$ is D–optimum for the linear model whose terms are of the form $f_i(x_1)g_j(x_2)$ on the space $X_1 \times X_2$. The same is true for models in higher dimensions and with no interactions if $f_0 = 1$ and $g_0 = 1$. For the case of Fourier regression this last restriction is not necessary (see Schwabe, 1994b). The same result for the other criteria has been proved e.g. by Rafajłowicz and Myszka (1992) and Schwabe (1994).

The *additive model* $F(2; m_1, m_2; 1)$ is expressed by

$$\begin{aligned} E(Y(x_1, x_2)) &= \mu(x_1, x_2) \\ &= \theta_0 + \sqrt{2}\sum_{r=1}^{m_1} \sin(2\pi r x_1)\theta_{1,r} + \sqrt{2}\sum_{r=1}^{m_1} \cos(2\pi r x_1)\phi_{1,r} \\ &\quad + \sqrt{2}\sum_{s=1}^{m_2} \sin(2\pi s x_2)\theta_{2,s} + \sqrt{2}\sum_{s=1}^{m_2} \cos(2\pi s x_2)\phi_{2,s} \end{aligned} \tag{2}$$

Let ξ_1 and ξ_2 be D–optimum in the marginal models (1), then $\xi_1 \times \xi_2$ is D–optimum for the additive model. Unfortunately, $\xi_1 \times \xi_2$ has at least $(2m_1 + 1)(2m_2 + 1) >> 2m_1 + 2m_2 + 1$ supporting points.

Using the standard formulae, which form an orthogonal transformation not affecting the D–, A–, E– and IMSE criteria, the product Fourier model $F(2; m_1, m_2; 2)$ can be written in more transparent form. Thus

$$\begin{aligned} E(Y(x_1, x_2)) &= \mu(x_1, x_2) \\ &= \theta_0 + \sqrt{2}\sum_{r=1}^{m_1} \sin(2\pi r x_1)\theta_{1,r} + \sqrt{2}\sum_{r=1}^{m_1} \cos(2\pi r x_1)\phi_{1,r} \\ &\quad + \sqrt{2}\sum_{s=1}^{m_2} \sin(2\pi s x_2)\theta_{2,s} + \sqrt{2}\sum_{s=1}^{m_2} \cos(2\pi s x_2)\phi_{2,s} \\ &\quad + 2\sum_{r=1}^{m_1}\sum_{s=1}^{m_2} \sin(2\pi r x_1)\sin(2\pi s x_2)\theta_{rs} \\ &\quad + 2\sum_{r=1}^{m_1}\sum_{s=1}^{m_2} \sin(2\pi r x_1)\cos(2\pi s x_2)\lambda_{rs} \end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\cos(2\pi r x_1)\sin(2\pi s x_2)\tau_{rs} \\
&+ 2\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\cos(2\pi r x_1)\cos(2\pi s x_2)\phi_{rs}
\end{aligned} \tag{3}
$$

is written as

$$
\begin{aligned}
&E(Y(x_1,x_2)) = \\
&\quad \theta_0 + \sqrt{2}\sum_{r=1}^{m_1}\sin(2\pi r x_1)\theta_{1,r} + \sqrt{2}\sum_{r=1}^{m_1}\cos(2\pi r x_1)\phi_{1,r} \\
&\quad + \sqrt{2}\sum_{s=1}^{m_2}\sin(2\pi s x_2)\theta_{2,s} + \sqrt{2}\sum_{s=1}^{m_2}\cos(2\pi s x_2)\phi_{2,s} \\
&\quad + \sqrt{2}\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\sin(2\pi(r x_1 + s x_2))\theta^{+}_{rs} \\
&\quad + \sqrt{2}\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\sin(2\pi(r x_1 - s x_2))\theta^{-}_{rs} \\
&\quad + \sqrt{2}\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\cos(2\pi(r x_1 + s x_2))\phi^{+}_{rs} \\
&\quad + \sqrt{2}\sum_{r=1}^{m_1}\sum_{s=1}^{m_2}\cos(2\pi(r x_1 - s x_2))\phi^{-}_{rs}
\end{aligned} \tag{4}
$$

If ξ_1^* and ξ_2^* have minimal support then $\xi_1^* \times \xi_2^*$ also has minimal support.

2.4 Higher dimensional models

All of section 1.3 generalises to d–dimensions. Thus the additive model, $F(d; m_1, \ldots, m_d; 1)$ is expressed by the formula

$$
\begin{aligned}
E(Y(x_1,...,x_d)) &= \mu(x_1,...,x_d) \\
&= \theta_0 + \sqrt{2}\sum_{k=1}^{d}\sum_{r_k=1}^{m_k}\sin(2\pi r_k x_k)\theta_{k,r_k} + \sqrt{2}\sum_{k=1}^{d}\sum_{r_k=1}^{m_k}\cos(2\pi r_k x_k)\phi_{k,r_k}
\end{aligned} \tag{5}
$$

If we introduce the sets $A_\ell = \{\alpha \in \{-1,1\}^\ell; \alpha_1 = 1\}$ of all ℓ–dimensional multi–indices from $\{-1,1\}$ with unit first entry, we can write the *product type model* (complete interactions) $F(d; m_1, \ldots, m_d; d)$ in the following compact form:

$$
\begin{aligned}
&E(Y(x_1,...,x_d)) = \mu(x_1,...,x_d) \\
&= \quad \theta_0 + \sqrt{2}\sum_{\ell=1}^{d}\sum_{k_1<...<k_\ell}\sum_{r_{k_1}=1}^{m_{k_1}}...\sum_{r_{k_\ell}=1}^{m_{k_\ell}}\sum_{\alpha\in A_\ell}\sin(2\pi(\alpha_1 r_{k_1}x_{k_1}+...+\alpha_\ell r_{k_\ell}x_{k_\ell})) \\
&\qquad\qquad \times\, \theta^{\alpha}_{k_1...k_\ell,r_{k_1}...r_{k_\ell}} \\
&\qquad + \sqrt{2}\sum_{\ell=1}^{d}\sum_{k_1<...<k_\ell}\sum_{r_{k_1}=1}^{m_{k_1}}...\sum_{r_{k_\ell}=1}^{m_{k_\ell}}\sum_{\alpha\in A_\ell}\cos(2\pi(\alpha_1 r_{k_1}x_{k_1}+...+\alpha_\ell r_{k_\ell}x_{k_\ell})) \\
&\qquad\qquad \times\, \phi^{\alpha}_{k_1...k_\ell,r_{k_1}...r_{k_\ell}} \qquad (6)
\end{aligned}
$$

The complete 2–factor interactions model, F(d; $m_1, \ldots, m_d$; 2) is

$$
\begin{aligned}
&E(Y(x_1,...,x_d)) = \mu(x_1,...,x_d) \\
&= \quad \theta_0 + \sqrt{2}\sum_{k=1}^{d}\sum_{r_k=1}^{m_k}\sin(2\pi r_k x_k)\theta_{k,r_k} + \sqrt{2}\sum_{k=1}^{d}\sum_{r_k=1}^{m_k}\cos(2\pi r_k x_k)\phi_{k,r_k} \\
&\qquad + \sqrt{2}\sum_{k=1}^{d}\sum_{\ell=k+1}^{d}\sum_{r=1}^{m_k}\sum_{s=1}^{m_\ell}\sin(2\pi(rx_k+sx_\ell))\theta^{+}_{k\ell,rs} \\
&\qquad + \sqrt{2}\sum_{k=1}^{d}\sum_{\ell=k+1}^{d}\sum_{r=1}^{m_k}\sum_{s=1}^{m_\ell}\sin(2\pi(rx_k-sx_\ell))\theta^{-}_{k\ell,rs} \\
&\qquad + \sqrt{2}\sum_{k=1}^{d}\sum_{\ell=k+1}^{d}\sum_{r=1}^{m_k}\sum_{s=1}^{m_\ell}\cos(2\pi(rx_k+sx_\ell))\phi^{+}_{k\ell,rs} \\
&\qquad + \sqrt{2}\sum_{k=1}^{d}\sum_{\ell=k+1}^{d}\sum_{r=1}^{m_k}\sum_{s=1}^{m_\ell}\cos(2\pi(rx_k-sx_\ell))\phi^{-}_{k\ell,rs} \qquad (7)
\end{aligned}
$$

and the compact form for the complete M–factor interactions model, $F(d; m_1, \ldots, m_d; M)$ is

$$
\begin{aligned}
&E(Y(x_1,...,x_d)) = \mu(x_1,...,x_d) \\
&= \quad \theta_0 + \sqrt{2}\sum_{\ell=1}^{M}\sum_{k_1<...<k_\ell}\sum_{r_{k_1}=1}^{m_{k_1}}...\sum_{r_{k_\ell}=1}^{m_{k_\ell}}\sum_{\alpha\in A_\ell}\sin(2\pi(\alpha_1 r_{k_1}x_{k_1}+...+\alpha_\ell r_{k_\ell}x_{k_\ell}))
\end{aligned}
$$

$$
\times\, \theta^{\alpha}_{k_1 \dots k_\ell, r_{k_1} \dots r_{k_\ell}}
$$

$$
+ \sqrt{2} \sum_{\ell=1}^{M} \sum_{k_1 < \dots < k_\ell} \sum_{r_{k_1}=1}^{m_{k_1}} \dots \sum_{r_{k_\ell}=1}^{m_{k_\ell}} \sum_{\alpha \in A_\ell} \cos(2\pi(\alpha_1 r_{k_1} x_{k_1} + \dots + \alpha_\ell r_{k_\ell} x_{k_\ell}))
$$

$$
\times\, \phi^{\alpha}_{k_1 \dots k_\ell, r_{k_1} \dots r_{k_\ell}} \qquad (8)
$$

Again if the designs ξ_k are D–optimum in the marginal models (1) then the product type design $\xi_1 \times \dots \times \xi_d$ is D–optimum for all the above models (see for example Schwabe, 1993,1994b). The number of supporting points for $\xi_1 \times \dots \times \xi_d$ is at least $\prod_{k=1}^{d}(2m_k + 1)$ which is attained by a uniform design on a $(2m_1 + 1) \times \dots \times (2m_d + 1)$ equidistant grid in each component. The number of supporting points is much larger than the number of parameters, in general. This design has minimal support only in the complete interaction model (6). Note that for any of the present Fourier models the information matrix equals the identity matrix.

Model	Number of parameters
$F(1; m; 1)$	$2m + 1$
$F(2; m_1, m_2; 1)$	$2(m_1 + m_2) + 1$
$F(2; m_1, m_2; 2)$	$(2m_1 + 1)(2m_2 + 1)$
$F(d; m_1, \dots, m_d; 1)$	$1 + 2\sum_{k=1}^{d} m_k$
$F(d; m_1, \dots, m_d; d)$	$\prod_{k=1}^{d}(2m_k + 1)$
$F(d; m_1, \dots, m_d; 2)$	$1 + 2\sum_{k=1}^{d} m_k + 4\sum_{j=1}^{d-1} m_j \sum_{k=j+1}^{d} m_k$
$F(d; m_1, \dots, m_d; M)$	$1 + 2\sum_{k=1}^{d} m_k + 2^2\sum_{j=1}^{d-1} m_j \sum_{k=j+1}^{d} m_k + \cdots$ $\cdots + 2^M \sum_{j_1=1}^{d-M+1} m_{j_1} \cdots m_{j_{M-1}} \sum_{j_M=j_{M-1}+1}^{d} m_{j_M}$

3 Lattice designs and optimality

The main purpose of the paper is to show that optimality can be achieved using lattice designs for complete Fourier models without having to take product designs. They have a structure which is cyclic, but for which the aliasing theory is very different from classical polynomial models. The new theory can be considered as a theory of orthogonal fractions for the Fourier models. We shall give examples of orthogonal (optimum) families of designs or rather optimum *design–model* pairs.

3.1 A first single generator design

Let us consider $F(2; m_1, m_2; 2)$. As already pointed out a particular D–optimum design is given by $\xi_1^* \times \xi_2^*$ which is uniform on the $(2m_1+1) \times (2m_2+1)$ grid $\{(\frac{i}{2m_1+1}, \frac{j}{2m_2+1}); i = 0, ..., 2m_1, j = 0, ..., 2m_2\}$.

Fix ξ_2^* and look for different D–optimum designs in the following way. For any supporting point x_2 of ξ_2^* we can take any D–optimum marginal design $\xi_{1|x_2}^*$ without changing the information matrix. Now we can consider $\xi_{1|x_2}^*$ as a conditional design (conditioned on x_2). Further explanation of this concept see Cook and Thibodeau (1980) and, more generally, Kurotschka (1984). Then $\xi_{1|\cdot}^*$ is a conditional probability measure and $\xi_{1|x_2}^* \times \xi_2^*$ defines a new design in $[0,1)^d$ which is itself D–optimum.

In particular, let $\xi_{1|x_2}^*$ given $x_2 = \frac{j}{2m_2+1}$ be uniform on $\{\frac{i}{2m_1+1} + \frac{j}{(2m_1+1)(2m_2+1)}; i = 0, ..., 2m_1\}$ then $\xi^* = \xi_{1|x_2}^* \times \xi_2^*$ is uniform on

$$\{(\frac{1}{2m_1+1}(i + \frac{j}{2m_2+1}), \frac{j}{2m_2+1}); i = 0, ..., 2m_1, j = 0, ..., 2m_2\}$$

After a rearrangement of the design points we see that ξ^* is uniform on

$$\{(\frac{k}{n}, \frac{k(2m_1+1) \bmod n}{n}); k = 0, ..., n-1\}$$

where $n = (2m_1+1)(2m_2+1)$ is the number of supporting points, coinciding with the number of parameters. This concept can be generalized to d dimensions.

3.2 Lattice designs

We consider only lattices generated by a single integer generator which therefore are one–dimensional objects. Let $(g_1, \ldots, g_d)$ be a d–dimensional vector with positive integer components and n a positive integer. The vector generates a finite number of line segments in the d–dimensional hyper–cube $[0,1)^d$, i. e. the set $\{(tg_1 \bmod 1, ..., tg_d \bmod 1)\,;\, t \in \mathbb{R}\}$ consists of a finite number of line segments intersecting $[0,1)^d$. The generated designs will be supported on these line segments and are hereafter referred to as lattice designs.

The lattice design generated by $(g_1, \ldots, g_d)$ and with n supporting points is the uniform design supported on the grid

$$\begin{aligned} &\left\{\left(\frac{k}{n}g_1 \bmod 1, ..., \frac{k}{n}g_d \bmod 1\right)\,;\, k = 0, ..., n-1\right\} \\ = \quad &\left\{\left(\frac{kg_1 \bmod n}{n}, ..., \frac{kg_d \bmod n}{n}\right)\,;\, k = 0, ..., n-1\right\} \end{aligned}$$

The cyclic structure of a lattice design suggests some affinity with trigonometric functions. Indeed, because of the periodicity of the trigonometric functions we have

$$\sin(2\pi r_i \frac{kg_i \bmod n}{n}) = \sin(2\pi r_i \frac{kg_i}{n}) \text{ and } \cos(2\pi r_i \frac{kg_i \bmod n}{n}) = \cos(2\pi r_i \frac{kg_i}{n})$$

More generally, we have for all $\alpha \in A_\ell$

$$\begin{aligned} &\sin\left(2\pi\left(\alpha_1 r_{k_1}\frac{kg_{k_1} \bmod n}{n} + ... + \alpha_\ell r_{k_\ell}\frac{kg_{k_\ell} \bmod n}{n}\right)\right) \\ = \quad &\sin\left(2\pi\left(\alpha_1 r_{k_1}\frac{kg_{k_1}}{n} + ... + \alpha_\ell r_{k_\ell}\frac{kg_{k_\ell}}{n}\right)\right) \\ &\cos\left(2\pi\left(\alpha_1 r_{k_1}\frac{kg_{k_1} \bmod n}{n} + ... + \alpha_\ell r_{k_\ell}\frac{kg_{k_\ell} \bmod n}{n}\right)\right) \\ = \quad &\cos\left(2\pi\left(\alpha_1 r_{k_1}\frac{kg_{k_1}}{n} + ... + \alpha_\ell r_{k_\ell}\frac{kg_{k_\ell}}{n}\right)\right) \end{aligned}$$

If g_1 and n are mutually prime we may assume without loss of generality that $g_1 = 1$ (see Niederreiter, 1992 page 126). This can be achieved by a rearrangement of the generated design points. Note that the design described in Section 1.2 for $F(1; m; 1)$ is a lattice–design generated by $g = 1$ with at least $2m + 1$ supporting points.

4 General ideas and main statements

4.1 General ideas

A key observation in this paper is that a model with terms in higher dimensions can be "tricked" into a model in one–dimension by exploiting the structure of the lattice. We explain this with a two dimensional example. Consider the model $F(2; 1, 1; 2)$

and the lattice with generator $(1, 5)$ and $n = 13$ supporting points.

The four ways in which the variables are "combined" are

$$x_1, \qquad x_2, \qquad x_1 + x_2, \qquad x_1 - x_2$$

Associating x_1 to $g_1 = 1$, x_2 to $g_2 = 5$ and multiplying by $k \in \{0, \ldots, 12\}$ we generate the column vectors

0	0	0	0
1	5	6	−4
2	10	12	−8
3	15	18	−12
4	20	24	−16
5	25	30	−20
6	30	36	−24
7	35	42	−28
8	40	48	−32
9	45	54	−36
10	50	60	−40
11	55	66	−44
12	60	72	−48

Since the lattice is generated by a $(\bmod 13)$ reduction these columns can be replaced by

0	0	0	0
1	5	6	9
2	10	12	5
3	2	5	1
4	7	11	10
5	12	4	6
6	4	10	2
7	9	3	11
8	1	9	7
9	6	2	3
10	11	8	12
11	3	1	8
12	8	7	4

Identical columns would be produced by a *one–dimensional model* with only the terms $r = 1$, $r = 5$, $r = 6$, $r = 9$ and equally spaced observations, i.e. the model $F(2; 1, 1; 2)$ is converted into the 1-dimensional non–complete "model" with regression function

$$\begin{aligned}(\, 1, \quad & \cos(2\pi x), \cos(2\pi 5x), \cos(2\pi 6x), \cos(2\pi 9x), \\ & \sin(2\pi x), \sin(2\pi 5x), \sin(2\pi 6x), \sin(2\pi 9x) \,)\end{aligned}$$

For this last model the uniform design with 13 equispaced supporting points is D–optimum. Since the information matrix for $F(2; 1, 1; 2)$ evaluated in the $(1, 5)$ design is equal to the information matrix for the non–complete model evaluated in the 13 points uniform design, we conclude that the given lattice design is D–optimum for $F(2; 1, 1; 2)$. On the lattice the function $\cos(2\pi(-4)z)$ is equal to $\cos(2\pi 9z)$ and hence to $\cos(2\pi 4z)$ and the function $\sin(2\pi 9z)$ is equal to $-\sin(2\pi 4z)$. As the minus implies an orthogonal transformation not affecting the D–criterion -4 $(9 \pmod 1)3)$ can be replaced by 4.

Moreover, notice that in considering the array $(1, 5, 6, 9)$ it is critical that in the augmented array $(0\,1,\ 5,\ 6,\ 9,\ 13 - 1 = 12,\ 13 - 5 = 8,\ 13 - 6 = 7,\ 13 - 9 = 4)$ there are no repetitions. Note that this fact causes the trigonometric functions to be orthogonal on the lattice. We could alternatively use the negative (inverse) elements in the cyclic group $\mathcal{G}_{13}$ giving the sequence $(-6,\ -5,\ -4,\ -1,\ 1,\ 4,\ 5,\ 6)$

We formalise this technique in the theorems of the next subsection.

4.2 Statements

Theorem 1 *For the model $F(1;m;1)$ the lattice design generated by g (one–dimensional vector) and with n support points ($n \geq 2m+1$) is D–optimum if and only if $2m+1$ is the cardinality of*

$$g\mathcal{N} \qquad \textit{where} \quad \mathcal{N} := \{r;\ r = -m, ..., -1, 0, 1, ..., m\}$$

considered as a subset of the cyclic group $\mathcal{G}_n = (\{0, 1, ..., n-1\}, +)$.

Proof This follows from the observation that the appropriate columns of the information matrix are orthogonal. Hence the information matrix becomes the identity which is known to be the information matrix belonging to a D–optimum design. $\square$

We remark that the condition in the last theorem is satisfied if and only if all the parameters of the model are identifiable, that is estimable.

If all the elements in the array are mutually different in $\mathbb{Z}$, they are also different in $\mathcal{G}_n$ where $n = 2mg + 1$, i. e. twice the maximum element in the array itself plus 1. While this is an upper bound for the sample size, a lower bound is given by the number of parameters in the model. These rules are not limited to the one–dimensional situation and hold in general (see later).

The next result allows us to prove similar conditions for more complicated models in which not all of the lower order functions are present straightforwardly from Theorem 1.

Proposition 1 *For the non–complete one–dimensional model of m–th order*

$$E(Y(x)) = \mu(x) = \theta_0 + \sqrt{2}\sum_{j=1}^{q} \sin(2\pi r_j x)\theta_{r_j} + \sqrt{2}\sum_{j=1}^{q} \cos(2\pi r_j x)\phi_{r_j} \qquad (9)$$

where $r_j \neq r_k$ for all $j, k = 1, \ldots, q$, $q \leq m$ and $r_j \in \{1, \ldots, m\}$ $\forall j = 1, \ldots, q$, the lattice design generated by g and with n support points is D–optimum if and only if $2q+1$ is the cardinality of the set

$$g\mathcal{N} \qquad \textit{with} \quad \mathcal{N} := \{r_j;\ j = -q, ..., -1, 0, 1, ..., q\}$$

considered as a subset of the cyclic group $\mathcal{G}_n = (\{0, 1, ..., n-1\}, +)$.

The above result follows again immediately from the orthogonality of the trigonometric functions on the given lattice. The first strong result is that we can have D–optimum lattice designs for the model (4) $F(2; m_1, m_2; 2)$.

Theorem 2 *For the model $F(2; m_1, m_2; 2)$ the lattice design generated by (g_1, g_2) and with n support points is D–optimum if and only if all sums*

$$rg_1 + sg_2; \quad r \in \mathcal{N}_1,\, s \in \mathcal{N}_2$$

are different to each other (in the cyclic group $\mathcal{G}_n$).

We omit the proofs of this and the next results, which can be deduced directly from Theorem 1 by imbedding the multi–dimensional model into a non–complete one–dimensional one of higher order. More generally we state

Theorem 3 *For the model $F(d; m_1, \ldots, m_d; d)$ the lattice design generated by $(g_1, \ldots, g_d)$ and with n support points is D–optimum if and only if all the sums*

$$r_1g_1 + ... + r_dg_d; \quad r_1 \in \mathcal{N}_1, ..., r_d \in \mathcal{N}_d$$

are different to each other (in the cyclic group $\mathcal{G}_n$).

For an M–factor interactions model with $M \leq d$, we have to be careful with zeros and suppress their multiple influence in some way. So let

$$\mathcal{N}_k' = \mathcal{N}_k \setminus \{0\} = \{r;\, r = -m_k, ..., -1, 1, ..., m_k\} \qquad k \in \{1, \ldots, d\}$$

Theorem 4 *For the model $F(d; m_1, \ldots, m_d; M)$ the lattice design generated by $(g_1, ..., g_d)$ and with n support points is D–optimum if and only if the members of the following array are different to each other (in the cyclic group $\mathcal{G}_n$)*

$$\begin{array}{lll}
0, & & \\
r_kg_k, & r_k \in \mathcal{N}_k', & k = 1, ..., d \\
r_kg_k + r_\ell g_\ell, & r_k \in \mathcal{N}_k', r_\ell \in \mathcal{N}_\ell', & 1 \leq k < \ell \leq d \\
\vdots & & \\
r_{k_1}g_{k_1} + ... + r_{k_m}g_{k_m}, & r_{k_1} \in \mathcal{N}_{k_1}', ..., r_{k_m} \in \mathcal{N}_{k_m}', & 1 \leq k_1 < ... < k_m \leq d \\
\vdots & & \\
r_{k_1}g_{k_1} + ... + r_{k_M}g_{k_M}, & r_{k_1} \in \mathcal{N}_{k_1}', ..., r_{k_M} \in \mathcal{N}_{k_M}', & 1 \leq k_1 < ... < k_M \leq d.
\end{array}$$

In particular, for additive models there are only two rows in the array the members of which have to be different. For a 2–factor interaction model there are only three rows, and for complete interactions model we recover the conditions of Theorem 3.

Sometimes it is necessary to investigate situations in which only the main effects are involved.

Theorem 5 *For the parameters associated with the main effects for the model $F(d; m_1, \ldots, m_d; M)$ the lattice design generated by $(g_1, \ldots, g_d)$ and with n support points is D–optimum if and only if the members in the first two rows*

$$\begin{array}{l} 0, \\ r_k g_k, \quad r_k = -m_k, ..., -1, 1, ..., m_k, \quad k = 1, ..., d \end{array}$$

of the array

$$\begin{array}{lll}
0, & & \\
r_k g_k, & r_k = -m_k, ..., -1, 1, ..., m_k, & k = 1, ..., d \\
r_k g_k + r_\ell g_\ell, & r_k = -m_k, ..., -1, 1, ..., m_k, & \\
 & r_\ell = -m_\ell, ..., -1, 1, ..., m_\ell, & 1 \leq k < \ell \leq d \\
\vdots & & \\
r_{k_1} g_{k_1} + ... + r_{k_m} g_{k_m}, & r_{k_1} \in \mathcal{N}'_{k_1}, ..., r_{k_m} \in \mathcal{N}'_{k_m}, & 1 \leq k_1 < ... < k_m \leq d \\
\vdots & & \\
r_{k_1} g_{k_1} + ... + r_{k_M} g_{k_M}, & r_{k_1} \in \mathcal{N}'_{k_1}, ..., r_{k_M} \in \mathcal{N}'_{k_M}, & 1 \leq k_1 < ... < k_M \leq d
\end{array}$$

are different to each other (in the cyclic group $\mathcal{G}_n$) and, additionally, the members in the others rows are all different to the members in the first two rows (in $\mathcal{G}_n$).

This can be generalised if we are interested in the effects up to a certain order, say S.

Theorem 6 *To identify the parameters up to the order S for the model $F(d; m_1, \ldots, m_d; M)$ the lattice design generated by $(g_1, \ldots, g_d)$ and with n support points is D–optimum if and only if the members in the first $S+1$ rows of the array in Theorem 5 are different to each other (in the cyclic group $\mathcal{G}_n$) and, additionally, the members in the last rows are all different from the members in the first $S+1$ rows in $\mathcal{G}_n$.*

We have obtained the results in Section 5 as follows: first we start from a generator and check whether all the members in the above array are different in the set of all positive integers. If this is true then they are also different in the cyclic group $\mathcal{G}_n$ with $n = 2N_1 + 1$ where N_1 is the largest member in the array. We call this choice of n the *Upper Law*.

After that we look for smaller sizes n, maybe even minimal, such that the members remain different within the corresponding cyclic group $\mathcal{G}_n$. For example, a closer investigation of the array being considered for additive models shows that there are large gaps already at the beginning. To reduce the size we can match the larger members of the array into suitable gaps in the upper half. Because of the symmetry of the array with respect to zero the corresponding negative members will automatically fit into gaps in the lower half. Usually the largest gap will be between the maximum member N_1 and the next to maximum one, say N_2, and usually $N_1 > N_2 + 1$. So a size $N_1 + N_2 + 1$ is suitable and gives a reduction of, at most, g_d. A similar rule holds for the other models. We refer to it with the terms *Generalised Upper Law*.

If we are interested only in the main effects we may proceed as follows. If N_3 is the largest member in the first two rows of the array associated with the main effects and N_1 is the largest member in the whole array a straightforward *Main Generalised Upper Law* states that a size $N_1 + N_3 + 1$ is suitable. Obviously any size larger than the one given by the *Upper Law* resp. the *Main Generalised Upper Law* is suitable.

We note that because of the diagonal structure of the information matrix the D–optimum designs obtained are also optimum with respect to any (convex) criterion based on the eigenvalues of the information matrix. In particular, they are also A– and E–optimum, i.e. they minimize the averaged variance for the single parameters respect to the largest eigenvalue of the corresponding covariance matrix. Moreover, those designs which are D–optimum for the whole parameter vector result in a minimum value for the integrated mean squared error

$$\int_0^1 \dots \int_0^1 (\hat{\mu}(t_1, \dots, t_d) - \mu(t_1, \dots, t_d))^2 \, dt_d \dots dt_1$$

where $\hat{\mu}$ is the best linear unbiased predictor of the response function μ.

4.3 Increasing the dimension

In order to guarantee both identifiability and D–optimality, at least the entries $g_1, \ldots, g_d$ of a generator $(g_1, \ldots, g_d)$ have to be different for every Fourier model. In the next section we search for ordered generators such that $g_1 < \ldots < g_d$.

Below we list the different generators prototypes that will be discussed.

1. Power generator and generalised power generator
$$(g_1, \ldots, g_d) = (1, p_1, p_1 p_2, \ldots, \prod_{k=1}^{d-1} p_k)$$
where the p_k are defined case by case
2. Fibonacci type generator i.e. the g_d are defined by a non–homogeneous linear recursion
3. Linear generator
4. One–step or 'myopic' generator which considers as the next generator the smallest integer such that the condition in the suitable theorem of the previous section is satisfied.

4.4 An algorithm for the one–step strategy

We present an algorithm for an iterative sequence of generator components obtained by the one–step strategy from the theorems. This algorithm reduces the time–consuming search to a minimum. Indeed we have only to look for the smallest positive integer not belonging to a given set. Also, this algorithm gives a reasonable bound on the increase of the generator components and, hence, of the required size by the *Upper Law* as the dimension d increases.

For the generator $(g_1, \ldots, g_d)$ denote by $\mathcal{A}_d^{(M)}$ the set of all members in the array

$$
\begin{array}{lll}
0, & & \\
r_k g_k, & r_k \in \mathcal{N}'_k, & k = 1, ..., d \\
r_k g_k + r_\ell g_\ell, & r_k \in \mathcal{N}'_k, r_\ell \in \mathcal{N}'_\ell, & 1 \le k < \ell \le d \\
\vdots & & \\
r_{k_1} g_{k_1} + ... + r_{k_s} g_{k_s}, & r_{k_1} \in \mathcal{N}'_{k_1}, ..., r_{k_s} \in \mathcal{N}'_{k_s}, & 1 \le k_1 < ... < k_s \le d \\
\vdots & & \\
r_{k_1} g_{k_1} + ... + r_{k_S} g_{k_M}, & r_{k_1} \in \mathcal{N}'_{k_1}, ..., r_{k_M} \in \mathcal{N}'_{k_M}, & 1 \le k_1 < ... < k_M \le d
\end{array}
$$

associated with a d–factor model with M–factor interactions. Further assume that we are interested in all the parameters up to the S–factor interactions, $1 \le S \le M$, and that the generator $(g_1, ..., g_d)$ is suitable, i. e. the members in the first $S+1$ rows of the array (including the first row containing only 0) are all different to each other, and the members in the remaining last $M-S$ rows are different from those in the first $S+1$ rows, but not necessarily different to each other (within the set $\mathbb{Z}$ of all integers). (Note that $\mathcal{A}_d^{(S)}$ collects the members of the first $S+1$ rows of the array and that $\mathcal{A}_d^{(0)} = \{0\}$.)

The general algorithm follows

Step 1 Start with $d = 1$ and $g_1 = 1$. Then (g_1) is suitable and $\mathcal{A}_1^{(S)} = \{-m_1, ..., -1, 0, 1, ..., m_1\}$, $1 \le S \le M$.

Step 2 If the generator $(g_1, ..., g_d)$ is suitable, then determine the set of all sums $\mathcal{A}_d^+ = \mathcal{A}_d^{(M-1)} + \mathcal{A}_d^{(S)}$ of members of the first $M-1$ and S rows of $\mathcal{A}_d^{(M)}$ respectively. Then include also all integers for which any multiple up to order m_{d+1} is included in $\mathcal{A}_d^+$, i. e. $\mathcal{A}_d^{++} = \bigcup_{r=1}^{m_{d+1}} \frac{1}{r} \cdot \mathcal{A}_d^+ \cap \mathbb{Z}$.

Step 3 Let g_{d+1} be a positive integer not included in $\mathcal{A}_d^{++}$.

Then $(g_1, ..., g_d, g_{d+1})$ is suitable and

$$
\mathcal{A}_{d+1}^{(S)} = \mathcal{A}_d^{(S)} \cup \bigcup_{r \in \mathcal{N}'_{d+1}} \{r n_{d+1} - \mathcal{A}_d^{(S-1)}\} \cup \bigcup_{r \in \mathcal{N}'_{d+1}} \{r n_{d+1} + \mathcal{A}_d^{(S-1)}\},
$$

for $S = 1, ..., M$.

Step 4 Set $d := d + 1$ and go to **Step 2**.

For $S = 1$, we are only interested in the main effects. For $S = 2$ we are interested in main effects plus two–factor interactions and so on up to $S = M-1$, where we are interested in all but the M–factor interactions (*Resolution 2M*) and for $S = M$ we are interested in all parameters (*Resolution 2M+1*).

In the particular situation of equal order of Fourier regression in all factors, the constructions in the algorithm guarantee that $g_d = O(d^{M+S-1})$, as $d \to \infty$, because the cardinality of $\mathcal{A}_d^{++}$ increases, at most, as d^{M+S-1}. Hence the iterative sequence of generator components and the corresponding sizes increase polynomially, which is to be compared to an exponential growth for *power generators* and *Fibonacci type* generators. Note that the number of all parameters increases polynomially like d^M and the number of parameters of interest increases like d^S. Using the one–step algorithm it may happen that we obtain a better rate than $O(d^{M+S-1})$.

We wish to find out classes of lattice designs for which the number of supporting points increases polynomially with the dimension.

5 Examples and Constructions

5.1 Complete interactions model

For the complete model $F(d; m_1, \ldots, m_d; d)$ the lattice design generated by the *generalised power generator*

$$(g_1, ..., g_d) = (1, p_1, p_1 p_2, ..., \prod_{k=1}^{d-1} p_k)$$

where $p_k = 2m_k + 1$, i.e. the number of parameters in the kth marginal models, is D–optimum and with minimal support.

Indeed, by induction one shows the integers

$$r_1 g_1 + ... + r_d g_d, \quad r_k \in \mathcal{N}_k, \quad k = 1, ..., d$$

are all different. The largest is

$$N_1 = \sum_{k=1}^{d} m_k g_k = \sum_{k=1}^{d} m_k \prod_{i=1}^{k-1} p_i$$

and by the *Upper Law* they are also different in the cyclic group $\mathcal{G}_n = (\{0, 1, ..., n-1\}, +)$ where $n = 2N_1 + 1 = \prod_{k=1}^{d} p_k$, that is the number of parameters in the model. In the particular case $m_1 = ... = m_d = m$ the proposed generator becomes the *power generator*

$$(g_1, ..., g_d) = (1, 2m+1, (2m+1)^2, ..., (2m+1)^{d-1})$$

Here the one–step generator coincides with the generalised power generator.

5.2 Additive Model

Every design D–optimum in a (large) Fourier model is also D–optimum in every submodel because of the diagonal structure of the information matrix. Moreover in the present situation the arrays of integers to be checked become smaller. Hence the generalised power generator with $p_k = 2m_k + 1$ generates a D–optimum design for the additive model $F(d; m_1, \ldots, m_d; 1)$. In the particular case $m_1 = ... = m_d = m$ we have again the *power generator* with base $2m+1$ and its size by the *Upper Law* is $2m(2m+1)^{d-1}+1$. It increases exponentially with the dimension and with number of parameters linearly.

Noticing that for these last model and generators in the array there is a considerable gap at the start between m and $2m+1$, we obtain the *power generator* with minimum base $m+1$

$$(g_1, ..., g_d) = (1, m+1, (m+1)^2, ..., (m+1)^{d-1}).$$

The *Upper Law* gives a size $n = 2m(m+1)^{d-1} + 1$ which is approximately $1/2^{d-1}$ times the size for the previous generator.

More generally for $F(d; m_1, \ldots, m_d; 1)$ we consider the *generalised power generator*

$$(g_1, ..., g_d) = (1, m_1+1, (m_1+1)(m_2+1), ..., \prod_{k=1}^{d-1}(m_k+1))$$

and a size $n = 2m_d \prod_{k=1}^{d-1}(m_k+1) + 1$.

For the model $F(d; 1, \ldots, 1; 1)$ the lattice design generated by

$$(g_1, ..., g_d) = (1, 2, 3, ..., d-1)$$

and with a minimum size $2d-1$ is minimal and D–optimum. This example suggests an iterative strategy for adding dimensions. Suppose the $(d-1)$–dimensional generator $(g_1, ..., g_{d-1})$ is such that the members of the array

$$\begin{array}{l} 0, \\ r_k g_k \quad r_k \in \mathcal{N}'_k, \quad k = 1, ..., d-1 \end{array}$$

are all different to each other in $\mathbb{Z}$, then we choose the next generator g_d greater than the largest member of the above array. The resulting generator $(g_1, ..., g_d)$ produces an array in which again all members are different from each other. Obviously, the largest member at each step will be $m_d g_d$ such that $\mathrm{g}_d \geq m_{d-1} g_{d-1} + 1$. As we want to choose g_d as small as possible in order to reduce the gaps we obtain the following recursion scheme

$$\begin{array}{l} g_1 = 1 \\ g_d = m_{d-1} g_{d-1} + 1 \end{array}$$

and a size $2m_d g_d + 1$ by the *Upper Law*. For the models $F(d; m, \ldots, m; 1)$ $(m \geq 2)$ we get

$$\begin{array}{l} g_1 = 1 \\ g_d = \sum_{i=0}^{d-1} m^i = (m^d - 1)/(m-1) \end{array}$$

and a size $n = 2mg_d + 1 = (2m^{d+1} - m - 1)/(m-1)$. This size is smaller than the size of the previous power generator designs. However, it is still exponentially increasing in the dimension d.

The *one–step strategy* says that given a $(d-1)$–dimensional generator $(g_1, ..., g_{d-1})$ such that the members of the array

$$\begin{array}{l} 0, \\ r_k g_k \quad r_k \in \mathcal{N}'_k, \quad k = 1, ..., d-1 \end{array}$$

are all different to each other in $\mathbb{Z}$, we choose the next generator g_d as the smallest integer such that adding

$$r_d g_d \quad r_d \in \mathcal{N}'_d,$$

the increased array has still all members different to each other. For the model $F(d; 2, \ldots, 2; 1)$ this strategy defines the following sequence of generator components

$$(1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 25, 27, 28, 29, ...).$$

We notice that all odd numbers are included in the sequence. Hence $g_d \leq 2d-1$ and the size $n \leq 8d-3$ increases linearly in the dimension d.

Again for the model $F(d;2,\ldots,2;1)$, the last observation suggests an alternative iterative sequence of generator components

$$\begin{aligned} g_1 &= 1 \\ g_d &= 2d-1 \end{aligned}$$

with a linearly increasing size $n = 8d-3$. The corresponding array

$$\begin{array}{l} 0, \\ g_k, 2g_k, \quad k = 1, ..., d \end{array}$$

has all different entries as all members of the form $2g_k$ are even and are also different to each other.

With a similar argument this can be readily extended to the model $F(d;m,\ldots,m;1)$ by

$$\begin{aligned} g_1 &= 1 \\ g_d &= (d-1)m+1 \end{aligned}$$

Then the corresponding array

$$\begin{array}{l} 0, \\ g_k, 2g_k, ..., mg_k \quad k = 1, ..., d \end{array}$$

has all different entries. Hence, the size $n = 2(d-1)m^2 + 2m + 1$ by the *Upper Law* increases linearly in d.

In the general case of unequal orders m_k, arranging the factors in such a way that $m_1 \geq m_2 \geq \cdots \geq m_d$, we use the following generating procedure

$$\begin{aligned} g_1 &= 1 \\ g_d &= (d-1)m_1+1 \end{aligned}$$

This model is a submodel of $F(d;m_1,\ldots,m_1;1)$, so the entries in the corresponding array are different to each other and the generated design will be D–optimum as long as the condition $n \geq 2\max_k\{m_k g_k\}+1$ from the *Upper Law* is satisfied. These are referred to as *linear generators*.

We summarize some results in self–explanatory tables, to be found at the end of the paper: “Dim” stands for “dimension”, “gen” for “generator”, N_1 is the largest number and N_2 is the second–largest number in the array to be checked. By UL we mean the number of supporting points given by the *Upper Law* and Gen UL by the *Generalised Upper Law*. Finally "Min.Size" is the minimum number of support points for the generator in the row.

In most situations the one–step generator is preferable to the mechanical linear generator with respect to the minimum attainable size n. The only exceptions we found were for $F(d; 2, \ldots, 2; 1)$ in case $d \geq 5$.

The last additive model we consider is $F(2; m_1, m_1; 1)$. The linear generator $(1, m_1 + 1)$ is also the one–step generator. By the *Upper Law* the size is $2m_1^2 + 2m_1 + 1$ and by the *Generalized Upper Law* it is $2m_1^2 + m_1$. A complete compression gives $m_1^2 + 2m_1 + 2$.

5.3 Models with two–factor interactions

As $F(d; m_1, \ldots, m_d; 2)$ is a submodel of the complete model, the generalised power generator with $p_k = 2m_k + 1$ produces a D–optimum lattice design. The *Upper Law* gives a size

$$2(m_d g_d + m_{d-1} g_{d-1}) + 1 = 2 \prod_{k=1}^{d-2} (2m_k + 1) \left[2m_d m_{d-1} + m_d + m_{d-1}\right] + 1$$

which is slightly smaller than the size for a product type design.

When analysing only the main effect, the *Main Generalised Upper Law* gives a design size

$$2m_d g_d + m_{d-1} g_{d-1} + 1 = \prod_{k=1}^{d-2} (2m_k + 1) \left[4m_d m_{d-1} + 2m_d + m_{d-1}\right] + 1$$

which is again slightly smaller than the size required for making inference on the whole parameter vector (see below).

We continue listing models and techniques, distinguishing inference on main effects and inference on the whole parameter vector. These are the models and the techniques

$F(d; 1, \ldots, 1; 2)$	minimum power generator
$F(d; 2, \ldots, 2; 2)$	Fibonacci type generator
$F(d; m, \ldots, m; 2)$	one–step generator

At first we consider $F(d;1,\ldots,1;2)$. To make inference on the main effects with the generator $(g_1,...,g_d)$ we have to check the entries of the array

$$\begin{array}{ll} 0, & \\ g_k, -g_k & k=1,...,d \\ g_k+g_\ell, g_k-g_\ell, -g_k+g_\ell, -g_k-g_\ell, & 1\le k<\ell\le d \end{array}$$

and $N_1 = g_d + g_{d-1}$ and $N_3 = g_d$.

The minimum power generator is $(1,3,\ldots,3^{d-1})$ and from the *Main Generalised Upper Law* the size is $2g_d + g_{d-1} + 1 = 7\cdot 3^{d-2}+1$. The minimum size greater than N_1 is $g_d + g_{d-1} + 1 = 4\cdot 3^{d-2}+1$, but stronger compression is possible.

An iterative sequence of generator components is given by the recursion

$$\begin{aligned} g_1 &= 1 \\ g_d &= 2g_{d-1}+1 = 2^d - 1, \quad d\ge 2 \end{aligned}$$

and the size defined by the *Main Generalised Upper Law* is $5\cdot 2^{d-1} - 2$ increasing exponentially.

For the linear type generator, equal to the one–step algorithm and defined by $g_d = 2d-1$, the size $6d-4$, by the *Main Generalised Upper Law*, increases linearly in the dimension.

Similar results for $F(d;m,\ldots,m;2)$, $m\ge 2$, follow later.

Turning to inference on the whole parameter, we observe that applying the *Upper Law* to the minimum power generator which has base 3, the size becomes $2(g_d+g_{d-1})+1 = 8\cdot 3^{d-2}+1$.

A Fibonacci type generator is defined by the sequence

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 3 \\ g_d &= 2g_{d-1}+g_{d-2}+1 \end{aligned}$$

Hence $g_d = \frac{1}{4}((1+\sqrt{2})^{d+1} + (1-\sqrt{2})^{d+1} - 2)$ by the formula for non–homogeneous linear recursions (see for example Biggs, 1989, pp. 250). This gives an approximated size $2(g_d+g_{d-1})+1 \approx \frac{1}{\sqrt{2}}(1+\sqrt{2})^{d+1}$ by the *Upper Law*.

Finally, the one–step sequence of generator components begins with

$$(1,3,8,18,30,43,67,90,122,161,...)$$

and the corresponding sizes

$$*, 9, 23, 53, 97, 147, 221, 315, 425, 567, ...$$

obtained by the *Upper Law*.

Now consider the model $F(d; 2, \ldots, 2; 2)$, for which $N_1 = 2g_d + 2g_{d-1}$ and $N_3 = 2g_d$. We start by making inference on the parameters associated with the main effects. The minimum power generator is given by

$$(g_1, ..., g_d) = (1, 5, 5^2, ..., 5^{d-1})$$

by the *Main Generalised Upper Law* we get a size $4g_d + 2g_{d-1} + 1 = 22 \cdot 5^{d-2} + 1$

An iterative sequence of generator components is given by

$$\begin{aligned} g_1 &= 1 \\ g_2 &= 5 \\ g_d &= 2g_{d-1} + 2g_{d-2} + 1, \quad d \geq 3 \end{aligned}$$

Hence $g_d = \frac{2}{3}((1+\sqrt{3})^d + (1-\sqrt{3})^d - \frac{1}{2})$ and the size given by the *Main Generalised Upper Law* equals approximately $(4 + \frac{8}{\sqrt{3}})(1+\sqrt{3})^{d-1}$ and increases exponentially in the dimension.

The one–step iterative sequence of generator components starts as follows,

$$(1, 5, 13, 17, 37, 41, 49, 53, 109, 113, 121, 125, ...) \tag{10}$$

After a careful inspection of this sequence it was found to be compatible with the generating law

$$\begin{array}{ll} g_1 = 1 & \\ g_d = g_{d-1} + 4\alpha_{d-1} & d \geq 2 \\ \alpha_1 = 1 & \\ \alpha_{2^j} = 3\alpha_{2^{j-1}} - 1 & j \geq 2 \\ \alpha_k = \alpha_{2^j} & \text{for } k = 2^j q, \quad q \text{ odd} \end{array}$$

Both the one–step method and the iteration produce the further terms

$$145, 149, 157, 161, 325, 329.$$

However, it is not clear, yet, whether the generating law produces indeed the one–step sequence for dimensions larger than 18. The difference sequence α_k, $k = 1, 2, ...$ gives

$$\begin{aligned}&1,2,1,5,1,2,1,14,1,2,1,5,1,2,1,41,\\&1,2,1,5,1,2,1,14,1,2,1,5,1,2,1,122,\\&1,2,1,5,1,2,1,14,1,2,1,5,1,2,1,41,\\&1,2,1,5,1,2,1,14,1,2,1,5,1,2,1,365,...\end{aligned}$$

which shows a nice *self similar* (*fractal*) structure. Standard methods yield the solution $\alpha_{2^j} = \frac{1}{2}(3^j + 1)$ and $g_{2^j+1} = 4 \cdot 3^j + 1$. This implies that g_d increases like d^γ where $\gamma = \frac{\log 3}{\log 2} \approx 1.5850$, a number which is reminiscent of results in fractal dimension. Hence we can take sizes n which increase like d^γ as the dimension d increases.

For inference on the whole parameter vector for the minimum power generator we obtain the size $4(g_d + g_{d-1}) + 1 = 24 \cdot 5^{d-2} + 1$ by the *Upper Law*.

We present the following iterative sequence of generator components

$$\begin{aligned}g_1 &= 1\\ g_2 &= 5\\ g_d &= 4g_{d-1} + 2g_{d-2} + 1, \quad d \geq 3\end{aligned}$$

Hence $g_d = \frac{1}{30}\left((3+2\sqrt{6})(2+\sqrt{6})^d + (3-2\sqrt{6})(2-\sqrt{6})^d - 6\right)$ and the support size is approximately equal to $\frac{1}{5}(4+\sqrt{6})(2+\sqrt{6})^d$ by the *Upper Law*.

We end these considerations with the one–step sequence of generator components

$$(1, 5, 23, 60, 77, 173, 222, 409, 535, 634, 935, 1182, 1361, 1497, ...) \tag{11}$$

and the corresponding sizes

$$*, 25, 113, 333, 549, 1001, 1581, 2525, 3777, 4677, 6277, 8469, 10173, 11533, ...$$

obtained by the *Upper Law*.

Finally we investigate $F(d; m, \ldots, m; 2)$, for which $N_1 = m(g_d + g_{d-1})$ and $N_3 = mg_d$. Let us start by making inference on main effects with the minimum power generator given by $(1, (2m+1), \ldots, (2m+1)^{d-1})$. By the *Main Generalised Upper Law* we get a size $m(2g_d + g_{d-1}) + 1 = m(4m + 3)(2m+1)^{d-2} + 1$. We present the following iterative sequence

$$
\begin{aligned}
g_1 &= 1 \\
g_2 &= 2m + 1 \\
g_d &= m(g_{d-1} + g_{d-2}) + 1, \quad d \geq 3
\end{aligned}
$$

By the linear non–homogeneus recursion formula we obtain the solution

$$
\begin{aligned}
g_d \;\; = \;\; & \tfrac{m}{2m-1}\times \\
& \times \left((1 - \tfrac{m-2}{\sqrt{m^2+4m}})(\tfrac{m+\sqrt{m^2+4m}}{2})^d \; + (1 + \tfrac{m-2}{\sqrt{m^2+4m}})(\tfrac{m-\sqrt{m^2+4m}}{2})^d - \tfrac{1}{m}) \right)
\end{aligned}
$$

and by the *Generalised Upper Law* the support size is

$$
n \approx \tfrac{m^2}{2m-1}(3 + \tfrac{5m+2}{\sqrt{m^2+4m}})(\tfrac{m+\sqrt{m^2+4m}}{2})^{d-1}
$$

The sequences of sizes increases exponentially in the dimension.

In generalisation of the case $m = 2$ we can obtain another iterative sequence of generator components which is increasing at most polynomially:

$$
\begin{aligned}
& g_1 = 1, & \\
& g_d = g_{d-1} + 2m\alpha_{d-1}, & d \geq 2; \\
& \alpha_1 = 1, & \\
& \alpha_{2^j} = (m+1)\alpha_{2^{j-1}} - 1, & j \geq 2 \\
& \alpha_k = \alpha_{2^j}, & \text{for } k = 2^j q, \quad q \text{ odd}
\end{aligned}
$$

The difference sequence again shows a *self similar* structure. After some algebra we obtain $\alpha_{2^j} = \frac{(m-1)(m+1)^j+1}{m}$ and $g_{2^j} = 2m(m+1)^j + 1$. This implies that g_d increases like d^γ where $\gamma = \frac{\log(m+1)}{\log 2}$.

Turning to inference on the whole parameter vector, for the minimum power generator we obtain the size $2m(g_d + g_{d-1}) + 1 = 4m(m+1)\cdot(2m+1)^{d-2} + 1$ by the *Upper Law.* An iterative sequence of generator components is

$$
\begin{aligned}
g_1 &= 1, \\
g_2 &= 2m + 1, \\
g_d &= m(2g_{d-1} + g_{d-2}) + 1, \quad d \geq 3.
\end{aligned}
$$

Hence

$$
\begin{aligned}
g_d = \; & \tfrac{1}{3m-1}\times \\
\times & \left(\left(\sqrt{\tfrac{m}{m+1}} + \tfrac{1}{2}\right)(m + \sqrt{m(m+1)})^d - \left(\sqrt{\tfrac{m}{m+1}} - \tfrac{1}{2}\right)(m - \sqrt{m(m+1)})^d - 1 \right)
\end{aligned}
$$

with the approximated size

$$n = \frac{m}{3m-1}\left(2+\sqrt{\frac{m+1}{m}}\right)\left(m+\sqrt{m(m+1)}\right)^d$$

As for the additive models we summarise the results in self–explanatory tables.

We conclude with some figures representing the behaviour of the one–step generators for the two factors interactions models, both for inference on the main effects and on the whole parameters vector.

For the main effects in $F(d; 2, ..., 2; 2)$ Figure 1 shows the logarithm of α_d, the first differences of the generators divided by 4, for the sequence (10) against the dimension d up to 180. The *self similar* structure mentioned in the text is evident. Figure 2 shows the logarithm of the generator of the same sequence against $\log(d)$ and gives an estimated slope of 1.7.

For inference on the whole parameter vector in $F(d; 2, ..., 2; 2)$ Figure 3 shows the logarithm of the first differences for the one–step sequence (11) against the dimension d. Figure 4 shows the second differences against the dimension d and Figure 5 is $\log(g_d)$ against $\log(d)$ giving an estimated slope of $\gamma \approx 2.72$.

The two log log plots indicate polynomial growth d^γ of the size n in d with γ approximately equal to the estimated slopes. Note that the estimate 1.7 for the model $F(d; 2, ..., 2; 2)$ only slightly over-estimates $\gamma = \frac{\log 3}{\log 2} \approx 1.5850$.

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Dim	gen	N_1	N_2	UL	Gen UL	Min.Size
2	(1,3)	6	3	13	10	10
3	(1,3,5)	10	6	21	17	14
4	(1,3,5,7)	14	10	29	25	18
5	(1,3,5,7,9)	18	14	37	33	22
d	$(1,3,\ldots,2d-1)$	$4d-2$	$4d-6$	$8d-3$	$8d-7$	$4d+2$

Table 1: Linear type generator for the model $F(d;2,\ldots,2;1)$

Dim	gen	N_1	N_2	UL	Gen UL	Min.size
2	(1,3)	6	3	13	10	10
3	(1,3,4)	8	6	17	15	13
4	(1,3,4,5)	10	8	21	19	17
5	(1,3,4,5,7)	14	10	29	25	23
6	(1,3,4,5,7,9)	18	14	37	33	29
7	(1,3,4,5,7,9,11)	22	18	45	41	34
8	(1,3,4,5,7,9,11,12)	24	22	49	47	37
9	(1,3,4,5,7,9,11,12,13)	26	24	53	51	41
10	(1,3,4,5,7,9,11,12,13,15)	30	26	61	57	46

Table 2: One–step generator for the model $F(d;2,\ldots,2;1)$

Dim	gen	N_1	N_2	UL	Gen UL	Min.Size
2	(1,4)	12	8	25	21	17
3	(1,4,7)	21	14	43	36	27
4	(1,4,7,10)	30	21	61	52	36
5	(1,4,7,10,13)	39	30	79	70	45
d	$g_d = 3d-2$	$9d-6$	$9d-15$	$18d-11$	$18d-20$	$9d$

Table 3: Linear generator for the model $F(d;3,\ldots,3;1)$

Dim	Gen	N_1	N_2	UL	Gen UL	Min.size
2	(1,4)	12	8	25	21	17
3	(1,4,5)	15	12	31	28	21
4	(1,4,5,7)	21	15	43	37	32
5	(1,4,5,7,9)	27	21	55	49	38
6	(1,4,5,7,9,11)	33	27	67	61	46

Table 4: One–step generator for the model $F(d;3,\ldots,3;1)$

Dim	1	2	3	4	5	6	7	8	9	10	11	12	13	14
gen	1	4	5	7	9	11	13	16	17	19	20	23	25	28
2gen	2	8	10	14	18	22	26	32	34	38	40	46	50	56
3gen	3	12	15	21	27	33	39	48	51	57	60	69	75	84

Dim	15	16	17	18	19
gen	29	31	35	36	37
2gen	58	62	70	72	74
3gen	87	93	105	108	111

Table 5: Array to be checked for the one–step generator in model $F(d; 3, \ldots, 3; 1)$

Dim	gen	N_1	N_2	UL	Gen UL	Min.size
2	(1,5)	20	15	41	36	26
3	(1,5,9)	36	27	73	64	43
4	(1,5,9,13)	52	39	105	92	58
5	(1,5,9,13,17)	68	52	137	121	74
d	$g_d = 4d - 3$	$16d - 12$	$16d - 28$	$32d - 23$	$32d - 39$	

Table 6: Linear generator for the model $F(d; 4, \ldots, 4; 1)$

Dim	gen	N_1	N_2	UL	Gen UL	Min.size
2	(1,5)	20	15	41	36	26
3	(1,5,6)	24	20	49	45	31
4	(1,5,6,7)	28	24	57	53	37
5	(1,5,6,7,11)	44	33	89	78	60
6	(1,5,6,7,11,13)	52	44	105	97	69

Table 7: One–step generator for the model $F(d; 4, \ldots, 4; 1)$

$F(d;1,\ldots,1;2)$	$N_1 = g_d + g_{d-1}$	$N_3 = g_d$
MAIN EFFECTS		Main gen UL
Min power gen	$g_d = 3^{d-1}$	$7 \cdot 3^{d-2} + 1$
Fibonacci type	$g_d = 2^d - 1$	$5 \cdot 2^{d-1} + 2$
One–step	$g_d = 2d - 1$	$6d - 4$
ALL EFFECTS		UL
Min power gen	$g_d = 3^{d-1}$	$8 \cdot 3^{d-2} + 1$
Fibonacci type	$\begin{cases} g_1 = 1 \\ g_2 = 3 \\ g_d = 2g_{d-1} + g_{d-2} + 1 \end{cases}$	$\approx \frac{1}{\sqrt{2}}(1+\sqrt{2})^{d+1}$
One–step	$(1, 3, 8, 18, 30, 43, \ldots)$	$(*, 9, 23, 53, 97, 147, \ldots)$

$F(d;2,\ldots,2;2)$	$N_1 = 2(g_d + g_{d-1})$	$N_3 = 2g_d$
MAIN EFFECTS		Main Gen UL
Min power gen	$g_d = 5^{d-1}$	$22 \cdot 5^{d-2} + 1$
Fibonacci type	$\begin{cases} g_1 = 1 \\ g_2 = 5 \\ g_d = 2g_{d-1} + 2g_{d-2} + 1 \end{cases}$	$\approx 2(1+\sqrt{3})^{d+1}$
One–step	$\begin{cases} g_1 = 1 \\ g_d = g_{d-1} + 4\alpha_{d-1} \\ \alpha_1 = 1 \\ \alpha_{2^j} = 3\alpha_{2^{j-1}} - 1 \text{ for } j \geq 2 \\ \alpha_k = \alpha_{2^j} \text{ for } k = 2^j q, q \text{ odd} \end{cases}$	growth: d^γ with $\gamma = \frac{\log 3}{\log 2}$
ALL EFFECTS		UL
Min power gen	$g_d = 5^{d-1}$	$24 \cdot 5^{d-2} + 1$
Fibonacci type	$\begin{cases} g_1 = 1 \\ g_2 = 5 \\ g_d = 4g_{d-1} + 2g_{d-2} + 1 \end{cases}$	$\approx 2(2+\sqrt{6})^d$
One–step	$(1, 5, 23, 60, 77, \ldots)$	$(*, 25, 113, 333, 549, \ldots)$

$F(d; m, \dots, m; 2)$	$N_1 = m(g_d + g_{d-1})$	$N_3 = m g_d$
MAIN EFFECTS		Main Gen UL
Min power gen	$g_d = (2m+1)^{d-1}$	$m(4m+3)(2m+1)^{d-2}+1$
Fibonacci type	$\begin{cases} g_1 = 1 \\ g_2 = 2m+1 \\ g_d = m(g_{d-1}+g_{d-2})+1 \end{cases}$	$\approx c(m)^{\dagger} \left(\frac{m+\sqrt{m^2+4m}}{2}\right)^{d-1}$
"One–step" (self-similar)	$\begin{cases} g_1 = 1 \\ g_d = g_{d-1} + 2m\alpha_{d-1} \\ \alpha_1 = 1 \\ \alpha_{2^j} = (m+1)\alpha_{2^{j-1}} - 1 \text{ for } j \geq 2 \\ \alpha_k = \alpha_{2^j} \text{ for } k = 2^j q, q \text{ odd} \end{cases}$	growth: d^{γ} with $\gamma = \frac{\log(m+1)}{\log 2}$
ALL EFFECTS		UL
Min power gen	$g_d = (2m+1)^{d-1}$	$4m(m+1)(2m+1)^{d-2}+1$
Fibonacci type	$\begin{cases} g_1 = 1 \\ g_2 = 2m+1 \\ g_d = m(2g_{d-1}+g_{d-2})+1 \end{cases}$	$\approx c'(m)^{\ddagger} \left(m+\sqrt{m(m+1)}\right)^{d}$

† $c(m) = \frac{m^2}{2m-1}\left(3 + \frac{5m+2}{\sqrt{m^2+4m}}\right)$ and $\lim_{m\to\infty} c(m) = +\infty$; ‡ $\lim_{m\to\infty} c'(m) = 1$

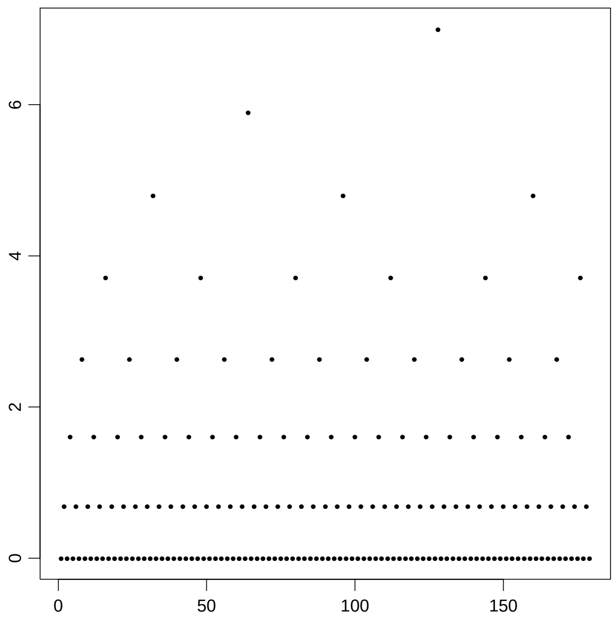

Figure 1: Main effect of the model $F(d; 2 \ldots 2; 2)$ -log α_d against d

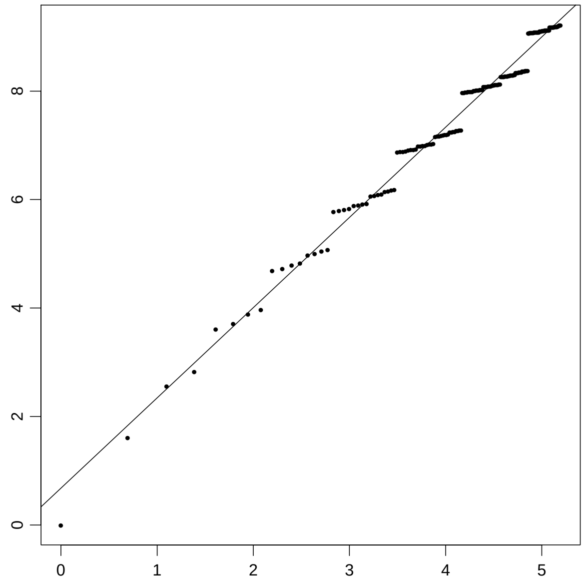

Figure 2: Main effect of the model $F(d; 2 ... 2; 2)$ -log g_d against $\log(d)$

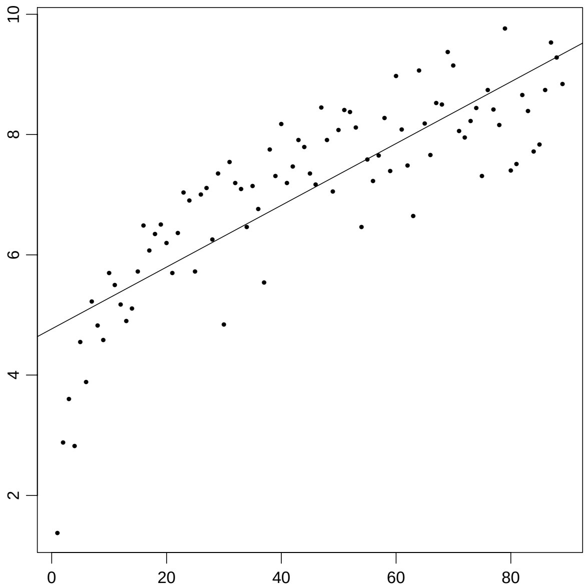

Figure 3: All effects of the model $F(d; 2 ... 2; 2)$ -log$(g_d - g_{d-1})$ against d

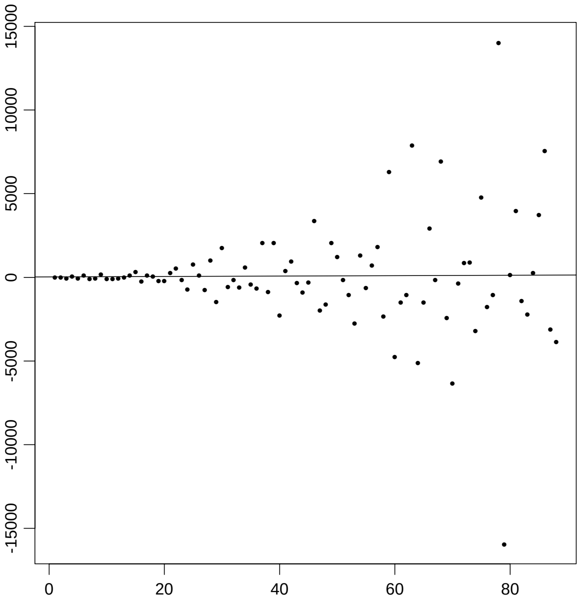

Figure 4: All effects of the model $F(d;2...2;2)$ -second differences against d

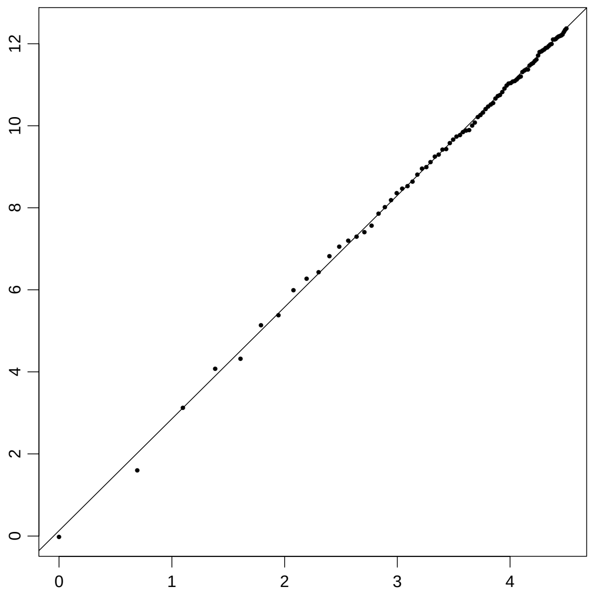

Figure 5: All effects of the model $F(d;2...2;2)$ -$\log g_d$ against $\log d$