Semilinear Elliptic Problems in Unbounded Domains: Solutions in Weighted Sobolev Spaces

Klaus Pflüger
Institut für Mathematik I
Freie Universität Berlin

Preprint Nr. 21 A 95

Abstract

The semilinear elliptic equation $Lu = f(x,u)$ in an unbounded domain $\Omega \subset I\!R^N$ is studied. We use several embedding theorems of weighted Sobolev spaces to prove the solvability of this equation under natural growth conditions on f and on the coefficients of the elliptic operator L. This paper gives a unified approach to such problems and generalizes some results in [1], [9] and [13].

email: pflueger@math.fu-berlin.de

1 Introduction

We study the problem

$$\begin{aligned} Lu &= f(x,u) \qquad \text{in } \Omega, && (1)\\ u &= 0 \qquad\qquad \text{on } \partial\Omega, && (2)\\ \lim_{|x|\to\infty} u(x) &= 0 && (3) \end{aligned}$$

where Ω is an unbounded domain in $I\!\!R^N$ (or $\Omega = I\!\!R^N$) and L is a uniformly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{N} \partial_j(a_{ij}(x)\partial_i u) + b(x)u$$

with sufficiently smooth coefficients satisfying $a_{ij} = a_{ji}$ (here $\partial_i = \partial/\partial x_i$).

In the case, when this problem is radially symmetric (for example if the coefficients of L and the nonlinearity f do not depend on x and $\Omega = I\!\!R^N$), the solvability in the subcritical case is well known (see [2]). If the coefficients are asymptotically constant, the well-known concentration-compactness priciple of P. L. Lions [6] can be applied to obtain nontrivial solutions. In the present paper we study the case where non of these conditions is satisfied. Our approach via weighted Sobolev spaces generalizes some previous results of [1], [9] and [13].

Corresponding to L we define the bilinear form

$$\langle u, v\rangle_L = \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i u \partial_j v + b(x)uv\, dx$$

and the energy functional

$$J(u) = \frac{1}{2}\langle u, u\rangle_L - \int_\Omega F(x,u)\, dx\,,$$

where $F(x,u) = \int_0^u f(x,t)dt$ is the primitive of f.

Now a weak solution of (1), (2) is obtained as a critical point of J in an appropriate function space. The main result of this paper is the solvability of this problem in certain weighted Sobolev spaces (Theorem 3.4). The exact asymptotic behaviour of these solutions will be studied in a forthcoming paper.

2 Weighted Sobolev Spaces

For $x \in \mathbb{R}^N$, $\rho > 0$ let $B(x,\rho)$ be the open ball in $\mathbb{R}^N$ of radius ρ, centered at x. For an (unbounded) domain $\Omega \subset \mathbb{R}^N$ denote $\Omega_n = \Omega \cap B(0,n)$, and $\Omega^n = \operatorname{int}(\Omega \setminus \Omega_n)$. We always assume that there exists an $\tilde{n} \geq 2$ such that

$$\Omega^{\tilde{n}} = \{x \in \mathbb{R}^N \mid |x| > \tilde{n}\}.$$

For a positive function σ, we define the weighted p-norm $(1 \leq p < \infty)$

$$\|u\|_{p,\Omega,\sigma} = \left(\int_\Omega |u(x)|^p \sigma(x)\, dx \right)^{1/p}$$

and denote by $L^q(\Omega; w)$ the space of all measurable functions u such that $\|u\|_{q,\Omega,w}$ is finite. The weighted Sobolev space $W^{1,p}(\Omega; v_0, v_1)$ is defined as the space of all functions $u \in L^p(\Omega, v_0)$ such that all derivatives $\partial u/\partial x_i$ belong to $L^p(\Omega, v_1)$. The corresponding norm is defined by

$$\|u\|_{1,p,\Omega,v_0,v_1} = \left(\int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0(x)\, dx \right)^{1/p}.$$

Let r be a positive function defined on $\Omega^{\tilde{n}}$, satisfying

$$\begin{aligned} r(x) &\leq |x|/3 && \text{for a. e. } x \in \Omega^{\tilde{n}} \\ c_r^{-1} \leq r(y)/r(x) &\leq c_r && \text{for a. e. } x \in \Omega^{\tilde{n}},\ y \in B(x, r(x)), \end{aligned}$$

where $c_r \geq 1$ is a constant.

Then the following holds (cf. Kufner/Opic [5], Theorem 18.12/20.5):

Theorem 2.1 *Let $1 \leq p \leq q < \infty$, $\frac{N}{q} - \frac{N}{p} + 1 \geq 0$ and assume that there are constants $0 \leq k_0 \leq K_0$, $0 \leq c_0 \leq C_0$, $0 \leq c_1 \leq C_1$, and positive measurable functions b_0, b_1, defined in $\Omega^{\tilde{n}}$, such that the weight functions v_0, v_1, w satisfy*

$$k_0 v_0(x) \leq v_1(x) r^{-p}(x) \leq K_0 v_0(x) \qquad \text{for a. e. } x \in \Omega^{\tilde{n}}, \tag{4}$$

$$c_0 b_0(x) \leq w(y) \leq C_0 b_0(x) \qquad \text{for a. e. } x \in \Omega^{\tilde{n}},\ y \in B(x, r(x)), \tag{5}$$

$$c_1 b_1(x) \leq v_1(y) \leq C_1 b_1(x) \qquad \text{for a. e. } x \in \Omega^{\tilde{n}},\ y \in B(x, r(x)). \tag{6}$$

Assume that the embedding $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$ is compact for every $n \in I\!N$. Then the embedding

$$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$$

is compact if and only if

$$\lim_{n \to \infty} \mathcal{B}_n = 0\,, \quad \text{where} \quad \mathcal{B}_n = \sup_{x \in \Omega^n} \frac{b_0^{1/q}(x)}{b_1^{1/p}(x)}\, r^{(\frac{N}{q} - \frac{N}{p} + 1)}(x)\,.$$

Furthermore this embedding is continuous if $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$ is continuous for every $n \in I\!N$ and $\lim_{n \to \infty} \mathcal{B}_n < \infty$.

In this paper we are mainly concerned with the following examples.

Example 2.2 *(1) Let β and α be real numbers and define*

$$v_0(x) = (1 + |x|)^{\beta - p}\,,\ v_1(x) = (1 + |x|)^{\beta}\,,\ w(x) = (1 + |x|)^{\alpha}\,.$$

If we choose $b_0 = w$, $b_1 = v_1$ and $r(x) = (1 + |x|)/6$, we see that the assumptions of Theorem 2.1 are satisfied, if

$$\frac{N}{q} - \frac{N}{p} + 1 > 0 \qquad \text{and} \qquad \frac{\alpha}{q} - \frac{\beta}{p} + \frac{N}{q} - \frac{N}{p} + 1 < 0\,,$$

therefore $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ is compact.

(2) Define $v_0(x) = v_1(x) = e^{\beta |x|}$, $w(x) = e^{\alpha |x|}$.

If we choose $b_0 = w$, $b_1 = v_1$ and $r(x) = 1$, the assumptions of Theorem 2.1 are satisfied, if

$$\frac{N}{q} - \frac{N}{p} + 1 > 0 \qquad \text{and} \qquad \frac{\alpha}{q} - \frac{\beta}{p} < 0\,.$$

Another useful tool for the study of partial differential equations in unbounded domains is the Hardy-inequality. For this we assume that Ω has the property, that $x \in \Omega$ implies $tx \in \Omega$ for every $t \geq 1$. Furthermore the weight functions v_0, v_1 are assumed to be radial, i. e. of the form $v_0(|x|)$, $v_1(|x|)$ and are bounded from below and above on each compact subinterval in $(0, \infty)$. Let p' be defined by $(1/p) + (1/p') = 1$. Then the following holds ([5], Theorem 21.8):

Theorem 2.3 (Hardy-inequality) *Assume that there are constants $k, t_0 > 0$ such that*

$$\begin{aligned} v_0(t) &\geq kv_1(t)t^{-p} \qquad \textit{for a. e. } t > t_0, \\ \mathcal{B}_R(v_0(t)t^{N-1}, v_1(t)t^{N-1}, p) &< \infty \,, \end{aligned}$$

where

$$\mathcal{B}_R(v_0(t)t^{N-1}, v_1(t)t^{N-1}, p) = \sup_{0<x<\infty} \left\|(v_0(t)t^{N-1})^{1/p}\right\|_{p,(0,x)} \left\|(v_1(t)t^{N-1})^{-1/p}\right\|_{p',(x,\infty)} .$$

Then there is a constant $C \geq 0$ such that

$$\|u\|_{p,\Omega,v_0} \leq C \, \|\nabla u\|_{p,\Omega,v_1}$$

holds for every $u \in W^{1,p}(\Omega; v_0, v_1)$.

Example 2.4 *We set $p = 2$ and*

$$v_0(|x|) = (1 + |x|)^{\beta-2} \,, \; v_1(|x|) = (1 + |x|)^{\beta} \,, \quad \beta \in I\!R$$

and obtain $v_0(t) \geq \frac{1}{4}v_1(t)t^{-2}$ for $t \geq 1$ and $\mathcal{B}_R(v_0(t)t^{N-1}, v_1(t)t^{N-1}, 2) < \infty$ for $0 \leq \beta \leq 2$. As a consequence of the Hardy-inequality we get the equivalence of the norms

$$\|u\|_{1,2,\Omega,v_0,v_1} \qquad \textit{and} \qquad |||u|||_{1,2,\Omega,v_1} = \left(\int_\Omega |\nabla u(x)|^2 v_1(x) \, dx \right)^{1/2}$$

in $W^{1,2}(\Omega; v_0, v_1)$.

In the sequel we write $\|\cdot\|_{v_0,v_1} = \|\cdot\|_{1,2,\Omega,v_0,v_1}$ and $\|\cdot\|_{q,w} = \|\cdot\|_{q,\Omega,w}$ for the norms in $W^{1,2}(\Omega; v_0, v_1)$ and $L^q(\Omega; w)$, respectively.

3 Existence results

In this section we always assume that the weight functions v_0, v_1, w are choosen such that the embedding $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ is compact for $2 < q < 2N/(N-2)$ and continuous for $2 \le q \le 2N/(N-2)$. By S_q we denote the corresponding embedding constant, $\|u\|_{v_0,v_1} \le S_q \|u\|_{q,w}$. We assume that L defines a continuous bilinear form $\langle \cdot, \cdot \rangle_L$ on $W^{1,2}(\Omega; v_0, v_1)$ and satisfies the ellipticity condition (with some constant $K > 0$)

$$\langle u, u \rangle_L \ge 2K \|u\|^2_{v_0,v_1} \qquad \text{for every} \quad u \in W^{1,2}(\Omega; v_0, v_1)\,. \tag{7}$$

We make the following assumptions on f.

f1 f is a Caratheodory function on $\Omega \times I\!R$, $f(x,0) = 0$, $\lim_{t\to 0} f(x,t)/t = 0$ uniformly in x and there exist an open, nonempty subset $O \subset \Omega$ and a number $R > 0$ such that $F(x,u) > 0$ for every $u \ge R$, $x \in O$.

f2 $|f(x,t)| \le f_0(x) + f_1(x)|t|^{q-1}$ for $x \in \Omega$, $t \in I\!R$, where $2 < q < 2N/(N-2)$ and f_0, f_1 are positive functions satisfying

$$f_0 \in L^{q/(q-1)}(\Omega; w^{1/(1-q)})\,, \quad f_0(x) \le Cw\,, \quad f_1(x) \le Cw$$

with appropriate constant C.

f3 There exists a $\theta \in I\!R$, $0 \le \theta < 1/2$ such that $F(x,t) \le \theta f(x,t)t$ for a. e. $x \in \Omega$.

Under these assumptions we obtain the following results.

Lemma 3.1 *The Nemytski operator N_f associated to f is bounded and continuous as operator $N_f : L^q(\Omega; w) \to L^{q/(q-1)}(\Omega; w^{1/(1-q)})$.*

The Nemytski operator N_F associated to F is bounded and continuous, $N_F : L^q(\Omega; w) \to L^1(\Omega)$.

Proof. For $u \in L^q(\Omega; w)$ we get $(q' = q/(q-1))$

$$\int_\Omega |f(x,u)|^{q'} w^{1/(1-q)} dx \;\le\; C \left(\int_\Omega f_0(x)^{q'} w^{1/(1-q)} dx + \int_\Omega f_1(x)^{q'} |u|^{q'(q-1)} w^{1/(1-q)} dx \right)$$

$$
\begin{aligned}
&\le C + C\int_\Omega |u|^q w^{q/(q-1)} w^{1/(1-q)} dx \\
&= C + C\|u\|_{q,w}^q .
\end{aligned}
$$

Therefore, N_f is bounded. The continuity now follows from standard properties of Nemytski operators.

In the same way we obtain (using Hölder's inequality)

$$
\begin{aligned}
\int_\Omega |F(x,u)|dx &\le \int_\Omega f_0(x)|u|dx + \int_\Omega f_1(x)|u|^q dx \\
&\le \left(\int_\Omega f_0(x)^{q'} w^{1/(1-q)} dx\right)^{1/q'} \left(\int_\Omega |u|^q w\, dx\right)^{1/q} + C\int_\Omega |u|^q w\, dx \\
&\le C\|u\|_{q,w} + C\|u\|_{q,w}^q .
\end{aligned}
$$

This completes the proof. □

Lemma 3.2 *The functional J is Fréchet-differentiable in $W^{1,2}(\Omega; v_0, v_1)$, and the derivative is given by*

$$
\langle J'(u), v\rangle = \langle u, v\rangle_L - \int_\Omega f(x,u)v\, dx. \tag{8}
$$

Proof. Since (8) is the Gateaux derivative of J in u in direction v, we only have to show the continuity of J', i. e. the continuity of the mapings $u \mapsto \langle u, \cdot\rangle_L$ and $u \mapsto \int_\Omega u \cdot dx$ in the dual norm. The first mapping is clearly continuous since we assumed that $\langle \cdot, \cdot\rangle_L$ is a continuous bilinear form. The second mapping is regarded as composition of the following mappings:

$$
W^{1,2}(\Omega; v_0, v_1) \overset{i}{\hookrightarrow} L^q(\Omega; w) \xrightarrow{N_f} L^{q/(q-1)}(\Omega; w^{1/(1-q)}) \xrightarrow{j} \left(W^{1,2}(\Omega; v_0, v_1)\right)',
$$

where $\langle j(\varphi), v\rangle = \int_\Omega \varphi v\, dx$. Since we know that i and N_f are continuous, it suffices to show that j is a bounded operator. By Hölder's inequality we get

$$
\begin{aligned}
\int_\Omega |\varphi v| dx &\le \left(\int_\Omega |\varphi|^{q'} w^{1/(1-q)} dx\right)^{1/q'} \left(\int_\Omega |v|^q w\, dx\right)^{1/q} \\
&\le C\|\varphi\|_{q/(q-1), w^{1/(1-q)}} \|v\|_{v_0, v_1} ,
\end{aligned}
$$

which proves the lemma. $\square$

Lemma 3.3 *J satisfies the Palais-Smale condition on $W^{1,2}(\Omega;v_0,v_1)$.*

Proof. First we show that every Palais-Smale sequence is bounded.

Let $u_k \in W^{1,2}(\Omega;v_0,v_1)$ be such a sequence, i. e. $J'(u_k) \to 0$, $J(u_k) \leq M$. From $|\langle J'(u_k), u_k/\|u_k\|_{v_0,v_1}\rangle| \leq \varepsilon$ for k large, we obtain

$$-\langle u_k, u_k\rangle_L + \int_\Omega f(x,u_k)u_k\,dx \leq \varepsilon\|u_k\|_{v_0,v_1}\,.$$

Multiplying this equation with θ from f3 and adding this to the inequality $\frac{1}{2}\langle u_k,u_k\rangle_L - \int_\Omega F(x,u_k)\,dx \leq M$, we obtain

$$(\frac{1}{2}-\theta)\langle u_k,u_k\rangle_L + \int_\Omega \theta f(x,u_k)u_k - F(x,u_k)\,dx - \theta\varepsilon\|u_k\|_{v_0,v_1} \leq M\,. \tag{9}$$

From f3 we get

$$(\frac{1}{2}-\theta)\langle u_k,u_k\rangle_L - \theta\varepsilon\|u_k\|_{v_0,v_1} \leq M\,.$$

Now (7) implies that $\langle u,u\rangle_L/\|u\|_{v_0,v_1} \to \infty$ as $\|u\|_{v_0,v_1} \to \infty$, which shows that $\|u_k\|_{v_0,v_1}$ must be bounded.

Therefore, we get a subsequence (again denoted by u_k) which is convergent in $L^q(\Omega;w)$. For this subsequence we get

$$\begin{aligned}
\langle u_n-u_k, u_n-u_k\rangle_L &\leq |\langle u_n, u_n-u_k\rangle_L|\;|\langle u_k, u_n-u_k\rangle_L| \\
&\leq 2\varepsilon\|u_n-u_k\|_{v_0,v_1} + \\
&\quad \left|\int_\Omega f(u_n)(u_n-u_k)dx\right| + \left|\int_\Omega f(u_k)(u_n-u_k)dx\right|
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
\left|\int_\Omega f(u_n)(u_n-u_k)dx\right| &\leq \int_\Omega |f(u_n)w^{-1/q}|\,|(u_n-u_k)w^{1/q}|\,dx \\
&\leq \left(\int_\Omega |f(u_n)w^{-1/q}|^{q'}dx\right)^{1/q'} \left(\int_\Omega |u_n-u_k|^q w\,dx\right)^{1/q}.
\end{aligned}$$

Since $\|u_n - u_k\|_{q,w}$ tends to zero and $\int\limits_\Omega |f(u_n)|^{q'} w^{1/(1-q)} dx$ is bounded (by Lemma 3.1), it follows that $\|u_n - u_k\|_{v_0,v_1}$ tends to zero. $\square$

By means of the Mountain Pass Lemma (cf. [10]) we can now prove

Theorem 3.4 *If f satisfies conditions f1–f3, and L defines a continuous bilinear form on $W^{1,2}(\Omega; v_0, v_1)$, which satisfies the ellipticity condition (7), then there exists a non-trivial positive solution of (1), (2).*

Proof. From f1–f2 we obtain to every $\varepsilon > 0$ a constant $C_\varepsilon \geq 0$ such that $F(x,t) \leq \varepsilon f_0(x) t^2 + C_\varepsilon f_1(x)|t|^q$, $2 < q < 2N/(N-2)$. Consequently

$$\begin{aligned} J(u) &\geq \frac{1}{2}\langle u, u\rangle_L - \int\limits_\Omega \left(\varepsilon f_0(x) u^2 + C_\varepsilon f_1(x)|u|^q\right) dx \\ &\geq K\|u\|^2_{v_0,v_1} - \varepsilon C\|u\|^2_{q,w} - C_\varepsilon C\|u\|^q_{q,w} \\ &\geq (K - \varepsilon\tilde{C})\,\|u\|^2_{v_0,v_1} - \tilde{C}_\varepsilon\|u\|^q_{v_0,v_1}\,. \end{aligned}$$

For ε small we can now choose a $\rho > 0$ such that $J(u) \geq \delta > 0$ if $\|u\|_{v_0,v_1} = \rho$.

On the other hand, by f1 and f3, we can choose a positive $v \in W^{1,2}(\Omega; v_0, v_1)$, $v \geq R > 0$ in $O \subset \Omega$, such that $|F(x, \lambda v)| \geq h(x)|\lambda v|^q$, $q > 2$ for $\lambda \geq 1$ with $h > 0$ in O and get

$$J(\lambda v) \leq \frac{1}{2}\lambda^2\langle v, v\rangle_L - |\lambda|^q \int\limits_O h(x)|v|^q\, dx + c\,.$$

For $\lambda \to \infty$ the right hand side of this inequality tends to $-\infty$.

Therefore, we have shown, that the functional J satisfies the Palais-Smale condition, $J(0) = 0$, $J(u) \geq \delta > 0$ if $\|u\|_{v_0,v_1} = \rho$ (small enough) and $J(v) \leq 0$ for some v, $\|v\|_{v_0,v_1} > \rho$. The Mountain Pass Lemma (cf. [10]) now implies the existence of a critical point of J.

To show the existence of a positve solution, we consider the truncated function

$$\tilde{f}(x,u) = \begin{cases} f(x,u) & , u \geq 0 \\ 0 & , u < 0 \end{cases} .$$

With the same arguments as above we get a solution $\tilde{u}$ of the corresponding problem. Let $\tilde{u}^+$, $\tilde{u}^-$ denote the positive and negative parts of $\tilde{u}$, i. e. $\tilde{u} = \tilde{u}^+ - \tilde{u}^-$. Since $\tilde{f}(x, \tilde{u}^-) = 0$ we get

$$\langle \tilde{u}^-, \tilde{u}^- \rangle_L = \langle \tilde{u}, \tilde{u}^- \rangle_L = \int_\Omega \tilde{f}(x, \tilde{u}) \tilde{u}^- \, dx = 0\,,$$

which shows $\tilde{u}^- = 0$. Since $\tilde{u}$ is also a solution of the original problem (1), (2), the proof is complete. □

Remark 3.5 *The condition* $\lim_{t \to 0} f(x,t)/t = 0$ *in f1 can be ommitted, if we assume instead the following*

f4 *Let* K *be as in (7),* C *from f2 and* S_q *denote the embedding constant. Assume that* $f_0 \in L^2(\Omega, w^{-1})$ *and there exists a number* ρ, $0 < \rho < \left(K/CS_q^q\right)^{1/(q-2)}$ *(*q *from f2), such that*

$$\|f_0\|_{2,w^{-1}} \leq \frac{K\rho - CS_q^q \rho^{q-1}}{2S_2}\,.$$

Then the statement of Theorem 3.4 remains true.

In fact, the only difference in the proof concerns the estimate $J(u) \geq \delta > 0$. For $\|u\|_{v_0,v_1} = \rho$ (ρ from f4) we now obtain

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\langle u, u \rangle_L - C\|u\|_{q,w}^q - \int_\Omega f_0(x) u \, dx \\
&\geq K\|u\|_{v_0,v_1}^2 - CS_q^q \|u\|_{v_0,v_1}^q - \|f_0\|_{2,w^{-1}} S_2 \|u\|_{v_0,v_1} \\
&\geq K\rho^2 - CS_q^q \rho^q - S_2 \rho \frac{K\rho - CS_q^q \rho^{q-1}}{2S_2} \\
&= \frac{1}{2}(K\rho^2 - CS_q^q \rho^q) \\
&= \delta \;>\; 0\,.
\end{aligned}$$

Condition f4 allows to consider also nonhomogeneous problems under some more restrictive assumptions on f. For example, if $f(x,u) = f_0(x) + f_1(x)u^{q-1}$, f satisfies f1 – f2 (except $\lim_{t \to 0} f(x,t)/t = 0$), f_0 satisfies f4 and additionally $f_0, f_1 \in L^1(\Omega)$, then f3 can be replaced by the weaker condition

f3′ There exists a $\theta \in I\!R$, $0 \le \theta < 1/2$ and $R \ge 0$ such that $F(x,t) \le \theta f(x,t)t$ for every $x \in \Omega$, $t \ge R$.

The only difference in the arguments above now concerns the proof of Lemma 3.3. Let u_k be a Palais-Smale sequence for J. From the estimates

$$\begin{aligned} \int\limits_{\Omega\cap\{u_k(x)<R\}} |\theta f(x,u_k)u_k|dx &\le C\int\limits_{\Omega} |f_0(x)|+|f_1(x)|dx \le C_1\,, \\ \int\limits_{\Omega\cap\{u_k(x)<R\}} |F(x,u_k)|dx &\le C_2\,, \end{aligned}$$

we obtain $\int\limits_{\Omega} \theta f(x,u_k)u_k - F(x,u_k)\,dx \ge -(C_1+C_2)$. Then (9) leads to

$$(\frac{1}{2}-\theta)\langle u_k,u_k\rangle_L - \theta\varepsilon\|u_k\|_{v_0,v_1} - (C_1+C_2) \le M\,.$$

Again we conclude that $\|u_k\|_{v_0,v_1}$ must be bounded.

We remark that f3′ can be satisfied for example if $f_0 \le f_1$, with $\theta > 1/q$ and $R \ge [(1-\theta)/(\theta-1/q)]^{1/(q-1)}$.

4 Examples

We consider the problem

$$\begin{aligned} -\Delta u + b(x)u &= f(x,u) \qquad \text{in } \Omega, &(10)\\ u &= 0 \qquad \text{on } \partial\Omega, &(11)\\ \lim_{|x|\to\infty} u(x) &= 0\,, &(12) \end{aligned}$$

where $c(x) \ge 0$ is in $L^\infty(\Omega)$.

For the first example we consider the weight functions

$$v_0(x) = (1+|x|)^{-2}\,, \quad v_1(x) = 1\,, \quad w(x) = (1+|x|)^{-\alpha} \quad (\alpha > 0)$$

and we assume that f satisfies f1–f3 from Section 3. To get compactness of the embedding $W^{1,2}(\Omega;(1+|x|)^{-2},1) \hookrightarrow L^q(\Omega;(1+|x|)^{-\alpha})$, we have to choose α such that

$$-\frac{\alpha}{q}+\frac{N}{q}-\frac{N}{p}+1<0\,, \tag{13}$$

where $q \in (2, 2N/(N-2))$ is defined by f2 (see Example 2.2(1) with $\beta = 0$). This inequality is satisfied for $\alpha > [2N - q(N-2)]/2$, in particular for $\alpha = 2$. On the other hand, if $0 < \alpha < 2$, then (13) holds only for

$$q-1>\frac{N+2-2\alpha}{N-2}$$

(this is exactly condition (3) in [9]). Furthermore, the Hardy-inequality (Example 2.4) shows that the norms $\|u\|_{v_0,v_1}$ and $|||u|||_{1,2,\Omega,v_1} = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2}$ are equivalent. Consequently, if $0 \le b(x) \le Cv_0(x)$, then the norm $\langle\cdot,\cdot\rangle_L^{1/2}$ defined by $L = -\Delta + b$ is equivalent to $\|\cdot\|_{v_0,v_1}$. Therefore, $\langle\cdot,\cdot\rangle_L$ is a continuous bilinear form on $W^{1,2}(\Omega;(1+|x|)^{-2},1)$, which satisfies the ellipticity condition (7). Theorem 3.4 now shows that there is a positive solution of problem (10)–(12). Note that this includes also the so-called zero mass case ($b = 0$). This result is essentially the same as in [9] and [1], although the assumptions on f are somewhat different.

For the next example we assume $b(x) \ge b_0 > 0$ and choose

$$v_0 = v_1 = 1\,, \quad w(x) = e^{-\alpha|x|} \quad (\alpha > 0).$$

From Example 2.2(2) we obtain compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega; e^{-\alpha|x|})$ for every $\alpha > 0$, $2 < q < 2N/(N-2)$. Clearly, $L = -\Delta + b$ defines a continuous bilinear form and satisfies (7). If f satisfies f1–f3, then Theorem 3.4 again gives a positive solution to (10)–(12).

Remark that if f is given as a function of the form

$$f(x,t)=\sum_{i=1}^{m} g_i(x)t^{k_i}\,, \qquad 1<k_1<\ldots<k_m<(N+2)/(N-2)\,,$$

then assumption f2 reduces to $|g_i(x)| \le Ce^{-\alpha|x|}$, $i = 1,\ldots,m$. In particular, if $\sup_{x\in\Omega} |g_i(x)e^{\alpha|x|}| < \infty$, then f2 holds. Therefore, our assumptions are somewhat weaker that those in [13], where (10)–(12) was solved by a fixed point method in a weighted subspace of $C^0(\mathbb{R}^N)$.

Of course, instead of $-\Delta + b$ we can take any other elliptic operator in (10), which defines a continuous bilinear form and satisfies (7).

References

[1] W. Allegretto, L. S. Yu. *Positive L^p-solutions of subcritical nonlinear problems.* J. Differ. Equations 87, 340–352 (1990)

[2] H. Berestycki, P. L. Lions. *Nonlinear scalar field equations I,II.* Arch. Ration. Mech. Anal. 82, 313 – 375 (1983)

[3] V. Benci, D. Fortunato. *Weighted Sobolev spaces and the nonlinear Dirichlet problem in unbounded domains.* Ann. Mat. Pura Appl. 121, 319 – 336 (1979)

[4] W.–Y. Ding, W.–M. Ni. *On the existence of positive entire solutions of a semilinear elliptic equation.* Arch. Ration. Mech. Anal. 91, 283 – 308 (1986)

[5] A. Kufner, B. Opic. *Hardy-type inequalities.* Pitman Res. Notes. Math. 219, Longman, Harlow (1990).

[6] P. L. Lions. *The concentration-compactness method in the calculus of variations. The locally compact case. I, II.* Ann. Inst. H. Poincaré, Anal. Non Linéaire 1, 109 – 145, 223 – 283 (1984).

[7] J. Nečas. *Les méthodes directes en théorie des équations elliptiques.* Academia, Prague (1967)

[8] E. S. Noussair, C. A. Swanson. *Positive solutions of semilinear elliptic problems in unbounded domains.* J. Differ. Equations 57, 349 – 372 (1985)

[9] E. S. Noussair, C. A. Swanson. *An $L^q(\mathbb{R}^n)$ - theory of subcritical semilinear elliptic problems.* J. Differ. Equations 84, 52 – 61 (1990)

[10] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations.* Reg. Conf. Ser. Math. 65 (1986)

[11] M. Struwe. *Variational methods.* Springer, Berlin etc. (1990)

[12] M. M. Vainberg. *Variational methods for the study of nonlinear operators.* Holden Day, San Francisco (1964)

[13] P. Volkmann, A. Chaljub-Simon. *Existence of ground states with exponential decay for semilinear elliptic equations in $\mathbb{R}^n$.* J. Differ. Equations 76, 374 – 390 (1988)