SERIE A — MATHEMATIK

Weakly Pancyclic Graphs

Stephan Brandt[†]*
Ralph Faudree[‡]**
Wayne Goddard[§]***

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Abstract

In generalizing the concept of a pancyclic graph, we say that a graph is 'weakly pancyclic' if it contains cycles of every length between the length of a shortest and a longest cycle. In this paper it is shown that in many cases the requirements on a graph which ensure that it is weakly pancyclic are considerably weaker than those required to ensure that it is pancyclic. This sheds some light on the content of a famous metaconjecture of Bondy. From the main result of this paper it follows that 2-connected nonbipartite graphs of sufficiently large order n with minimum degree exceeding $2n/7$ are weakly pancyclic; and that graphs with minimum degree at least $n/4 + 250$ are pancyclic, if they contain both a triangle and a hamiltonian cycle.

[†]FB Mathematik und Informatik, Freie Universität Berlin,
Graduiertenkolleg 'Alg. Diskr. Mathematik',
Arnimallee 2–6, 14195 Berlin, Germany
e-mail: brandt@math.fu-berlin.de
[‡]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
e-mail: faudreer@hermes.msci.memst.edu
[§] Department of Computer Science
University of Natal
Durban 4000, South Africa
e-mail: goddard@cs.und.ac.za
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1 Introduction

We investigate the set of cycle lengths occurring in nonbipartite graphs with large minimum degree. The *girth* of a graph G is the length of a shortest cycle, denoted $g(G)$, and the *odd girth* the length of a shortest odd cycle. The *circumference* is the length of a longest cycle and will be denoted by $K(G)$. A graph of order n is said to be *hamiltonian* if the circumference is n. A graph is called *weakly pancyclic* if it contains cycles of every length between the girth and the circumference. A graph is *pancyclic* if it is weakly pancyclic with girth 3 and circumference $n = |G|$.

The investigation of pancyclic graphs was initiated by Bondy [7], who established several sufficient conditions for a graph to be pancyclic. A special case of one of them (see Theorem 2.10 below) is the following extension of Dirac's famous condition for hamiltonicity [16].

Theorem 1.1 (Bondy [7]) *If graph G of order n has minimum degree $\delta(G) \geq n/2$, then G is pancyclic, or $n = 2r$ and $G = K_{r,r}$.*

This result is best possible: if the condition on the minimum degree is dropped to below $n/2$, then the graph can be bipartite and/or nonhamiltonian. Over the years the condition has been improved by replacing the minimum-degree condition by conditions requiring only a few vertices to have large degree.

Most of the known sufficient conditions for pancyclic graphs involving degree constraints can be derived from the following local condition.

Theorem 1.2 (Schmeichel & Hakimi [34]) *Let G be a graph of order n containing a hamiltonian cycle $x_1x_2 \ldots x_nx_1$ where the degree sum $d(x_1) + d(x_n) \geq n$. Then G is either pancyclic, bipartite, or missing only an $(n-1)$-cycle.*

In the latter case much more can be said about the structure of the graph. But these conditions always require some vertices of degree at least $n/2$.

The first paper to investigate conditions without this requirement is due to Amar et al. [2]. They showed the following.

Theorem 1.3 (Amar, Flandrin, Fournier & Germa [2]) *Let G be a nonbipartite hamiltonian graph of order $n \geq 102$. If $\delta(G) > 2n/5$ then G is pancyclic.*

Theorem 1.7 below shows that the conclusion is in fact true for all n. The result is best possible since there are nonbipartite graphs of minimum degree $\delta(G) = 2n/5$ without triangles (the lexicographic products $C_5[\overline{K_r}]$ of the 5-cycle with the empty graph on r vertices). This result was extended slightly by Shi [38] who showed that if G is a nonbipartite hamiltonian graph of order $n \geq 50$ and for all pairs of nonadjacent vertices u and v it holds that $d(u) + d(v) > 4n/5$, then G is pancyclic.

In this paper we introduce weakly pancyclic graphs as a relaxation of the concept of pancyclic graphs. Our main result is a sufficient degree condition for a nonbipartite graph to be weakly pancyclic.

Theorem 1.4 *Let G be a 2-connected nonbipartite graph of order n with minimum degree $\delta(G) \geq n/4 + 250$. Then G is weakly pancyclic unless G has odd girth 7, in which case it has every cycle from 4 up to its circumference except the 5-cycle.*

We make no attempt to obtain the best value of the additive constant. Since a shortest odd cycle in a graph with minimum degree $\delta > 2n/7$ has length at most 5 (easy exercise), we obtain the following consequence.

Corollary 1.5 *If G is a 2-connected graph of sufficiently large order n with $\delta(G) > 2n/7$, then G is weakly pancyclic.*

If we explicitly require the triangle and the hamiltonian cycle, it follows from Theorem 1.4 that the above minimum degree is sufficient to ensure that the graph is pancyclic.

Corollary 1.6 *If a graph G of order n with minimum degree $\delta(G) \geq n/4 + 250$ contains a triangle and a hamiltonian cycle, then it is pancyclic.*

Observe that this bound is much smaller than the sufficient minimum degree for a nonbipartite graph to be hamiltonian ($\delta \geq n/2$) or to contain a triangle ($\delta > 2n/5$).

Theorem 1.4 and Corollary 1.6 are best possible (up to the additive constant). Take two copies of $K_{m,m}$ intersecting in one vertex and join one vertex on the opposite side of the intersection vertex in one $K_{m,m}$ to such a vertex in the other $K_{m,m}$. This graph is hamiltonian, has minimum degree $m = (n+1)/4$ and contains a triangle, but it contains no even cycle of length more than $(n+1)/2$.

Corollary 1.5 is best possible, since the lexicographic product $C_7[\overline{K_r}]$ for $r \geq 2$ contains all possible cycles except the 3- and 5-cycle. Moreover, there are small order examples such as the Petersen graph (which contains no 7-cycle) which have $\delta(G) > 2n/7$ but are not weakly pancyclic.

For small-order graphs we can improve on the degree bound of Theorem 1.4.

Theorem 1.7 *Every nonbipartite graph G of order n with minimum degree $\delta(G) \geq (n+2)/3$ is weakly pancyclic.*

A proof is given in the first author's thesis [11]. Note that this result implies the truth of Theorem 1.3 for every n.

It is interesting to note that there is no connectivity requirement in Theorem 1.7. It is almost best possible since the graph formed by taking K_{m+1} and $K_{m,m}$ and identifying one vertex ($m \geq 3$) has minimum degree $m = n/3$ and all even cycles up to $2m$ but no odd cycle on more then $m + 1$ vertices. There is also a 2-connected 3-regular graph of order 8 which is not weakly pancyclic.

For triangle-free graphs with minimum degree exceeding $n/3$, the first author [10] showed that the exact value of the circumference of a graph G can be determined in terms of the order and independence number $\alpha(G)$.

Theorem 1.8 (Brandt [10]) *Let $G \neq C_5$ be a nonbipartite triangle-free graph of order n. If $\delta(G) > n/3$ then G is weakly pancyclic with girth 4 and circumference $\min\{2(n-\alpha(G)), n\}$.*

Graphs which show that Theorem 1.8 is best possible are given in [10].

In the next section we discuss some of the consequences of Theorem 1.4 and some related results about (weakly) pancyclic graphs. Thereafter, in Sections 3 through 7 we prove Theorem 1.4. Finally, in Section 8 we address algorithmic issues and in Section 9 we discuss possible future work.

2 Some Consequences and Related Results

Using Theorems 1.4 and 1.7 one can take known sufficient conditions for a graph to be hamiltonian and extend them to conditions for a graph to be pancyclic or to have all cycles but the triangle.

Nash-Williams-type results. These are degree conditions involving the independence number and the connectivity.

Corollary 2.1 *Let G be a 2-connected graph of order n. If $\delta(G) \geq \max\{\alpha(G), (n+2)/3\}$, then G contains all cycles between 4 and n, or $n = 2r$ and $G = K_{r,r}$.*

Nash-Williams showed that the hypothesis implies that the graph is hamiltonian. Since every bipartite graph with $\delta \geq \alpha$ is complete bipartite and balanced, the corollary follows from Theorem 1.7. An immediately consequence is that every 2-connected graph of order n with minimum degree $\delta \geq \max\{\alpha + 1, (n+2)/3\}$ is pancyclic.

Corollary 2.2 *Let G be a graph of order n with connectivity $\kappa(G) \geq 2$. If $\delta(G) \geq (n + \kappa(G))/3$, then G is pancyclic, or $n = 2r$ and $G = K_{r,r}$.*

Häggkvist and Nicoghossian proved that graphs satisfying the hypothesis of Corollary 2.2 obey the hypothesis of Corollary 2.1. Therefore such graphs contain all cycles between 4 and n unless $G = K_{r,r}$ ($n = 2r$). Now, suppose that G contains no triangle. Then by [10, Lemma 3.1] $\delta(G) = \kappa(G)$ and thus $\delta(G) \geq n/2$, so $G = K_{r,r}$.

Binding number. Another consequence of Theorem 1.7 is a result first proved by Shi [36] showing that graphs with binding number at least 3/2 are pancyclic:

Corollary 2.3 *Let G be a graph of order n. If for every subset $S \subseteq V(G)$ the cardinality of the neighborhood $|N(S)| \geq \min\{(3/2)|S|, n\}$, then G is pancyclic.*

Note that the condition implicitly requires $\delta(G) \geq (n+2)/3$. (Take $S = V(G) - N(v)$ for a vertex v of minimum degree.) This result was conjectured by Woodall [43] who verified that graphs satisfying the hypothesis are hamiltonian. For several years it was open whether the condition implies the triangle. This was verified in an earlier paper of Shi [35] (for a short proof see [19]). The remainder of the cycles are given by Theorem 1.7.

Regular graphs. Jackson [24] proved in 1980 that d-regular 2-connected graphs with $d \geq n/3$ are hamiltonian. This result has been refined and generalized by several authors. The current best result is due to Broersma et al. [13], who showed that 2-connected d-regular graphs of order n with $d \geq 2(n+7)/7$ are hamiltonian unless they contain three vertices which do not lie on a common cycle.

Together with Theorem 1.4 we derive the following:

Corollary 2.4 *Every 2-connected d-regular graph of sufficiently large order n with* $d \geq 2(n+7)/7$ *is pancyclic, unless it is triangle-free or it contains three vertices which do not lie on a common cycle.*

It was conjectured by Häggkvist (cf. [24]) that d-regular k-connected graphs with $d \geq n/(k+1)$ are hamiltonian. Though false in general, this is believed to hold for $k = 3$ (cf. [13]). This would imply that Corollary 2.4 holds even with the bound $d \geq n/4 + 250$.

Bondy's metaconjecture. Bondy [8] conjectured that

> almost any nontrivial condition on a graph which implies that the graph is hamiltonian also implies that it is pancyclic except for maybe a simple family of exceptional graphs.

The above results as well as other recent results shed some light on the metaconjecture.

Theorem 1.4 suggests that sufficient conditions for a nonbipartite graph to be weakly pancyclic might be significantly smaller than those for being hamiltonian. So once a hamiltonicity condition implies the triangle it follows that the graphs are pancyclic. Theorem 1.2 is further evidence that once a nonbipartite graph is hamiltonian the conditions to be pancyclic are weaker than that required to force the hamiltonian cycle in the first place. This goes some way to explaining the success of the metaconjecture.

But there are several conditions for graphs to be hamiltonian which apply to large classes of triangle-free graphs. Consider, for example, the family of graphs where G_i is the complement of the ith power of the cycle C_{3i+2} $(i \geq 1)$. So $G_1 = C_5$ and G_2 is an 8-cycle with the long chords. (Woodall [43] showed that the binding numbers of these triangle-free graphs approach 3/2 from below.) The lexicographic product $G_i[\overline{K}_s]$ $(s \geq 1)$ is a triangle-free r-regular graph with independence number $\alpha = r = s(i+1)$ and connectivity r. So it satisfies the famous hamiltonicity conditions given by Chvátal and Erdős [14] $(\alpha \leq \kappa)$, by Nash-Williams (cf. Corollary 2.1) (for $s \geq 2$), and by Jackson [24] (2-connected and r-regular with $r \geq n/3$). But these graphs are not pancyclic.

The graph $G_i[\overline{K}_s]$ is weakly pancyclic though, which is a consequence of the following more general result.

Theorem 2.5 (Lou [28]) *If a triangle-free graph G satisfies $\alpha(G) \leq \kappa(G)$, then G is weakly pancylic with girth 4 and circumference n, unless $G = K_{r,r}$ or $G = C_5$.*

And it is possible that a slight strengthening of the Chvátal–Erdős condition implies that the graph is pancyclic.

Conjecture 2.6 *If a graph G satisfies $\alpha(G) < \kappa(G)$, then it is pancyclic.*

Jackson and Ordaz [26] extracted this conjecture from a conjecture of Amar, Fournier and Germa [3]. Note that the claim in [26] that this conjecture is a consequence of Theorem 2.5 is incorrect.

Toughness. A graph is said to be t-tough if the removal of any cutset S of vertices yields at most $|S|/t$ components in $G - S$. Recently, Bauer, van den Heuvel and Schmeichel [5] constructed triangle-free graphs which are t-tough with t arbitrarily large. This refutes a conjecture of Chvátal [15] of the existence of a constant t_0 such that every t_0-tough graph is pancyclic. Their result was extended by Alon [1] who proved that there are graphs with arbitrarily large girth and toughness. We show here that there is also no sufficiently large value of toughness that will ensure that a graph is weakly pancyclic.

Lemma 2.7 (Brandt [11]) *For every $t > 1$ the line graph $L(G)$ of a $2t$-edge-connected graph G with girth $g(G)$ and maximum degree $\Delta(G)$ that satisfies $g(G) > \Delta(G) + 1$ is t-tough and has cycles of length $\Delta(G)$ and $g(G)$ but none in between.*

Proof: The line graph $L(G)$ is $2t$-connected, as every vertex cutset of $L(G)$ corresponds to an edge cutset of G. Since $L(G)$ is claw-free, its toughness is equal to one-half its connectivity by a result of Matthews and Sumner [30]. Since the edges incident with a vertex v in G form a clique in $L(G)$ there is a cycle of length $\Delta(G)$ in $L(G)$. The edges of a cycle of length $g(G)$ in G form a cycle in $L(G)$.

Now any subgraph of G on less than $g(G)$ edges is a forest. So every subgraph of $L(G)$ induced by less than $g(G)$ vertices is the line graph of a forest F, and hence has circumference $\Delta(F) \leq \Delta(G)$. This means that every cycle of length less than $g(G)$ in $L(G)$ has length at most $\Delta(G)$. □

To use Lemma 2.7 we need graphs of large connectivity whose girth exceeds the maximum degree. For infinitely many d, Lubotsky, Phillips and Sarnak [29] constructed graphs which are d-regular and have arbitrarily large girth. As these graphs are connected and vertex-transitive, the connectivity must be at least $2d/3$ (Watkins [42]). Thus we obtain the following result.

Corollary 2.8 (Brandt [11]) *There are t-tough graphs with t arbitrarily large which are not weakly pancyclic.*

If another famous conjecture of Chvátal [15] holds—saying that there is a constant t_0 such that every t_0-tough graph is hamiltonian—then this is a condition which seriously violates Bondy's metaconjecture, cited above.

As a step towards this, Zhan [44] and Jackson [25] showed that every 7-connected (or equivalently 3.5-tough) line graph is hamiltonian. This was recently generalized to claw-free graphs by Ryjáček [33]. (In fact, using a closure concept Ryjáček showed that any sufficient condition for line graphs which is closed under the addition of edges also suffices for claw-free graphs.) Perhaps it is true that claw-free graphs of sufficiently large connectivity have a large range of cycles.

In light of the demise of Chvátal's pancyclic conjecture, Bauer, van den Heuvel and Schmeichel conjectured the following.

Conjecture 2.9 (Bauer et. al. [5]) *If G is a t-tough graph of order n and $\delta(G) > n/(t+1)$, then G is pancyclic.*

Note that the hypothesis is sufficient for a graph to contain a triangle. (Consider $S = V(G) - N(v)$ for v a vertex of degree $\delta(G)$: the hypothesis implies that $G - S$ has fewer than $\delta(G)$ components.) Furthermore, Bauer et al. [4] have shown that a slightly weaker bound is sufficient for the graph to be hamiltonian. It follows from Corollary 1.6 that the conjecture is true at least when $t < 3 - O(1/n)$.

Minimum size. We turn now to lower bounds involving the number of edges in the graph. Bondy proved the following statement.

Theorem 2.10 (Bondy [7]) *Every hamiltonian graph G of order n and size $e(G) \geq n^2/4$ is pancyclic, or $n = 2r$ and $G = K_{r,r}$.*

Bollobás [6, Sec. III, Theorem 5.2] extended this to show that graphs with circumference K, order n and size $e(G) > K(2n-K)/4$ are weakly pancyclic (with girth 3).

The bound in Theorem 2.10 was improved for nonbipartite graphs.

Theorem 2.11 (Häggkvist, Faudree & Schelp [21]) *Every hamiltonian graph G of order n and size $e(G) > (n-1)^2/4 + 1$ is pancyclic or bipartite.*

The same size is sufficient without the hamiltonicity requirement to guarantee that the graph is weakly pancyclic with girth 3.

Theorem 2.12 (Brandt [12]) *Every nonbipartite graph G of order n and size $e(G) > (n-1)^2/4 + 1$ contains cycles of every length between 3 and the circumference of G.*

Even smaller size might suffice, if we only want the graph to be weakly pancyclic.

Conjecture 2.13 (Brandt [12]) *Every nonbipartite graph G of order n and size $e(G) > (n-1)(n-3)/4 + 4$ is weakly pancyclic.*

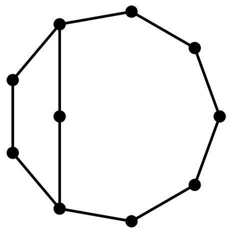

Figure 1: A 9-bicycle in a graph with odd girth 5

3 Proof of Main Theorem: Outline

Our approach to proving Theorem 1.4 is to start with a shortest cycle of a graph G and to enlarge the cycles we find. This process stops once we find a cycle C where $G - C$ consists of small order components. As C is not necessarily a longest cycle, we have to start again with a longest cycle and work in the opposite direction. The key to making this work is what we call a "bicycle", a subgraph representing two consecutive cycle lengths. This allows one to build up or down in steps of 1 or 2 vertices.

3.1 Bicycles

Let T be a shortest odd cycle of a nonbipartite graph G of order n and $2t+1$ be its length. By counting the edges joining T and $G - T$ we get that $\delta(G) \leq 2n/(2t+1)$ for $t > 1$, since T is an induced cycle and no vertex of G can have more than two neighbors in T. As we investigate graphs with $\delta \geq n/4$ we get $t \leq 3$, so T is a 3-, 5- or 7-cycle.

We define a pair of vertices in T as *antipodal* if they are at distance t apart on T. A *k-bicycle* of G is a subgraph consisting of a shortest odd cycle T and an internally disjoint path of length $k - t - 1$ joining a pair of antipodal vertices of T. So a k-bicycle contains both a k-cycle and a $(k-1)$-cycle (and a $(2t+1)$-cycle). See Figure 1 for an example.

The *length* of a k-bicycle is k, which can be different from its order $k + (t-3)/2$. We will refer to the path of length $k - t - 1$ (including its endvertices on T) as the *standard path* and call its vertices *standard vertices*. We say that:

a k-bicycle is maximal *if there is neither a $(k+1)$- nor a $(k+2)$-bicycle.*

Note that maximality is a property of the length of a bicycle. In general, there can be maximal bicycles of different lengths in a graph, but we will see that under the hypothesis of Theorem 1.4 this is not the case.

Bicycles with T being a triangle were introduced and used extensively by Amar et al. [2] (and thereafter in [36] and [38]). Some of the ideas used in our proof and the global approach were motivated by [2].

3.2 Proof of Theorem 1.4

Theorem 1.4 is a consequence of the following three results.

Proposition 3.1 *If G is a nonbipartite graph with $\delta \geq n/4 + 250$, then G contains a bicycle C with standard path of length 3 or 4, and cycles of every even length less than the length of C.*

Proposition 3.2 *If G is a 2-connected nonbipartite graph with $\delta \geq n/4 + 250$, then G misses no two consecutive bicycles between its shortest and longest.*

Proposition 3.3 *If G is a 2-connected nonbipartite graph with $\delta \geq n/4 + 250$ and circumference K, then G contains a k-bicycle for some $k \geq K - 9$ and every cycle length between k and K.*

We now derive Theorem 1.4 from the above three propositions.

Suppose that G has circumference K and odd girth $2t + 1$. We have to show that G contains even cycles of every length k between 4 and K and odd cycles of every length k between $2t + 1$ and K. Let g_0 and K_0 be the length of a shortest and longest bicycle, respectively. By Proposition 3.1 we have $g_0 \leq t + 5$. Let k be a cycle length in the indicated range. If $k < g_0 - 1$ and even then, by Proposition 3.1, G contains a cycle of length k. The only required odd cycle length in this range is $2t+1$, which the graph has. If $g_0 - 1 \leq k \leq K_0$ then, by Proposition 3.2, G contains a k- or $(k+1)$-bicycle, which both contain a k-cycle. Finally, by Proposition 3.3, G contains cycles of every length between K_0 and K. □

Most of the remainder of the paper is devoted to establishing these three propositions.

3.3 More propositions

Proposition 3.1 is straightforward. But the other two propositions require a lot more work.

In the investigation of cycles contained in a graph it has proven useful to examine for a cycle C the structure of $G - C$. The same features are useful for bicycles. Following Veldman [40] we call a (bi)cycle C a D_λ-(bi)cycle, if all components of $G - C$ have order less than λ. So a D_1-cycle is a hamiltonian cycle and a D_2-cycle is what is called a dominating cycle.

The proof of Proposition 3.3 when G is 3-connected uses the following result on the structure of longest cycles in G.

Theorem 3.4 (Jung [27]) *If G is a 3-connected graph with $\delta \geq (n + 6)/4$, then every longest cycle is a D_3-cycle.*

In Proposition 3.6 below we prove the same result for maximal bicycles (with a larger additive constant). The proof of Proposition 3.2 is a consequence of the following four results.

Proposition 3.5 *Let G be a graph with $\delta \geq n/4 + 250$ and connectivity $2 \leq \kappa < 80$. Then G contains a triangle and a bicycle of every length from 4 up to the circumference of G.*

Proposition 3.6 *Let G be a 3-connected graph with $\delta \geq n/4 + 250$. Then every maximal bicycle of G is a D_3-bicycle. If G is triangle-free then every maximal bicycle is a D_2-bicycle.*

Proposition 3.7 *Let G be a 3-connected graph with $\delta \geq n/4 + 250$. If C is a maximal bicycle of G then $|C| \geq \min\{n-2, 3\delta - 10, n - \alpha + \delta - 6\}$.*

Proposition 3.8 *Let G be a graph with $\delta \geq n/4 + 250$. If G contains a k-bicycle for some k in the range $k > \min\{3\delta, n - \alpha + \delta\} - 10$, then G contains a $(k-1)$- or a $(k-2)$-bicycle.*

Note that Propositions 3.5, 3.6, and 3.7 are also used in the proof of Proposition 3.3. The remainder of the proof is performed in the following four sections:

- In Section 4 we establish some basic lemmas.
- In Section 5 we prove Proposition 3.1.
- In Section 6 we prove Proposition 3.2 by proving Propositions 3.5, 3.6, 3.7 and 3.8.
- In Section 7 we prove Proposition 3.3.

That will complete the proof of Theorem 1.4.

4 Lemmas and Notation

For a vertex, v we define $N(v)$ to be the set of neighbors of v, and $d(v)$ to be the degree of v. If H is a fixed subgraph, then $N_H(v) = N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$.

We often assign a fixed orientation to a cycle C; the orientation is arbitrary unless explicitly specified. For a vertex a we denote by a^+ and a^- the successor and predecessor of a on C, and by a^{+i} and a^{-i} the i-th successor and predecessor of a on C, respectively. We denote by $C[a,b]$ the segment $aa^+ \dots b^-b$ of the cycle.

4.1 Double sweeps

An important tool is what we call a "double sweep". A version for cycles is described in the following lemma.

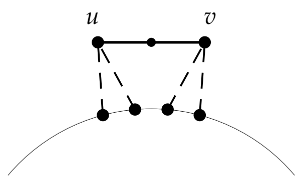

Figure 2: Double sweep of a cycle with $\varepsilon = 1$

Lemma 4.1 (Double sweep of a cycle) *Let G be a graph containing a k-cycle C of G which is oriented arbitrarily. Let u, v be vertices in $G - C$ both having no consecutive neighbors on C, and ε an integer. If $d_C(u) + d_C(v) > k/2$, then u has a neighbor z and v has a neighbor $z^{+\mu}$ for some μ in the range $\varepsilon \leq \mu \leq \varepsilon + 2$.*

Proof: For every $z \in C$, let e_z be the number of edges present in G among the four potential edges $uz, uz^+, vz^{+\varepsilon+1}, vz^{+\varepsilon+2}$. Clearly $\sum_{z \in C} e_z = 2(d_C(u) + d_C(v)) > k$, so there must be a z with $e_z \geq 2$. Since neither u nor v has consecutive neighbors on C, either z^+ or z is the desired neighbor of u. $\square$

See Figure 2. More useful will be the following technical-sounding extension to path systems.

Lemma 4.2 (Double sweep of paths) *Let $\mathcal{P} = \bigcup_{1 \leq i \leq r} P_i$ be a collection of r vertex-disjoint paths of a graph G, each of them oriented arbitrarily, with a total of k vertices. Suppose u, v are vertices in $G - \mathcal{P}$ both of which have at most c pairs of consecutive neighbors in $\mathcal{P}$, and let integer $\varepsilon \geq -1$ be given. If*

$$d_{\mathcal{P}}(u) + d_{\mathcal{P}}(v) > k/2 + r(\varepsilon + 2)/2 + 2c,$$

then there exists a path P_i where u has a neighbor z and v has a neighbor $z^{+\mu}$ for some μ in the range $\varepsilon \leq \mu \leq \varepsilon + 2$.

Proof: Link the paths P_i together to form a cycle C whose orientation agrees with the orientation of the paths, by inserting an auxiliary path of length $\varepsilon + 3$ ($\varepsilon + 3$ new edges and $\varepsilon + 2$ new vertices) between the endvertices of consecutive paths. So C has length $k + r(\varepsilon + 2)$. Now delete for every pair of consecutive neighbors of u and v an edge so that in the resulting graph neither u nor v has consecutive neighbors on C. This decreases the degree-sum of u and v by at most $2c$. By Lemma 4.1, u has a neighbor z in C such that v is adjacent to $z^{+\mu}$ for $\varepsilon \leq \mu \leq \varepsilon + 2$. By the construction of C, the vertices z and $z^{+\mu}$ are on the same path P_i, as required. $\square$

We call the ordered pair (u, v) the *beacons* in the double sweep.

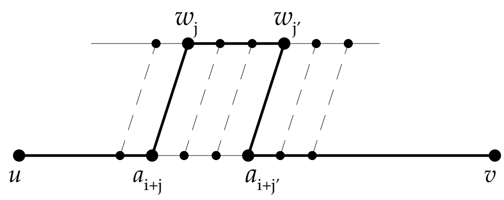

Figure 3: A comb sweep yields a $(k+2)$-path

4.2 Paths between specified vertices

Here we prove several lemmas on the length of paths between specified vertices given particular conditions. When dealing with bicycles, it simplifies the arguments if we leave the short cycle fixed and try to vary only the standard path. In many cases we have a path of length p joining two vertices u and v, and we want to find a (u,v)-path of length $p+1$ or $p+2$.

The basic tool in many of our arguments below is the first lemma: the "Comb Sweep Lemma".

Lemma 4.3 (Comb Sweep) *Let P be a (u,v)-path of length p in a graph G, and let Q be a path in $G-P$ of length q. If the number of edges between Q and P exceeds $p+q+1$, then u and v are joined by a path of length $p+2$ which uses only the vertices of P and Q.*

Proof: Suppose there is no (u,v)-path of length $p+2$ through vertices of P and Q. Say $P = a_0a_1\dots a_p$ and $Q = w_0w_1\dots w_q$. Consider the set $A_i = \{\, j : w_ja_{i+j} \in E(G)\,\}$ for $-q \le i \le p$. If $j,j' \in A_i$ where $j < j'$ then we obtain a (u,v)-path $a_0\dots a_{i+j}w_j\dots w_{j'}a_{i+j'}\dots a_p$ of length $p+2$. So $|A_i| \le 1$ for every i. But $\sum_i |A_i|$ is the sum of the degrees from Q to P. See Figure 3. $\square$

Lemma 4.4 *Let G be a graph containing two disjoint paths, one joining vertices u and x and one joining vertices v and y, that use all n vertices of G. If $d(x)+d(y) \ge n+1$ then G contains a hamiltonian (u,v)-path.*

Proof: The result is trivial if x and y are adjacent, so assume otherwise. Construct G' from G by adding the edge uv (if necessary). So, in G', x and y are joined by a hamiltonian (x,y)-path through uv. By the degree condition, there are two consecutive vertices z^- and z on the path, with $z \ne u$, such that x is adjacent to z and y to z^-. So we obtain a hamiltonian cycle through uv in G', and hence a hamiltonian (u,v)-path in G. $\square$

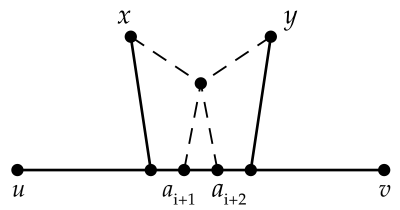

Figure 4: None of the four dashed edges is present

Lemma 4.5 *Assume $P = a_0a_1 \dots a_p$ ($p \geq 3$) is a (u,v)-path in a graph G. Suppose for some i ($0 \leq i \leq p-3$), x and y are distinct neighbors of a_i and a_{i+3}, resp., in $G-P$, and*

$$d(x) + d(a_{i+1}) + d(a_{i+2}) + d(y) \geq n + p + 5.$$

Then G contains a (u,v)-path of length $p+1$ or $p+2$.

Proof: The degree bound is chosen such that one of the following three possibilities must occur. The first possibility is that two of the four vertices have a common neighbor in $G-P$, but that trivially yields a longer (u,v)-path. See Figure 4. The second possibility is that the pair $\{x, a_{i+1}\}$ has degree-sum at least $p+3$ on the paths $a_0 \dots a_i x$ and $a_{i+1}a_{i+2} \dots a_p$. Then the result follows from Lemma 4.4 applied to the graph induced by the two paths. The third possibility is that the pair $\{y, a_{i+2}\}$ has degree-sum at least $p+3$ on the paths $a_0a_1 \dots a_{i+2}$ and $ya_{i+3} \dots a_p$. Again the result follows from Lemma 4.4. □

Lemma 4.6 *Let P be a (u,v)-path of length p in a graph G, and let x, y be vertices in $G-P$ with $d_P(x) + d_P(y) \geq p/2 + 3$.*

a) *If $xy \in E(G)$, then either G contains a (u,v)-path of length $p+1$ or $p+2$ using all but at most two edges of P, or for some $w \in P$ there are the four edges xw, yw, xw^{+3}, yw^{+3}.*

b) *If $xy \in E(G)$ and G is triangle-free, then G contains a (u,v)-path of length $p+2$ spanning $V(P) \cup \{x,y\}$.*

c) *If x and y have a common neighbor z in $G-P$, then G contains a (u,v)-path of length $p+1$ or $p+2$.*

Proof: **a)** Label each vertex of the path P with its degree to xy. (This is either 0, 1 or 2.) If there is a segment of P with labels of the form $(\geq 1, \geq 1)$, $(2, 0, \geq 1)$, or $(\geq 1, 0, 2)$, then a (u,v)-path of length $p+1$ or $p+2$ results. So assume otherwise. Partition the vertices of P into segments S_i each (apart from the first) starting with a vertex labelled 2 and containing no other vertex labelled 2. It is easily calculated that every segment S_i (except the last) contributes at most $|S_i|/2$ edges between xy

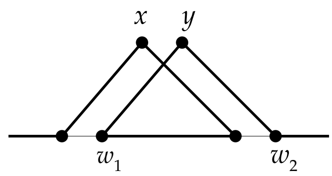

Figure 5: A $(k+2)$-path results anyway

and S_i, unless it is a $(2,0,0)$-segment. By the degree-sum there is a segment labelled $(2,0,0)$ which is not the last. This yields a segment of P of the form $(2,0,0,2)$, as required.

b) There exists a segment ww^+ of two successive vertices of P where there are two edges between xy and ww^+.

c) If x or y has a pair of consecutive neighbors on P then we are done. Otherwise, make a double sweep of P with beacons (x,y) and $\varepsilon = 1$ (and $c = 0$). By Lemma 4.2 there is a vertex w_1 of P such that $xw_1, yw_1^{+\mu} \in E(G)$. To avoid a path of length $p+1$ or $p+2$ via z, we get $\mu = 1$, and neither w_1 nor w_1^+ can have another neighbor in $\{x,y\}$. W.l.o.g. we may assume that w^{+2} has no neighbor in $\{x,y\}$, as if both vertices w_1^- and w_1^{+2} have a neighbor in $\{x,y\}$ then a (u,v)-path of length $p+1$ results. By ignoring the edge xw_1 and performing a double-sweep with the remaining edges, we get by the same reasoning a pair of edges $xw_2, yw_2^+ \in E(G)$ with the sets $\{w_1, w_1^+\}$ and $\{w_2, w_2^+\}$ disjoint. But then we obtain a path of length $p+2$ which does not use z. See Figure 5. □

Lemma 4.7 *Let G be a graph with $\delta \geq n/4 + 8$ containing a (u,v)-path P of length p and an (x,y)-path Q of length q. If $3 \leq q \leq p/6 + 2$ then G contains a (u,v)-path of length $p+1$ or $p+2$, or an (x,y)-path of length $q+1$ or $q+2$ in $G - P$.*

Proof: Assume that G contains no (u,v)-path of length $p+1$ or $p+2$. Let $Q = a_0 a_1 \ldots a_q$ and $A_i = \{a_i, a_{i+1}, a_{i+2}, a_{i+3}\}$.

First assume that $q \leq 6$. By Lemma 4.3 (with $P' = A_0$) we have $\sum_{w \in A_0} d_P(w) \leq p + 4$. So

$$\sum_{w \in A_0} d_{G-P-Q}(w) \geq 4\delta - p - 4 - 4q > n - (p+1) - (q+1) + 6 = |G - P - Q| + 6.$$

So there are two vertices a, a' of A_0 which have two common neighbors b, b' in $G - P - Q$. If a and a' are adjacent on Q then we are done, so assume that a and a' are at distance 2 or 3 on Q. Then the vertices of $B = \{b, b', a_1, a_2\}$ are on a common path of length at most 5 in $G - P$. So by Lemma 4.3, it follows that $\sum_{w \in B} d_P(w) \leq p + 6$. Therefore

$$\sum_{w \in B} d_{G-P-Q}(w) \geq 4\delta - p - 6 - 4q - 2 > n - (p+1) - (q+1),$$

and so two vertices in B must have a common neighbor in $G-P-Q$. It can easily be verified that this implies an (x,y)-path of length $q+1$ or $q+2$ in $G-P$, as required.

So we may assume that $q \geq 7$. By Lemma 4.3 there are at most $p+q+1$ edges between P and Q. Consider the edges $a_{2j-1}a_{2j}$ for $1 \leq j \leq \lfloor (q-1)/2 \rfloor$. For one of these edges, say $a_{i+1}a_{i+2}$, we have $d_P(a_{i+1}) + d_P(a_{i+2}) \leq 2(p+q+1)/(q-2)$.

If $d_P(a_i) + d_P(a_{i+3}) > (p+5)/2$, then proceed as follows. We know that there is an (a_i, a_{i+3})-path of length 4 or 5 in $G-P$, as we have already verified this Lemma for $q=3$. By performing a double-sweep (Lemma 4.2) with beacons (a_i, a_{i+3}) of P with $\varepsilon = 3$ (and $c=0$), we obtain a pair of independent edges joining a_i, a_{i+3} to P whose ends on P have distance $3 \leq \mu \leq 5$. If $3 \leq \mu \leq 4$ then we obtain a (u,v)-path of length $p+2$ or $p+1$ through the path $a_i a_{i+1} a_{i+2} a_{i+3}$ in $G-P$; if $\mu = 5$ then we obtain a (u,v)-path of length $p+1$ or $p+2$ through the path of length 4 or 5 joining a_i and a_{i+3} in $G-P$; either way a contradiction.

So assume $d_P(a_i) + d_P(a_{i+3}) \leq (p+5)/2$. We get $\sum_{w \in A_i} d_P(w) < p/2 + 2p/(q-2) + 6$. Hence

$$\sum_{w \in A_i} d_{G-P-Q}(w) > 4\delta - \frac{p}{2} - \frac{2p}{q-2} - 6 - 4q \geq n - (p+1) - (q+1) + 6,$$

as $7 \leq q \leq p/6 + 2$ and the expression $4q + 2p/(q-2)$ is maximized at an extremal value of q. Again, this implies that there are two vertices of A_i at distance 2 or 3 in Q which have two common neighbors b, b' in $G-P-Q$. As b and b' are joined by a path of length 2 in $G-P$, we obtain by Lemma 4.6c that $d_P(b) + d_P(b') < p/2 + 3$. So for $B = \{b, b', a_{i+1}, a_{i+2}\}$ we obtain

$$\sum_{w \in B} d_{G-P-Q}(w) > 4\delta - \frac{p}{2} - \frac{2p}{q-2} - 7 - 4q - 2 \geq n - (p+1) - (q+1).$$

Hence there must be two vertices in B which have a common neighbor in $G-P-Q$. Again we obtain an (x,y)-path of length $q+1$ or $q+2$ in $G-P$, as required. □

Lemma 4.8 *Let G be a graph of order n with $\delta \geq n/4 + 7$ containing a (u,v)-path P of length $p \geq \min\{3n/4, 5n/4 - \alpha\}$. If $G-P$ contains a path of length at least 3 then G contains a (u,v)-path of length $p+2$. If $G-P$ contains a path of length 2 then G contains a (u,v)-path of length $p+1$ or $p+2$.*

Proof: Let Q be a longest path of $G-P$ and let q be its length. If $q=2$ then by Lemma 4.6c applied to the endvertices of Q we obtain a (u,v)-path of length $p+1$ or $p+2$. So we may assume that $q \geq 3$.

We note that $q \leq p - n/2$. For, if $p \geq 3n/4$ this follows from $p+q < n$; and if $p \geq 5n/4 - \alpha$ then this follows from $n - \alpha \geq (p+q)/2$, as only $(r+2)/2$ vertices of an independent set can be in a path of length r.

Observe that a vertex of $G-P-Q$ is adjacent to neither endvertex of Q and it cannot be adjacent to two consecutive vertices of Q. If q is odd then let $Q' = V(Q)$.

If q is even then let Q' be all the vertices of Q except for the neighbor of one endvertex. Let $q' = |Q'|$. Then the number of edges between Q' and the vertices of P is bounded by

$$e(Q', P) \geq q'(\delta - q) - (n - p - q - 2)(q'/2 - 1).$$

Since $p \geq n/2 + q$, it follows that

$$e(Q', P) - (p + q + 1) \geq q'(\delta - n/4 + 1) - 4q - 3 > 0.$$

As there are more than $p+q+1$ edges joining Q to P, by Lemma 4.3 (Comb sweep) a (u, v)-path of length $p + 2$ results. □

Lemma 4.9 *Let G be a graph with $\delta \geq n/4+3$ containing a (u, v)-path P of length p but no (u, v)-path of length $p + 1$ or $p + 2$. If there is an (x, y)-path of length at least 3 in $G - P$, then there is an (x, y)-path of length q in $G - P$ for some q with $3 \leq q \leq 9$.*

Proof: Let $Q = a_0a_1 \dots a_q$ be a shortest (x, y)-path among the (x, y)-paths in $G - P$ of length at least 3. Suppose $q \geq 10$. Consider the initial segment $Q' = a_0a_1 \dots a_9$ and the set $A = \{a_0, a_3, a_6, a_9\}$. By Lemma 4.3 there are at most $p+10$ edges joining (Q' and hence) A to P. So, by the minimum degree bound, two vertices in A have a common neighbor in $G - P$. If the common neighbor is not in Q' then we replace the segment between the respective vertices of A by their common neighbor—this contradicts the minimality of Q. If the common neighbor is in Q' then we use a shortest chord of Q' to obtain a shorter (x, y)-path of length at least 3—again a contradiction. □

The following result is a special case of a result of Bondy and Jackson.

Theorem 4.10 (Bondy & Jackson [9]) *If G is a 3-connected graph with minimum degree δ, then any pair u, v of vertices are joined by a path of length at least $2\delta - 2$.*

4.3 Paths and bipaths of all lengths

For graphs with minimum degree exceeding $n/2$, we obtain stronger results. Let a *bipath* be any subgraph obtainable by removing an edge from the standard path of a bicycle. The standard paths of a bipath are analogously defined as for a bicycle; note that a bipath has two of them. A bipath obtained from a bicycle of length k in the indicated way is said to have length $k - 1$.

A graph G is said to be *panconnected* if every pair of vertices in G are joined by a path of length p for all $2 \leq p \leq n - 1$. It is *bipath-connected* if every pair of vertices in G are joined by a bipath of length k for all $3 \leq p \leq n - 1$. By definition bipath-connected graph cannot be triangle-free.

Theorem 4.11 (Faudree & Schelp [17]) *If G is a graph with $\delta \geq n/2+1$, then G is panconnected.*

Lemma 4.12 *If G is a graph with $\delta \geq n/2+3$, then it is bipath-connected.*

Proof: Let u and v be any pair of vertices. There exist vertices $x, y \neq v$ such that uxy forms a triangle. By Theorem 4.11 the graph $G' = G - \{u, x\}$ is panconnected. So in G' there are (v, y)-paths of lengths 2 through $n-3$, and thus bipaths in G joining u and v for every length p with $4 \leq p \leq n$. Since u and v have a common neighbor w and u and w have a common neighbor $z \neq v$, we also obtain the bipath of length 3. □

5 Short Bicycles

In this section we prove Proposition 3.1.

It is well-known that a graph which satisfies $\delta^2 - \delta + 1 > n$ has a 4-cycle. The remainder of the proof of Proposition 3.1 is given by the following lemma. Recall that every nonbipartite graph with $\delta > n/4$ has an odd cycle of length at most 7.

Lemma 5.1 *Let G be a nonbipartite graph of order n with minimum degree $\delta \geq n/4 + 250$.*
a) If G has a triangle then G contains a 5- or a 6-bicycle.
b) If G is triangle-free then G contains a 7-bicycle.

Proof: **a)** Say G has triangle T. Suppose there are four vertices in $G - V(T)$, each of which has two neighbors in T. By the degree requirement two of these must have a common neighbor in $G - V(T)$, which implies a 6-bicycle.

Otherwise, there must be a set S of more than $3n/4$ vertices which have a neighbor in T. Hence each vertex in S must have more than 200 neighbors in S. If there is an edge in S whose endvertices are adjacent to different vertices in T then we have found a 5-bicycle. If there is no such edge then there must be a path of length 4 in the neighborhood of a vertex $x \in V(T)$, which again implies a 5-bicycle.

b) Fix a shortest cycle T. By the minimality of T, any vertex is adjacent to at most two vertices of T and these two must be distance two apart on T.

First suppose $T = C_5$. Since there are at least $5n/4 + 990$ edges joining T to $G - V(T)$, there are at least 16 vertices in $G - V(T)$ each of which has two neighbors on T. Four of these must have the same two neighbors on T. By the degree requirement two of the four have a common neighbor in $G - V(T)$, so we obtain a 7-bicycle.

Finally suppose $T = C_7$. Since there are more than $7n/4 + 1000$ edges joining T to $G - V(T)$, there is a set S of at least $3n/4$ vertices in $G - V(T)$ each of which has two neighbors on T. Within S there must be an edge, which implies a 7-bicycle. □

6 Lengthening and Shortening Bicycles

In this section we prove Propositions 3.5, 3.6, 3.7 and 3.8 in succession. It is then straightforward to deduce Proposition 3.2.

In the proofs of the propositions we investigate graphs with minimum degree $\delta \geq n/4 + 250$. Moreover we may assume that

$$n > 988,$$

since for $n \leq 988$ we have $\delta \geq n/2 + 3$, so by Lemma 4.12 the graph is bipathconnected. So for any edge e we get bicycles of every length between 4 and n through e, which implies the truth of all the indicated propositions.

6.1 Proof of Proposition 3.5

Here we investigate the case when the connectivity of the graph is less than 80. We start with a lemma which provides, starting from a cycle C, bicycles of many consecutive lengths, and, in particular, a bicycle containing all vertices of C.

Lemma 6.1 *Let G be a graph of order n with $\delta \geq n/4 + 250$. Suppose G contains a vertex-cutset S with $|S| \leq 81$ and H is a component of $G - S$ with order less than $n/2$. If C is a cycle which has an edge in common with H, then G contains bicycles C^- and C^+ with the following properties:*

- *C^- and C^+ contain all edges of $E(C) \setminus E(H)$,*
- *C^- has only paths and bipaths of length at most 3 in common with H,*
- *C^+ satisfies $V(C^+) = V(C) \cup V(H)$.*

Moreover, G contains bicycles of every length k for $|C^-| \leq k \leq |C^+|$.

Proof: Let $v_1, w_1, v_2, w_2, \ldots, v_r, w_r$ be the endvertices of the paths of length at least 1 in the intersection of C and H. Let X_0 be the set of endvertices of the paths which H has in common with C (i.e. both endvertices of paths of length at least 1 and the single vertices of the length-0 paths). Then $|X_0| < 2|S|$. Define iteratively sets X_j for $1 \leq j < r$, where X_j is obtained from X_{j-1} by adding a common neighbor $y_j \notin X_{j-1}$ of v_j and w_j. Such a common neighbor exists, as for each j the graph $H_j = H - (X_{j-1} \setminus \{v_j, w_j\})$ satisfies $\delta(H_j) > |H_j|/2 + 3$. By the same count, H_r is bipath-connected by Lemma 4.12.

For $j < r$ replace the (v_j, w_j)-segment of C by $v_j y_j w_j$. By replacing the (v_r, w_r)-segment by a bipath of H_r of every length p for $3 \leq p \leq |H_r| - 1$, we obtain the indicated bicycles C^- ($p = 3$), C^+ ($p = |H_r| - 1$), and bicycles of every length between $|C^-|$ and $|C^+|$. $\square$

In order to prove Proposition 3.5, we need to show that if G has connectivity $2 \leq \kappa < 80$, then G contains a triangle and a bicycle of every length from 4 up to the circumference of G.

Let S be a smallest cutset of G and let G_1 be a smallest order component of $G - S$. Certainly, G_1 is 3-connected and contains a triangle, as any graph H with

$\delta(H) > |H|/2$ has these properties. As $\delta(G_1) \geq |G_1|/2+3$ we can apply Lemma 4.12 to two adjacent vertices of G_1 and obtain that G_1 and therefore G contains k-bicycles for every k with $5 \leq k \leq |G_1|$. Consider the graph $G' = G - S - G_1$.

First assume that G' is 3-connected. We show next that G contains k-bicycles for every k with $n/4 + \kappa + 3 \leq k \leq \min\{n - \kappa, 3(\delta - \kappa)\}$.

Since G is 2-connected there are two vertex-disjoint paths P_v, P_w, each linking a vertex of G_1 to a vertex of G', which have no further vertex in common with $V(G_1) \cup V(G')$. Let v_1, w_1 be the endvertices of P_v, P_w in G_1, and v_2, w_2 be the endvertices in G'.

Let q denote the length of a longest (v_2, w_2)-path in G'. By Theorem 4.10 it follows that $q \geq \min\{|G'| - 1, 2(\delta - \kappa) - 3\}$. Next we show that there cannot be a gap of $n/4$ consecutive lengths of (v_2, w_2)-paths in G'. That is, for every r with $n/4 \leq r \leq q$ there is a (v_2, w_2)-path in G' of length ℓ satisfying $r - n/4 < \ell \leq r$.

Otherwise there is a maximum r for which the condition is violated. Then H contains a (v_2, w_2)-path P of length $r + 1$. Let P' be a subpath of P of length $n/4$ centered at the center of P. The endvertices x, y of P' have only one neighbor each in P', and at most 2 common neighbors in $P - P'$ because every short chord of P yields a suitable shorter (v_2, w_2)-path. Therefore

$$d_{G'-P}(x) + d_{G'-P}(y) \geq 2(\delta - \kappa) - r + n/4 - 5 > |G'| - r - 1.$$

Thus x and y have a common neighbor in $G' - P$. So we obtain a (v_2, w_2)-path of length $r - n/4 + 2$ avoiding the interior of P', a contradiction.

Since G_1 is bipath-connected, we obtain bicycles of every length between $n/4 + \kappa + 3$ and $q + |G_1| + 3$ by a suitable combination of (v_1, w_1)-bipath in G_1 and (v_2, w_2)-path in G'. It follows that G has every bicycle up to $\min\{n - \kappa, 3(\delta - \kappa)\}$.

Now take a longest cycle C of G. We know that $|C| \geq \min\{n - \kappa, 3(\delta - \kappa)\}$, since G contains a bicycle of at least that length. So C shares an edge with both components of $G - S$, in particular with G_1. By Lemma 6.1 we obtain k-bicycles, for every k in the range of $|C^-| \leq |C| - |G_1| + 4|S| \leq k \leq |C| = |C^+|$, and since $|C| - |G_1| + 4|S| \leq \min\{n - \kappa, 3(\delta - \kappa)\}$ for $n > 988$, we have found all the required bicycles.

So we may assume that G' is not 3-connected. Hence there is a set T, with $0 \leq |T| \leq 2$, such that $G' - T$ consists of two components G_2 and G_3. Both of these components are 3-connected and have order less than $n/2$.

Take a longest cycle C of G. As G is 2-connected the length of C is at least $\min\{2\delta, n\}$ (Dirac [16]). So C shares an edge with at least two of the three components of $G - (S \cup T)$. Starting from C we will construct k-bicycles, for every k in the range $5|S \cup T| \leq k \leq |C|$.

Apply Lemma 6.1 to the first component G_i which shares an edge with C. We obtain bicycles of every length from $|C|$ down to a bicycle C^-, which contains all edges of C outside this component, and has only paths and bipaths of lengths at most 3 in G_i. Now take C^- as the new cycle, and apply Lemma 6.1 to the second component which shares an edge with C (and hence with C^-). Continuing in the same way also for the third component (if this shares an edge with C), we obtain a short bicycle C', which has only paths and bipaths of length at most 3 in common with each component, and bicycles of every length between $|C'|$ and $|C|$. Since $|C'| \leq 5|S \cup T| \leq \delta - \kappa \leq |G_1|$ as $n > 988$, the proof is complete. □

6.2 Proof of Proposition 3.6

We need to show first that if G is 3-connected, then for any maximal bicycle C each component of $G - C$ has order 1 or 2.

For $3 \leq \kappa < 80$ this follows from Theorem 3.4 and Proposition 3.5; so we may assume that G has connectivity at least 80.

Let C be a maximal bicycle with standard path P of length p. Let u and v be the endvertices of P. Consider the graph G' obtained from G by deleting the vertices of C which are not on the standard path P. Observe that G' has at least $n - 5$ vertices. (Note that $G' - C = G - C$.)

For $3 \leq p < 495$ we can apply Lemma 4.5 to P in G', since every vertex on that path has two neighbors in $G' - C$, to show that no such bicycle can be maximal. So we may assume $p \geq 495$.

Let H be a largest order component of $G - C$. We will show that:

> *if H has order at least 3, then one can find a (u, v)-path of length $p + 1$ or $p + 2$ in G'.*

This will contradict the maximality of C.

Let Q be a longest path in H and x, y be its endvertices. As H has order at least 3, the path Q has length $q \geq 2$. If $q = 2$ then $d_P(x) + d_P(y) \geq 2\delta - 6 > p/2 + 3$, so by Lemma 4.6c we obtain a (u, v)-path of length $p + 1$ or $p + 2$. Hence we may assume that $q \geq 3$.

By Lemma 4.7 it follows that $q > p/6 + 2 \geq 80$. Since G is 80-connected, there must be 80 vertex-disjoint paths, each with one endvertex in C and one endvertex in Q and every interior vertex in $G - C - Q$. Call this collection $\mathcal{R}$. We will use two of these paths to find two independent edges ab and $a'b'$ with $a, a' \in P$ and $b, b' \in H$ such that a and a' are distance μ apart on P, for μ in a suitable range, and b and b' are connected by a path of length μ or $\mu - 1$ in H. This will yield the desired (u, v)-path.

Orient P and Q. Consider segments of P of length $p/6$. Since at least 75 of the paths in $\mathcal{R}$ have an endvertex on P, there must exist such a segment, say P', where there are endvertices of at least 13 of the paths. Let A be the set of the first three endvertices of paths ending in P' and A' the set of the last two endvertices. (This ensures that a vertex of A and a vertex of A' are at least distance 9 apart on P.)

Let B and B' be the respective sets of neighbors on $\mathcal{R}$ of A and A'. We claim that there exist vertices $b \in B$ and $b' \in B'$ which are joined by a path of length at least 3 in H. For, let $z_1, \ldots, z_5$ be the endvertices of the five paths on Q in order; then among the pairs (z_1, z_4), (z_1, z_5), (z_2, z_5) there must be a pair where one vertex belongs to a path ending in $a \in A$ and the other to a path ending in $a' \in A'$: the respective neighbors b and b' of a and a' on these paths suffice. Observe also that a and a' have distance μ for $9 \leq \mu \leq p/6$ in P.

Now, assuming that there is no (u,v)-path of length $p+1$ or $p+2$, we obtain by Lemma 4.9 that b and b' are joined by a path of some length ℓ for $3 \leq \ell \leq 9$ in $G' - P$. By repeated application of Lemma 4.7 there are (b,b')-paths of length $r-1$ or r whenever $\ell \leq r \leq p/6+2$; so, in particular, for $r = \mu$. By replacing the (a,a')-path of length μ in P by the (a,a')-path of length $\mu+1$ or $\mu+2$ in $G' - P$ through b and b', we obtain a (u,v)-path of length $p+1$ or $p+2$. This is a contradiction.

Hence we have shown that C is a D_3-bicycle.

It remains to show that if G is triangle-free then C is a D_2-bicycle. But suppose there is a component $\{x,y\}$ of $G - C$; then by Lemma 4.6b we obtain a bicycle of length 2 more than C. This is a contradiction. □

6.3 Proof of Proposition 3.7

We need to show that one can bound the length k of a maximal bicycle from below by $\min\{n-2, 3\delta-10, n-\alpha+\delta-6\}$. This follows from Proposition 3.6 using standard "long-cycle" arguments.

We may assume that $k < n-2$ otherwise we are done. First assume that there is a k-bicycle C which is not dominating. By Proposition 3.6 the components of $G-C$ have cardinality at most 2. Consider a component of two vertices x and y. If there is a segment S of 3 vertices on the standard path P of C such that $d_S(x)+d_S(y) > 2$, then a bicycle contradicting the maximality of C results. So

$$2(\delta-3) \leq d_P(x) + d_P(y) \leq 2(k+1)/3.$$

Hence $k \geq 3\delta - 10$.

So we may assume that every bicycle of length k is dominating. Fix $v \in V(G-C)$. We will show that: $N_P^+(v) \cup V(G-C)$ is an independent set, where P is the standard path of C.

There is no edge between vertices of $G-C$. By the maximality of k, certainly v cannot have two consecutive neighbors on P. Also, if there exist neighbors a and b of v such that $a^+b^+ \in E(G)$, then there is a $(k+1)$-bicycle.

Finally, suppose that there is some vertex $w \in V(G-C)$ such that $va, wa^+ \in E(G)$. Then perform a double sweep (Lemma 4.2) of P with v and w as beacons and $\varepsilon = 1$ $(c = 0)$. We obtain a $(k+2)$- or $(k+1)$-bicycle, or a k-bicycle with an edge outside the bicycle, a contradiction (see Figure 6).

Hence

$$\alpha \geq \delta + n - k - 6,$$

so that $k \geq n - \alpha + \delta - 6$, as required. □

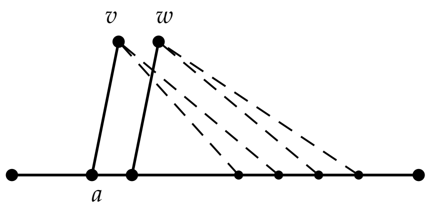

Figure 6: Any two of the edges yield a contradiction

6.4 Proof of Proposition 3.8

We start with a technical lemma.

Lemma 6.2 *Let G be a graph with $\delta \geq n/4 + 250$. Let $\mu \geq 2$ be an integer and assume G contains a k-bicycle C, and a path Q of length $q \geq \mu$ in $G - C$, where $k + q - \mu \geq \min\{3\delta, n - \alpha + \delta\} - 10$. If for every pair of edges $a^-a, a^{+r}a^{+r+1}$ of Q with $r \geq \mu$*

$$d_C(a^-) + d_C(a) + d_C(a^{+r}) + d_C(a^{+r+1}) \geq k + q + 11,$$

then G contains a $(k + q - \mu + 1)$- or a $(k + q - \mu + 2)$-bicycle.

Proof: Let P be the standard path of C and let p be its length. Set $G' = G - (V(C) \setminus V(P))$. For a fixed value of μ the proof uses induction on q.

If $q \leq \mu + 1$ then we can apply Lemma 4.8 to G' and P as $p \geq k - 4 \geq \min\{3\delta, n - \alpha + \delta\} - 15 \geq \min\{3|G'|/4, 5|G'|/4 - \alpha(G')\}$ and $\delta(G') > |G'|/4 + 7$. So we obtain a $(k + 1)$- or a $(k + 2)$-bicycle if $q = \mu = 2$ and a $(k + q - \mu + 2)$-bicycle otherwise.

Assume $q \geq \mu + 2$. Let $P = a_0a_1 \ldots a_p$ and $Q = b_0b_1 \ldots b_q$. Since the edges b_0b_1 and $b_{q-1}b_q$ satisfy the degree-sum condition, there are at least $k+\mu+3$ edges joining b_0, b_1, b_{q-1}, b_q to P. Consider the four potential edges $b_0a_i, b_1a_{i+1}, b_{q-1}b_{i+\mu}, b_qa_{i+\mu+1}$ for $-(\mu + 1) \leq i \leq p$. As every edge joining b_0, b_1, b_{q-1}, b_q to P occurs for exactly one i, there must be an i for which two of the four potential edges are present.

If the edges are incident with b_0 and b_1, or with b_{q-1} and b_q, then we can insert the edge b_0b_1 ($b_{q-1}b_q$, resp.) into C to obtain a $(k + 2)$-bicycle C' containing all vertices of C, and a path Q' of length $q - 2$ in $G - C$. The subgraphs C' and Q' satisfy the inductive hypothesis. Hence, by induction, we obtain a $(k+q-\mu+1)$- or $(k+q-\mu+2)$-bicycle. In the other four possible incidences, we obtain a $(k+q-\mu+1)$-bicycle immediately. □

We need to show that if G is nonbipartite and contains a k-bicycle C for some k in the range $k > \min\{3\delta, n - \alpha + \delta\} - 10$, then there is either a $(k - 1)$-bicycle or a $(k - 2)$-bicycle.

The above lemma suggests the following approach: Find a shorter bicycle which avoids a short subpath of the standard path of the original bicycle such that every pair of distant edges of the short subpath sends many edges to the new bicycle. There are three cases.

• *C is a spanning bicycle.*
Let aa^{+s} be a shortest chord of the standard path of C. So G contains a bicycle C' which avoids the induced path $Q = a^+ \dots a^{+s-1}$. If $s \le 3$ then C' is the desired bicycle of length $K-1$ or $K-2$; otherwise we obtain the bicycle by applying Lemma 6.2 to C' and Q with $\mu = 2$. (The degree sum to Q is at least $n+11$.)

• *Every vertex of $G-C$ has less than $k/6+6$ neighbors on C.*
It follows that $k > 3n/4$. Otherwise $k > n - \alpha + \delta - 10$ so that $\alpha > 5n/4 - k + 6$ and for a maximum independent set I we have $|I \cap V(G-C)| > 5n/4 - 3k/2 + 6$; thus every vertex of $I \cap V(G-C)$ has more than $\delta - (n-k-5n/4+3k/2-6) > k/6+6$ neighbors on C.

In particular, every vertex in $G-C$ has more than $n/4 - k/6 + 3$ neighbors in $G-C$. This implies that $n-k > n/10$ and that $\delta(G-C) > (n-k)/2 + 3$. By Theorem 4.11 the graph $G-C$ is panconnected.

As $n-k < n/4$, each vertex of $G-C$ has at least 250 neighbors in C. So there are at least $248n/10$ edges joining $G-C$ to the standard path P. This means that there is a segment S of P of length at most $k/10$ with more than $248n/100 > 4|S|$ edges joining S to $G-C$. Hence there are two independent edges ux, vy where $u, v \in V(G-C)$ and $x, y \in S$ where x and y are at some distance μ apart in S with $5 \le \mu < n/10$. We obtain a bicycle of length $k-1$ by replacing the subpath of S joining x and y by a path through ux, vy and $G-C$ that is one vertex shorter.

• *Some vertex v of $G-C$ has at least $k/6+6$ neighbors on C.*
Therefore v has $r \ge k/6+4$ neighbors on the standard path of C. Let $x_1, x_2, \dots, x_r$ be its neighbors in order, and let $y_i = x_{3i+1}$ for $0 \le i \le (r-1)/3$.

The vertices y_i split the standard path into more than $k/18$ segments $S_i = C[y_i, y_{i+1}]$. Let $\mathcal{S}$ be the set of such segments of length less than 36. Certainly $|\mathcal{S}| > k/36$. For every segment $S_i \in \mathcal{S}$ find a minimal subsegment S_i' with the properties that (i) S_i' has length at least 3, and (ii) the endvertices of S_i' have a common neighbor in $G-C$. As S itself satisfies (i) and (ii), S_i' is well defined. Define $\mathcal{S}' = \{ S_i' : S_i \in \mathcal{S} \}$.

Suppose that there is a set B of 14 vertices in $G-C$ each of which has a pair of consecutive neighbors in at least 14 segments of $\mathcal{S}'$. Let S' be the shortest segment in $\mathcal{S}'$: it has length at most 17. Then we obtain a bicycle C' of length k' with $k-1 \ge k' \ge k-15$ by replacing the interior path of S' by the common neighbor w in $G-C$ of the endvertices of S'. If $k-k' \le 2$ then we have found the desired bicycle. Otherwise we obtain the bicycle by inserting $k-k'-2 \le 13$ vertices of B (none of which is w) into distinct segments distinct from S'.

So, apart from at most 13 vertices, every vertex of $G-C$ has a pair of consecutive neighbors in at most 13 segments of $\mathcal{S}'$. So there are less than $13|\mathcal{S}'| + 13(n-k) < 13|\mathcal{S}'| + 13k$ pairs (x, S_i') where $x \in V(G-C)$ is adjacent to two consecutive vertices of $S_i' \in \mathcal{S}'$. Hence, as $|\mathcal{S}'| > k/36$, there is a segment where less than 481 vertices of $G-C$ have a pair. Call this segment S'.

Consider the bicycle C' of length $k' \geq k - 33$ obtained by replacing the interior path of S' by the common neighbor in $G - C$ of the endvertices of S'. Let Q be the interior path of length $k - k'$ of S' that is in $G - C'$.

Consider a pair of edges $a^- a, a^{+r} a^{+r+1}$ at distance $r \geq 3$ in Q. By the minimality of S' every vertex in $G - C$ is adjacent to at most two of the four vertices a^-, a, a^{+r}, a^{+r+1}, and if it is adjacent to two of the four, then the two neighbors are consecutive on S'. So the number of edges between $G - C$ and the foursome is at most $(n - k) + 481$. Thus

$$d_{C'}(a^-) + d_{C'}(a) + d_{C'}(a^{+r}) + d_{C'}(a^{+r+1}) \geq 4\delta - (n - k) - 481 - 132 > k + 36.$$

Thus we can invoke Lemma 6.2 with C', Q and $\mu = 3$ to obtain a bicycle of length $k - 1$ or $k - 2$. □

6.5 Proof of Proposition 3.2

Consider graphs with $\delta \geq n/4 + 250$.

Proposition 3.5 shows that for such graphs of connectivity 2 every maximal bicycle has length equal to the circumference of G.

For 3-connected graphs Proposition 3.7 shows that every maximal bicycle has length at least T where $T = \min\{n, 3\delta - 10, n - \alpha + \delta - 6\}$. Proposition 3.8 shows that there are no maximal bicycles of length between T and $K_0 - 1$ where K_0 is the length of the longest bicycle. Thus the only maximal bicycles are ones of maximum length.

7 Long Cycles and Bicycles

The aim of this section is to prove Proposition 3.3.

7.1 Creating a long bicycle

The following lemma is based on an idea from [2]. Call a vertex *insertible* into a path P (cycle C), if it is adjacent to two consecutive vertices of P (C, resp.).

Lemma 7.1 *Let P and P' be paths. If every vertex of P' is insertible in P, then there exists a path Q with the same endvertices as P that contains all vertices of P and P'.*

Proof: Let v and u be the endvertices of P' and suppose v is insertible at the edge yy^+ in P. Let w be the vertex of P' (not necessarily distinct from v) at largest distance in P' from v with the property that w is insertible at the same edge yy^+ in P. Insert $P'[v, w]$ into P at yy^+ to obtain a path P''. (That is, replace yy^+ by $yvv^+ \ldots wy^+$.) Every vertex of $P'[w^+, u]$ is insertible in P'', and so by induction we can insert the vertices of $P'[w, u]$ into P''. □

The most difficult task in this section is to prove that the graph contains a $(K-1)$-cycle where K is the circumference of the graph. The most complex parts of the proof occur in the following lemma, which is proved by using a kind of hopping technique for edges.

Lemma 7.2 *Suppose G is a graph with girth 3 and $\delta \geq n/4 + 250$, and among all cycles which share an edge with a triangle let C be a longest. If C is dominating then there is a bicycle of length at least $|C| - 1$.*

Proof: Assume $|C| = k$. If C shares two edges with a triangle then it is the bicycle we seek. Otherwise, consider the third vertex of a triangle that shares an edge with C. If it is outside C we obtain a contradiction of the maximality of C immediately. So we may assume that the third vertex of the triangle is in C.

Call a vertex of C *multi-insertible* if it is adjacent to at least 4 pairs of consecutive vertices of C. Suppose $C[u,v]$ is a segment of C which satisfies the following properties:

- u^+ is insertible in C, and
- every vertex (if any) of $C[u^{+2}, v^-]$ is multi-insertible.

If $C[u,v]$ contains a chord, then we obtain a k-bicycle as follows. Let xy be a shortest chord in $C[u,v]$, and let C' be the cycle obtained by using the edge xy and the path $C[y,x]$. Let $\{w, w^+\}$ be a consecutive pair of neighbors of x^+ on C and hence on C'. Then every vertex of $C[x^{+2}, y^-]$ is insertible into $C'[w^+, w]$. By Lemma 7.1 we obtain a path with endvertices w^+ and w which uses every vertex in C except x^+; this yields a k-bicycle. Thus we may assume that $C[u,v]$ has no chords.

Moreover, we may assume that no vertex in $G - C$ has two neighbors x and y in $C[u,v]$, since otherwise we obtain a $(k+1)$-bicycle by a similar reasoning. (Construct C' using the xy path of length 2 instead of the xy chord.)

Now take a segment $C[u,v]$ which satisfies the above two properties, and, additionally,

- neither u nor v is multi-insertible.

Such a segment exists, since there is some vertex with two consecutive neighbors on C (cf. the first paragraph of the proof), and by the above reasoning we may assume that on each side of every chord there is at least one vertex that is not multi-insertible.

Let ww^+ be a pair of consecutive neighbors of u^+. Let P be a longest path which contains $C[v,u]$ as a subpath and avoids the interior of $C[u,v]$. Let u' and v' be the endvertices of P. Since C is dominating, u' is either u or a neighbor of u in $G - C$; similarly v' is either v or a neighbor of v in $G - C$. By the above reasoning and the maximality of C and P, the endvertices of P have no neighbor in $G - P$, and each of them has at most 3 pairs of consecutive neighbors on $C[v,u]$.

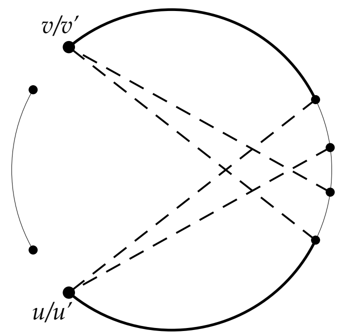

Figure 7: Double sweep yields cycle C'

Now perform a double sweep (Lemma 4.2) of the paths $C[v^+, w]$ and $C[w^+, u^-]$ with beacons (u', v') and $\varepsilon = 1$ $(c = 3,\ r = 2)$. As the result we obtain edges $u'z, v'z^{+\mu}$ for some μ with $1 \leq \mu \leq 3$. Hence there is a cycle C' in $\langle P \rangle$ (the subgraph induced by the vertex set of P) through ww^+ which misses only the interior vertices of $C[z, z^{+\mu}]$ (see Figure 7). By inserting u^+ at ww^+ and the vertices of $C[u^{+2}, v^-]$ at other places into this cycle (using Lemma 7.1), we obtain a bicycle. This bicycle has length at least $k - 1$, as required, unless $\mu = 3$, $u = u'$ and $v = v'$.

The latter case requires more work. It suffices to show that:

> *$\langle P \rangle$ contains a cycle through the edge ww^+ of length at least $|P| - 1$ such that every vertex of $C[u^{+2}, v^-]$ is multi-insertible into that cycle.*

Let e_0 denote the edge z^+z^{+2} of C disjoint from C'. Let us rename its endpoints to be y_0 and y_0^+. Now, we construct a sequence $(e_i = y_i y_i^+)$ of edges of C, one edge at a time, by taking for $e_{i+1} = y_{i+1}y_{i+1}^+$ any edge of C with the following properties:

- e_{i+1} is an edge of $C[v^+, w^-]$ or $C[w^{+2}, u^-]$;
- $y_{i+1}^- y_{i+1} y_{i+1}^+ y_{i+1}^{+2}$ is disjoint from $e_0, e_1, \ldots, e_i$; and
- the predecessor y_{i+1}^- and successor y_{i+1}^{+2} of e_{i+1} on C are both adjacent to both ends y_i and y_i^+ of e_i.

Let j_0 be the smallest index for which one of the following situations occurs:

(1) e_{j_0} has a neighbor in $G - P - u^+$;

(2) e_{j_0} has a neighbor in e_i for an $i < j_0$; or

(3) there is no suitable edge e_{j_0+1}.

Since C is finite such a j_0 must exist.

Let $\Gamma(i)$ denote the set of "neighboring edges" of the edges $e_0, e_1, \ldots, e_i$ in C, viz. $\{y_0y_0^-,\ y_0^+y_0^{+2},\ y_1y_1^-,\ \ldots,\ y_i^+y_i^{+2}\}$. We now show that $\langle P \rangle$ contains a series of spanning subgraphs:

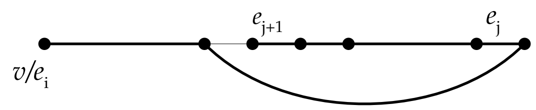

Figure 8: $P(j) \to P(j+1)$ & $P(i,j) \to P(i,j+1)$

(i) For $0 \leq j \leq j_0$ there is a spanning path $P(j)$ starting at v and ending with e_j which contains all edges of P except some of $\Gamma(j)$;

(ii) For $0 \leq j \leq j_0$ there is a cycle $C(j)$ which together with the vertex-disjoint edge e_j contains all vertices of P and all edges except some of $\Gamma(j)$;

(iii) For $0 \leq i < j \leq j_0$ there is a spanning bipath $P(i,j)$ starting with e_i followed by a common neighbor of the ends of e_i and ending with e_j, which contains all edges of P except some of $\Gamma(i)$.

To see (i), observe that the statement is true for $j = 0$, since $P(0) = v \ldots y_0^{-2} y_0^- u u^- \ldots y_0^+ y_0$ is a path containing all edges of P except for a neighboring edge of e_0. Now suppose we have found a path $P(j)$. Note that the segment $y_{j+1}^- y_{j+1} y_{j+1}^+ y_{j+1}^{+2}$ is a segment of $P(j)$ (though perhaps oriented the other way). Figure 8 shows how to construct $P(j+1)$ from $P(j)$ by using an edge from a neighbor of e_{j+1} to the end of $P(j)$.

To see (ii), observe that this is true for $j = 0$, since $C(0) = C' = v y_0^{+2} y_0^{+3} \ldots u y_0^- y_0^{-2} \ldots v$ is a cycle containing all edges of P except e_0 and its neighboring edges. Now suppose we have found a cycle $C(j)$. Note that the segment around e_{j+1} is a segment of $C(j)$. Figure 9 shows how to obtain $C(j+1)$ from $C(j)$ by simply replacing e_{j+1} with e_j.

To verify (iii), start with $C(i)$ and use the obvious way to find $P(i, i+1)$. To derive $P(i, j+1)$ from $P(i,j)$ we can refer to the same Figure 8 as above.

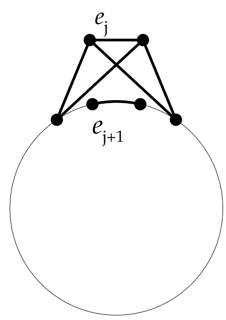

Figure 9: $C(j) \to C(j+1)$ & $C(j) \to P(j, j+1)$

Observe that every vertex of $C[u^{+2}, v^-]$ is still (multi-)insertible in $P(j_0)$, $C(j_0)$ and $P(i, j_0)$ (since we terminate the construction of the sequence if (1) occurs). We conclude the proof by showing how to find the desired bicycle from each of the three "accidents" (1) – (3).

Suppose that (1) occurs. The first possibility is that e_{j_0} has a neighbor in $C[u^{+2}, v^-]$. Take the first such neighbor z in $C[u^{+2}, v^-]$. Form a cycle of length at least $|P| - 1$ by starting at v, going along $P(j_0)$ to e_{j_0}, then to z (possibly skipping a vertex of e_{j_0}), and back to v along $C[z, v]$. By inserting u^+ at ww^+ and $C[u^{+2}, z^-]$ elsewhere (Lemma 7.1) this yields a bicycle of length at least $k - 1$, as required.

The other possibility is that e_{j_0} has a neighbor in $G - C$. Then take the cycle $C(j_0)$ and insert u^+ and $C[u^{+2}, v^-]$ into $C(j_0)$ as above. The result is a $(k-2)$-bicycle C'' with a component of cardinality at least 3 in $G - C''$. By Proposition 3.6 this bicycle is not maximal; hence, in particular, there must be a longer bicycle.

Assume that (2) occurs: e_{j_0} has a neighbor in e_i for an $i < j_0$. By the definition of $P(i, j_0)$ we can start it with either end of e_i since both are adjacent to the third vertex on the path. Hence adding the edge between e_{j_0} and e_i to $P(i, j_0)$ yields a bicycle in $\langle P \rangle$ of length at least $|P| - 1$, into which we can reinsert u^+ and $C[u^{+2}, v^-]$ as above to obtain a bicycle of length at least $k - 1$. So we may assume that neither (1) nor (2) occurs.

Suppose that (3) occurs: there is no suitable edge e_{j_0+1}. Apply Lemma 4.6a to e_{j_0} and the path $C(j_0)[w^+, w]$. Either we obtain a longer cycle in $\langle P \rangle$ using all but at most two edges of $C(j_0)$—in which case we obtain a bicycle of length at least $k - 1$ after inserting u^+ and $C[u^{+2}, v^-]$ as above—or for some $a \in C(j_0)$ both ends of e_{j_0} are adjacent to both a and a^{+3}. We know that neither a nor a^{+3} is a vertex of any e_i (since accident (2) did not occur). If both a^+ and a^{+2} are disjoint from every e_i, then the segment $aa^+a^{+2}a^{+3}$ must be a segment of C disjoint from every e_i, and so we could have taken a^+a^{+2} for e_{j_0+1}. Hence one of the vertices a^+, a^{+2} is an end of some e_i. Since $C(j_0)$ goes through e_i, in fact a^+a^{+2} is the edge e_i. By the construction of $C(j_0)$, the two neighbors of e_i in $C(j_0)$ are y_{i+1}^- and y_{i+1}^{+2}. That is, both ends of e_{j_0} are adjacent to both y_{i+1}^- and y_{i+1}^{+2}.

Finally, we go back to the original cycle C. We make a cyclic replacement of edge e_{i+2} by e_{i+1}, e_{i+3} by e_{i+2}, $\ldots$, e_{j_0} by e_{j_0-1} and e_{i+1} by e_{j_0}. That is, we replace the segment $y_{i+2}^- y_{i+2} y_{i+2}^+ y_{i+2}^{+2}$ by $y_{i+2}^- y_{i+1} y_{i+1}^+ y_{i+2}^{+2}$, etc. Note that both y_{i+1} and y_{i+1}^+ are adjacent to both y_{i+2}^- and y_{i+2}^{+2}, so we obtain a k-bicycle (see Figure 10). If $i + 1 = j_0$ we had a bicycle, anyway. $\square$

Now, let G be a nonbipartite graph with odd girth t_0. For $t < t_0$ define k_t as the length of a longest cycle which shares a segment of length at least t with a t_0-cycle. Clearly $K \geq k_0 \geq k_1 \geq \ldots \geq k_{t_0-1} \geq t_0$. Note that if $t_0 = 3$ then k_2 is the length of a longest bicycle.

Proposition 7.3 *Suppose G is a 3-connected nonbipartite graph with $\delta \geq n/4 + 250$, odd girth t_0 and circumference K. Then $k_0 = K$ and $k_t \geq k_{t-1} - 1$ for $t \geq 1$.*

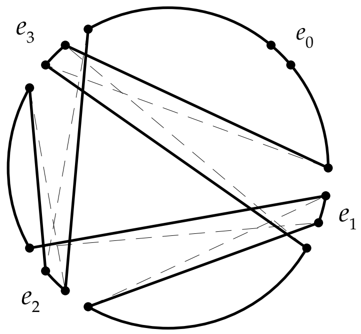

Figure 10: Cyclic replacement (dashed edges provide bicycle)

Moreover, G contains a dominating k_t-cycle sharing a segment of length t with a t_0-cycle, unless $t_0 = 3$ and $k_t = k_2$.

Proof: Let C be a cycle of length K. By Theorem 3.4 C is a D_3-cycle. In particular, C has a vertex in common with every t_0-cycle, and so $k_0 = K$.

If C is not dominating, i.e. there is an edge xy in $G-C$, then by Lemma 4.6a and the maximality of C, for some w on C there are the four edges xw, yw, xw^{+3}, yw^{+3}. This yields a triangle and a K-bicycle (by replacing the segment $ww^+w^{+2}w^{+3}$ of C by $wxyw^{+3}$). It follows that $k_2 = K$ and $t_0 = 3$, and the required conclusions are established.

Hence we may assume that C is dominating. We now proceed by induction on t. Assume that $t \geq 1$. We prove the case $t_0 \neq 3$ and the case $t_0 = 3$ and $t = 1$, and then appeal to the above lemma for the case $t_0 = 3$ and $t = 2$.

- *($t_0 \neq 3$), or ($t_0 = 3$ and $t = 1$).*

Take a dominating k_{t-1}-cycle C sharing a segment $a_0a_1 \dots a_{t-1}$ with a t_0-cycle T. If C shares a longer segment with a t_0-cycle then we are done; so, in particular, we may assume that C does not share an edge with a triangle. Fix an orientation of C where $a_1 = a_0^+$ if $t > 1$; otherwise any orientation will do.

Let a_t denote the other neighbor of a_{t-1} in T. If $a_t \in V(C)$ then set $P' = a_{t-1}^+a_{t-1}^{+2} \dots a_ta_{t-1} \dots a_0a_0^- \dots a_t^+$; otherwise set $P' = a_{t-1}^+a_{t-1}^{+2} \dots a_{t-1}^-a_{t-1}a_t$. The path P' includes all vertices of C and a_t, and shares the segment $S = a_0a_1 \dots a_t$ with T.

Let P be a longest path of G containing P' as a subpath and let u and v be its endvertices. Since C is dominating, P is at most two vertices longer than P'.

Perform a double sweep (Lemma 4.2) of the segment(s) of $P - S - \{u, v\}$ with beacons (v, u) and $\varepsilon = 1$ (and $c = 0$, $r \leq 2$). The result is a neighbor x of u such that v is adjacent to a vertex $x^{-\mu}$, where $1 \leq \mu \leq 3$. Hence we obtain a cycle C' of length at least $k_{t-1} - 2$ sharing S with T. Moreover, if C' has length $k_{t-1} - 2$ then C' misses both ends of the edge $e = x^{-2}x^-$ of C. Since a longest cycle is at least as long as a longest bicycle, we get $|C'| \geq \min\{n, 3\delta, n - \alpha + \delta\} - 20 \geq$

$\min\{3|G-S|/4, 5|G-S|/4-\alpha(G-S)\}+|S|$. If e has a neighbor in $G-C'$ then we can apply Lemma 4.8 to $G-S$ to extend the path $C'-S$ by at least one vertex. If e has no neighbor in $G-C'$ then we can extend $C'-S$ by at least one vertex by Lemma 4.6a. This implies that $k_t \geq k_{t-1}-1$.

Now consider a cycle C'' of length k_t sharing a segment S of length t with T. If follows from Proposition 3.7 that $k_t > \min\{n, 3\delta, n-\alpha+\delta\}-(10-t)$. By Lemma 4.8 applied to $C''-S$, $G-C''$ consists of independent vertices or edges. If there is an edge in $G-C''$ then, as in the second paragraph of the proof, it follows that $t_0 = 3$ and $k_t = k_2$.

- $t_0 = 3$ *and* $t = 2$.

Let C be a k_1-cycle sharing an edge with a triangle. As, by induction, C is dominating, Lemma 7.2 completes the proof. □

7.2 Proof of Proposition 3.3

We start with a lemma to handle the case of odd girth 7.

Lemma 7.4 *Let G be a graph with odd girth 7 and $\delta \geq n/4+250$. If C is a maximal k-bicycle, then G contains a $(k+1)$-cycle.*

Proof: Note that G is 3-connected by Proposition 3.5, so C is dominating by Proposition 3.6.

Let u and v be the endvertices of the length-2 path P of nonstandard degree-2 vertices in C. The removal of P from C leaves a $(k-1)$-cycle C'. If neither u nor v has two neighbors in $G-C$, then we make an "unusual" double sweep of C' with beacons u, v: we consider the edges of type $ua, va^+, va^{+2}, ua^{+3}$ (see Figure 11). Since u and v are joined by a path of length 2 outside C', and there is neither a 3- nor 5-cycle, up to symmetry the two edges must be ua and va^{+2}. This adds two vertices to the length of C'.

If one of u or v has two neighbors in $G-C$, say x and y, then we can do the same procedure with beacons x, y since C is dominating. □

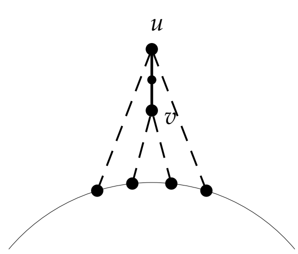

Figure 11: An unusual double sweep

In order to prove Proposition 3.3, we need to show that G contains a k-bicycle for some $k \geq K-9$ and all cycles between k and K (where K is the circumference of G).

If G has connectivity 2 then this is true by Proposition 3.5. So we may assume that G is 3-connected.

Let t_0 be the odd girth of G. Let C be a longest cycle that shares $t_0 - 1$ edges with an odd cycle T of length t_0. By Proposition 7.3 the length k of C is at least $K - t_0 + 1$. Further, there are all longer cycles up to length K. In particular, the conclusion holds for $t_0 = 3$, so we may assume that $t_0 \geq 5$.

Suppose a longest bipath P in G has length ℓ. Consider doing a double sweep (Lemma 4.2) of the interior of the standard paths of P with $\varepsilon = 1$ and beacons the endvertices of P ($c = 0$, $r \leq 2$). We obtain either an $(\ell+1)$- or ℓ-bicycle, or a $(\ell-1)$-bicycle that is not dominating. As $t_0 \geq 5$, by Proposition 3.6 only dominating bicycles can be maximal; so there is a bicycle of length at least ℓ. There are two cases.

- $t_0 = 5$.

Then let $aa^+ \dots a^{+4}$ be the path shared by T and C. The length ℓ of a longest bipath is at least $k-2$ (for example remove the edge a^-a from C). As by the above reasoning there is a bicycle of length at least ℓ, the proof is complete unless $\ell = k-2$.

In this case, neither a^{+2} nor a^{+3} has a neighbor in $G-C$. Let C' be the cycle of length $k-3$ obtained from C by using the chord aa^{+4}. We can apply Lemma 4.6b to the edge $a^{+2}a^{+3}$ and (a spanning path of) C' and so obtain a $(k-1)$-cycle.

- $t_0 = 7$.

Then let $aa^+ \dots a^{+6}$ be the path shared by T and C. The length ℓ of a longest bipath is at least $k-3$. By the above reasoning there is a bicycle of length at least ℓ. If $\ell > k-3$ we obtain the missing $(k-1)$-cycle by Lemma 7.4. So, we may assume that $\ell = k-3$.

By Lemma 7.4 we obtain a $(k-2)$-cycle but we still need to find a $(k-1)$-cycle. To avoid a $(k-2)$-bipath, none of a^{+2}, a^{+3} or a^{+4} has a neighbor in $G-C$. Also a^+ and a^{+5} have no common neighbor v in $G-C$, as $aa^+va^{+5}a^{+6}a$ would be a 5-cycle. Let C' be the cycle of length $k-5$ obtained from C by using the chord aa^{+6}. Then, by Lemma 4.6b, we can reinsert either the pair of edges a^+a^{+2} and $a^{+3}a^{+4}$ or the pair of edges $a^{+2}a^{+3}$ and $a^{+4}a^{+5}$ into (a spanning path of) C', according to which of a^+ or a^{+5} has less neighbors in $G-C$. □

8 Algorithmic Issues

In this section we discuss the complexity of determining whether a graph is weakly pancylic or not and of finding cycles of each length.

We note first that the proof of Theorem 1.4 is constructive, provided one is given a longest cycle. That is, each lemma provides a constructive procedure for finding the desired structure. Thus there is a polynomial-time procedure to construct a cycle of each desired length if a graph is weakly pancylic, 2-connected and has high minimum degree. Of course, it is necessary for the longest cycle to be given, since determining whether such a graph has a hamiltonian cycle is intractable [18].

Theorem 1.4 is also the basis of a simple algorithm for determining whether a graph with sufficiently high minimum degree is weakly pancylic or not. We offer the following result without proof. A manuscript is available from the authors.

Theorem 8.1 *If G is a graph with minimum degree $\delta \geq n/4+250$, then G is weakly pancyclic if and only if every block of order at least 5 contains a 5-cycle.*

Thus the test for weakly pancylic graphs requires determining the blocks and the presence of a 5-cycle, both of which can be performed in polynomial time. In general, though, determining whether a graph is weakly pancyclic or not is hard.

The decision problem Pancycle has input a graph and one outputs whether the graph is pancyclic or not. This problem is NP-hard in general, but it is in NP: a certificate of pancyclicness is simply a listing of a cycle of each length.

The decision problem WeakPan has input a graph and one outputs whether the graph is weakly pancyclic or not. There does not appear to be a simple certificate that a graph is weakly pancyclic, nor does there appear to be a simple certificate that a graph is not weakly pancyclic. In fact, the next theorem shows that this problem is in general both NP-hard and co-NP-hard, and hence unlikely to be in NP.

We say that a problem is NP-hard for graphs with minimum degree approaching αn if for all $\varepsilon > 0$ the problem is NP-hard for graphs with minimum degree at least $(\alpha - \varepsilon)n$. The results of this section show that WeakPan is polynomial-time computable for graphs with minimum degree more than $n/4 + 250$, NP-hard for graphs with minimum degree approaching $n/4$, and probably outside NP for graphs with minimum degree approaching $n/8$.

Theorem 8.2 WeakPan *is NP-hard and co-NP-hard. In fact it is NP-hard even for graphs with minimum degree approaching $n/4$, and co-NP-hard even for graphs with minimum degree approaching $n/8$.*

Proof: (1) The decision problem HamPath has input a graph and one outputs whether the graph has a hamiltonian path or not. HamPath is NP-complete even for graphs of even order n with minimum degree approaching $n/2$. (This can be established using techniques similar to those employed in [20].)

A reduction from HamPath to WeakPan is:

$$G \mapsto G' = (G + K_1) \cup K(n/2 + 1, n/2 + 1),$$

where $+$ denotes join and $\cup$ disjoint union. If G has a hamiltonian path, then it follows easily that $G + K_1$ is pancyclic and hence that G' is weakly pancyclic. If G' is weakly pancylic, then $G + K_1$ is hamiltonian so that G has a hamiltonian path.

(2) The decision problem HamBip has input a bipartite graph and one must output whether the graph has a hamiltonian cycle or not. This problem is NP-hard even for graphs with minimum degree approaching $n/4$ [41].

A reduction from HamBip to the complement of WeakPan is:

$$G \mapsto G' = G \cup K_{n-2}.$$

The resultant graph G' is weakly pancyclic if and only if (the bipartite graph) G on n vertices does not have a hamiltonian cycle. □

On the positive side we can say the following. The decision problem WeakPan is of the form: there exists a cycle of length c for c running from the girth to some value K, and for all ordered subsets of cardinality $K + d$ the resulting ordering is not a cycle for d running from 1 up to $n - K$. (Note that the girth is polynomial-time computable.) Thus there are boolean functions C and D such that:

$$G \in \text{WeakPan} \quad \text{iff} \quad \exists y C(y) \wedge \forall x D(x)$$

In particular, the problem WeakPan is in the intersection of the complexity classes Σ_2^P and Π_2^P. (For definition of these, see [18].)

9 Future Work

It would be interesting to see if the minimum-degree bound could be lowered even further for graphs of higher connectivity. The examples given in the introduction to show that Theorem 1.4 and Corollary 1.6 are best possible are only 2-connected. Perhaps one can prove a similar theorem for 3-connected graphs and minimum degree $n/6 + \mathcal{O}(1)$. This is a sort of limit because there are highly connected hamiltonian graphs with minimum degree $(n+3)/6$ which contain a triangle but no 5-cycle.

Another approach is to prove the equivalent of weakly pancyclic results for bipartite graphs. A bipartite version of the pancyclic results of Theorem 1.3 and Corollary 1.6 has been established.

Theorem 9.1 (Tian & Zang [39]) *If G is a hamiltonian bipartite graph on n vertices with minimum degree $\delta(G) > n/5 + 2$ then G contains cycles of every even length.*

However, Mitchem and Schmeichel [31] suggested that minimum degree larger than $(1 + \sqrt{2n-3})/2$ might suffice for a hamiltonian bipartite graph of order n to contain cycles of all even lengths. This was refuted by Shi [37]. But it remains possible that minimum degree more than $\sqrt{n+1} - 1$ might suffice.

The maximum number of edges in a nonbipartite graph of order n which is not weakly pancyclic has not been determined yet (see Conjecture 2.13). Also, instead of asking for many consecutive cycle lengths one might ask for a cycle of specific length. The question of how many edges a hamiltonian graph of order n can have, if it contains no k-cycle, has been investigated by Hendry and Brandt [23]. This question is solved for a few values of n and k, but for many cases there are interesting conjectures due to Hendry [23].

Finally, does every nonbipartite hamiltonian graph with $\delta \geq n/4 + 250$ contain a spanning bicycle? We have only shown above that it contains a bicycle of order at least $n - 7$.

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10 Appendix

Here we show that it can be tested in time polynomial in n whether a graph of order n with minimum degree at least $n/4+250$ is weakly pancyclic. This is an immediate consequence of the following theorem.

Theorem 10.1 *If G is a graph with minimum degree $\delta \geq n/4+250$, then G is weakly pancyclic if and only if every block of order at least 5 contains a 5-cycle.*

The proof of this theorem makes use of the following result due to Enomoto, van den Heuvel, Kaneko and Saito [Relative length of long paths and cycles with large degree sums, JGT 20 (1995), 213-225]. (In fact, a more refined statement is proved there.)

Theorem 10.2 *Let G be a 2-connected graph of order n. If the minimum degree $\delta(G) \geq (n+2)/3$, then a longest path of G has at most as many edges as a longest cycle.*

Proof: (of Theorem 10.1)

If G is 2-connected then this is a consequence of Theorem 1.4. So we may assume that there are at least two blocks.

Call a block B *large* if it contains a vertex which is not a cutvertex of G. Let b denote the number of large blocks. Since a large block has at least $\delta+1$ vertices, it follows that $b \leq 3$.

Claim 1 *If B is a large block and $|B| \leq n/2+1$, then B is weakly pancyclic with girth 3.*

Proof: Let S_B be the set of vertices of B that are cutvertices in G. It is not hard to argue that $|S_B| \leq 2$ (using $b \leq 3$). We have $\delta(B-S_B) > |B-S_B|/2+200$, so $B-S_B$ is pancyclic and panconnected (Theorems 2.10 and 4.11). We thus obtain all cycles from 3 up to $|B-S_B| \geq |B|-2$. We are done unless $|S_B| = 2$. But, since B is 2-connected, there must be two vertex disjoint paths linking $v \in S_B$ to $B-S_B$. Since $B-S_B$ is panconnected, the missing $(|B|-1)$-cycle results. □

If $b=3$, then it follows that every nontrivial block is weakly pancyclic with girth 3, and we are done.

Hence we may assume that $b=2$. If G is disconnected, or G contains a bridge, then we get the statement by induction. So we may assume that G is connected and there is a unique cut-vertex v in G.

Suppose that some large block B is bipartite. Then $B-v$ is bipartite with minimum degree exceeding $n/4+200$. Since any triangle-free graph with minimum degree $\delta \geq 2$ has a cycle of length at least 2δ (easy exercise), we obtain that $B-v$

has circumference $c(B-v) > n/2 + 400$. Obviously, G cannot contain a cycle of length $c(B-v) - 1$ so G is not weakly pancyclic.

For the remainder of the theorem, it suffices to show the following:

Claim 2 *In a 2-connected nonbipartite graph B, if every vertex has degree at least $n/3 + 100$ except for one vertex v, then B is weakly pancyclic (with girth at most 4).*

Proof: If $B - v$ has connectivity 1, then $B - v$ consists of two large blocks each of which is panconnected and pancyclic. This implies that B is weakly pancyclic. So we may assume that $B - v$ is 2-connected.

Now fix a shortest odd cycle T of B. It is easy to check that $|T| \leq 5$. *Suppose $B - v$ is triangle-free.* Then by Theorem 1.8 the circumference of $B - v$ is the largest it can be, viz. exactly $\min\{|B-v|, 2(|B-v| - \alpha(B-v))\}$. It follows from Theorem 1.4 that $B - v$ is weakly pancyclic unless $B - v$ is bipartite.

If $B - v$ is bipartite, it is easy to verify that any two vertices u and v are joined by paths of all lengths with the same parity as $dist(u, w)$ between the length of a shortest and a longest (u, w)-path. Hence B is weakly pancyclic. So we are done unless $c(B) = c(B-v) + 2 = 2(|B-v| - \alpha(B-v) + 1)$ and B is only missing a $c(B) - 1$ cycle.

In this case we can show that the graph contains a special k-bipath (called a 5-lasso in [10]), viz. a 5-cycle and a path of length $k - 5$ which has one endvertex on the cycle, with v sitting on the path on or near to the intersection vertex of path and cycle, where $k = c(B)$. We can prove along the lines of the proof of Theorem 4.1 in [10] that this implies that the graph is weakly pancyclic. [Omitted]

Suppose that $B - v$ contains a triangle. Choose a triangle T of $B - v$ and a longest cycle C of B such that a longest path P which C and T have in common is as long as possible (among all choices of T and C).

Since $B - v$ satisfies the hypothesis of Theorem 10.2 and contains a path of length at least $|C| - 2$, the circumference $c(B-v) \geq |C| - 2$. Moreover, by Theorem 1.4 the graph $B - v$ is weakly pancyclic with girth 3. So B is weakly pancyclic if and only if it contains a cycle of length $|C| - 1$.

Suppose that B does not contain a cycle of length $|C| - 1$. Then C contains v, and $c(B-v) = |C| - 2$.

Let v^- and v^+ be the predecessor and successor of v on C, respectively. Consider the auxiliary graph $B' = (B-v) + v^-v^+$. Taking the edges of C which are not incident with v and the edge v^-v^+, we obtain a cycle of length $|C| - 1$ in B'. Suppose this cycle is not dominating. Then there is a path of length at least $|C|$ in B', which implies a cycle of length at least $|C| - 1$ in $B - v$ by Theorem 10.2, a contradiction.

Hence, we have shown that every longest cycle C is a dominating cycle. So, in particular, T must have a vertex in common with C, and also P exists. If $|P| = 2$ then we found a contradictory cycle of length $|C| - 1$. So we may assume that $|P| \leq 1$.

We complete the proof by showing that B contains a (u, w)-path Q $(u, w \neq v)$ of length at least $|C| - 1$ which has a path P', $|P'| > |P|$, in common with a triangle T'

of $B-v$ with the following properties: neither u nor w has a neighbor in $B-V(Q)$ and u has at most four pairs of consecutive neighbors on Q.

The reason why this is sufficient is as follows. By easy counting there are vertices x, y in $Q-(V(T') \cup \{x\})$ such that $ux, wy \in E(B)$ and y is the first or second predecessor of x on Q. We obtain either a cycle of length at least $|C|$ which has more edges than C in common with a triangle—which contradicts the choice of C— or we obtain a cycle of length $|C|-1$. So the proof is complete if we can show the existence of Q.

If $|P|=0$ then, using the fact that C is dominating, there are at least two vertices $x, y \in V(T) \cap V(C)$. The predecessors x^-, y^- and successors x^+, y^+ of x, y on C are four distinct vertices ($|P|<2$), so w.l.o.g. $x^+, y^+ \neq v$. Taking a longest path which contains $x^+, \ldots, y^-, y, x, x^-, \ldots, y^+$ as a subpath we obtain the desired path Q (consecutive neighbors might be v^-v, vv^+, xy).

So suppose $|P|=1$. If all vertices of $B-v$ have a pair of consecutive neighbors on C, then we can take a triangle which has an edge in common with C and which uses a shortest chord xy in the path $C-v$ among all these triangles. We can insert the whole segment $C[x^+, y^-]$ into $C-C[x^+, y^-]$ to obtain a bicycle of length $|C|$ by Lemma 7.1. Otherwise, for a suitable orientation of C there is an edge xx^+ of C such that x has two consecutive neighbors yy^- on C and $x^+ \neq v$ has not. Again $y^+ \neq x^-$ and a longest superpath of one of the two paths $x^+, \ldots, y^-, y, x, x^-, \ldots, y^+$ and $x^+, \ldots, y^-, y, x, y^+, \ldots, x^-$ is the desired path Q. $\square$