A nonlinear Neumann problem for an indefinite superlinear elliptic equation

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Preprint Nr. 10 A 96

Abstract. We study the superlinear elliptic equation $-\Delta u = a(x)u^p$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with nonlinear Neumann boundary condition $\partial_{\mathsf{n}} u = b(x)u^q$ on $\partial\Omega$, where a and b may change sign. We derive existence and nonexistence results for positive solutions of this problem. In the case $p = q$ we use a constrained maximization technique, while for $p \neq q$ we apply the mountain-pass lemma. The results are based on ideas from a recent paper of Berestycki, Capuzzo-Dolcetta and Nirenberg [3].

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1 Introduction

In this paper we study existence of positive solutions to the following Neumann problem in a bounded domain $\Omega \subset I\!R^n$ $(n \geq 3)$

$$-\Delta u = a(x)u^p \quad \text{in} \quad \Omega\,, \tag{1}$$

$$\partial_{\mathsf{n}} u = b(x)u^q \quad \text{on} \quad \Gamma = \partial\Omega, \tag{2}$$

where $\partial_{\mathsf{n}} u$ denotes the outward normal derivative of u. We assume that a and b are continuous functions which may change sign, and the exponents p and q are subcritical, i. e. $1 < p < (n+2)/(n-2)$, $1 < q < n/(n-2)$.

Similar problems with homogeneous Dirichlet or Neumann conditions have been studied recently in [1] and [3]. Here we prove existence and nonexistence results for positve solutions in the case of nonlinear boundary conditions. The main results of this paper are the following.

Theorem 1 *If $1 < p = q < n/(n-2)$, then problem (1), (2) has a nontrivial positive solution if*

$$\int_\Omega a(x)\,dx + \int_\Gamma b(x)\,d\Gamma < 0 \quad \textit{and} \quad a(x) > 0 \quad \textit{somewhere in} \quad \Omega. \tag{3}$$

On the other hand, there exists no nontrivial positive solution if

$$\int_\Omega a(x)\,dx + \int_\Gamma b(x)\,d\Gamma \geq 0\,. \tag{4}$$

For the case $p \neq q$ we assume

$$\alpha = \int_\Omega a(x)\,dx < 0 \qquad \text{and} \qquad a(x) > 0 \quad \text{somewhere}, \tag{5}$$

$$\text{if} \quad p > q \quad \text{then} \quad b(x) \leq 0, \qquad \text{if} \qquad p < q \quad \text{then} \quad b(x) \equiv 0\,. \tag{6}$$

Theorem 2 *If $p \neq q$, then problem (1), (2) has a nontrivial positive solution if conditions (5) and (6) are satisfied.*

There is no nontrivial positive solution if

$$b(x) \geq 0 \quad \textit{and} \quad \int_\Omega a(x) \geq 0 \qquad \textit{or} \qquad a(x) \geq 0 \quad \textit{and} \quad \int_\Gamma b(x)d\Gamma \geq 0\,.$$

The proof of these theorems will be given in the next sections.

2 Necessary conditions

To obtain necessary conditions for the existence of a nontrivial solution, we assume that u is a positive solution of (1), (2). By the maximum principle and the Hopf lemma, we claim $u > 0$ in $\overline{\Omega}$. Multiplying (1) with u^{-p}, we obtain by partial integration

$$\begin{aligned}\int_\Omega a(x)\,dx &= \int_\Omega -\Delta u\, u^{-p} dx \\ &= \int_\Omega \nabla u \nabla(u^{-p})\,dx - \int_\Gamma b(x) u^q u^{-p}\, d\Gamma\,.\end{aligned}$$

This shows

$$\int_\Omega a(x)\,dx + \int_\Gamma b(x) u^{q-p}\, d\Gamma < 0\,. \tag{7}$$

In the same way, multiplying with u^{-q},we obtain

$$\int_\Omega a(x) u^{p-q}\,dx + \int_\Gamma b(x)\, d\Gamma < 0\,. \tag{8}$$

Therefore $\int_\Omega a(x)\,dx + \int_\Gamma b(x)\,d\Gamma < 0$ is a necessary condition in the case $p = q$ and we have proved the second part of Theorem 1.

From the inequalities above we derive that

$$\text{if} \qquad b(x) \geq 0 \qquad \text{then} \qquad \int_\Omega a(x)\,dx < 0 \tag{9}$$

is necessary, while

$$\text{if} \qquad a(x) \geq 0 \qquad \text{then} \qquad \int_\Gamma b(x)\,d\Gamma < 0 \tag{10}$$

is a necessary condition for existence. This proves the second part of Theorem 2.

On the other hand, direct integration of (1) leads to

$$\int_\Omega a(x) u^p\,dx = -\int_\Omega \Delta u\,dx = -\int_\Gamma \partial_{\mathsf{n}} u\, d\Gamma,$$

which shows

$$\int_\Omega a(x) u^p\,dx + \int_\Gamma b(x) u^q\, d\Gamma = 0\,. \tag{11}$$

Therefore, a or b must be positive somewhere.

3 Sufficient conditions in the case $p = q$

To obtain a nontrivial solution of (1), (2), we consider the following constrained maximization problem. Let

$$I(u) = \int_\Omega a(x)|u|^{q+1}\,dx + \int_\Gamma b(x)|u|^{q+1}\,d\Gamma\,.$$

Then we look for a solution of the variational problem

$$M = \max\{I(u) \mid u \in S\}\,, \quad S = \{u \in H^1(\Omega) \mid \int_\Omega |\nabla u|^2 = 1\}\,. \tag{12}$$

We assume that (3) from Theorem 1 holds. First of all we show that $M = \sup\{I(u) \mid u \in S\}$ is finite. For this we introduce a decomposition of $H^1(\Omega)$ similar to that in [3]: for $u \in H^1(\Omega)$ we set $t = |\Omega|^{-1}\int_\Omega u(x)\,dx$, the avarage of u in Ω, and $v = u - t$. Then $\int_\Omega v(x)\,dx = 0$ and we set

$$H^1(\Omega) = I\!R \oplus V\,, \quad \text{where} \quad V = \{v \in H^1(\Omega) \mid \int_\Omega v\,dx = 0\}\,. \tag{13}$$

From the Poincaré inequality $\|\varphi\|^2_{H^1} \le C\,(\int_\Omega |\nabla\varphi|^2 dx + |\int_\Omega \varphi\,dx|^2)$ for $\varphi \in H^1(\Omega)$, we see that

$$\|u\|^2 = |t|^2 + \|v\|_V^2\,, \quad \text{with} \quad \|v\|_V^2 = \int_\Omega |\nabla v|^2 dx$$

defines an equivalent norm on $H^1(\Omega)$.

Under the assumption that a is positive somewhere, we get $M > 0$, since we can choose a function with compact support in $\{x \in \Omega \mid a(x) > 0\}$, $\int_\Omega |\nabla u|^2 dx = 1$.

Now let $u_k \in S$ be a maximizing sequence, i. e. $I(u_k) \to M$, and let $u_k = t_k + v_k$ be the decomposition according to (13). Since $\int_\Omega |u_k|^2 dx = \int_\Omega |v_k|^2 dx = 1$ and the embedding $V \hookrightarrow L^{q+1}(\Omega)$ and trace $V \to L^{q+1}(\Gamma)$ are compact, v_k converges weakly in $H^1(\Omega)$ and consequently has a strongly convergent subsequence in $L^{q+1}(\Omega)$ and $L^{q+1}(\Gamma)$.

Next we show that $|t_k|$ is bounded. Assume that this is not true. Then we may assume that $|t_k|^{-1}v_k \to 0$ in $L^{q+1}(\Omega)$ and $L^{q+1}(\Gamma)$. Consequently, from (3) we get

$$\int_\Omega a(x)|1 + |t_k|^{-1}v_k|^{q+1}dx + \int_\Gamma b(x)|1 + |t_k|^{-1}v_k|^{q+1}dx \quad \longrightarrow$$
$$\int_\Omega a(x)\,dx + \int_\Gamma b(x)d\Gamma \quad < \quad 0.$$

From this we conclude

$$I(u_k) = |t_k|^{q+1}\left(\int_\Omega a(x)|1 + |t_k|^{-1}v_k|^{q+1}dx + \int_\Gamma b(x)|1 + |t_k|^{-1}v_k|^{q+1}dx\right) \to -\infty,$$

which is not possible. Therefore, t_k is bounded and contains a convergent subsequence. We may assume that u_k converges weakly in $H^1(\Omega)$ and strongly in $L^{q+1}(\Omega)$ and $L^{q+1}(\Gamma)$. The limit function u satisfies $I(u) = M$ and u is not a constant in view of (3). Hence $0 < \int_\Omega |\nabla u|^2 dx \leq 1$ and there exists $\mu \geq 1$ such that $\mu u \in S$. Since $I(\mu u) = \mu^{q+1} M \leq M$, we have $\mu = 1$. Further we claim $u \geq 0$, since we can choose a maximizing sequence of positive functions.

Consequently there exists a Lagrange multiplier $\lambda \in I\!R$ such that

$$\lambda 2 \int_\Omega \nabla u \nabla \varphi = \frac{1}{q+1}\left(\int_\Omega a(x) u^q \varphi \, dx + \int_\Gamma b(x) u^q \varphi \, d\Gamma\right) \tag{14}$$

for every $\varphi \in H^1(\Omega)$. Setting $\varphi = u$, we get

$$\lambda 2(q+1) \int_\Omega |\nabla u|^2 dx = I(u) = M > 0\,,$$

which shows $\lambda > 0$. Let $\tilde{\lambda} = \lambda 2(q+1)$. From (14) we obtain

$$-\tilde{\lambda} \Delta u = a(x) u^q \quad \text{in} \quad \Omega$$

and by partial integration

$$\tilde{\lambda} \int_\Gamma \partial_{\mathsf{n}} u \, \varphi \, d\Gamma = \int_\Gamma b(x) u^q \varphi \, d\Gamma$$

for every $\varphi \in H^1(\Omega)$. Therefore $w = \tilde{\lambda}^{-1/(q-1)} u$ is a solution of (1), (2) and the maximum principle implies $u > 0$ in Ω. This proves the first part of Theorem 1.

4 Sufficient conditions in the case $p \neq q$

In the case $p \neq q$ we want to apply the mountain-pass lemma to obtain a nontrivial solution. Throughout this section we assume that (5)–(6) hold. Define

$$J(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1} a(x)|u|^{p+1} dx - \int_\Gamma \frac{1}{q+1} b(x)|u|^{q+1} d\Gamma\,.$$

We will show that J satisfies the assumptions of the mountain-pass lemma (cf. [2] or [5]), that is:

(i) J satisfies the Palais-Smale condition.

(ii) There are numbers $\rho, \delta > 0$, such that $J(u) \geq \delta$ for every $u \in H^1(\Omega)$ with $\|u\| = \rho$.

(iii) $J(0) = 0$ and there exists $e \in H^1(\Omega)$, $\|e\| > \rho$, such that $J(e) < \delta$.

To this end we need the following lemma from [3]

Lemma 3 *There exists a positive real number $\eta > 0$ such that for all $t \in I\!R$, $v \in V$ satisfying $\|v\|_V \leq \eta|t|$ the estimate*

$$\int_\Omega a(x)|u(x)|^{p+1}dx = \int_\Omega a(x)|t + v(x)|^{p+1}dx \leq \frac{|t|^{p+1}}{2}\int_\Omega a(x)dx$$

holds.

To prove (i), let u_k be a Palais-Smale sequence in $H^1(\Omega)$ for J. Then for every $\varepsilon > 0$, $k \geq k_\varepsilon$ we have

$$|J(u_k)| = \left|\int_\Omega \frac{1}{2}|\nabla u_k|^2 - \frac{1}{p+1}a(x)|u_k|^{p+1}dx - \int_\Gamma \frac{1}{q+1}b(x)|u_k|^{q+1}d\Gamma\right| \leq M\,, \tag{15}$$

$$|\langle J'(u_k), u_k\rangle| = \left|\int_\Omega |\nabla u_k|^2 - a(x)|u_k|^{p+1}dx - \int_\Gamma b(x)|u_k|^{q+1}d\Gamma\right| \leq \varepsilon\|u_k\|\,. \tag{16}$$

Setting $\varepsilon = 1$, multiplying (16) with $1/(p+1)$ and adding, we obtain

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla u_k|^2dx + \left(\frac{1}{p+1} - \frac{1}{q+1}\right)\int_\Gamma b(x)|u_k|^{q+1}d\Gamma - \|u_k\| \leq M\,. \tag{17}$$

By (6) the middle term on the left hand side is nonnegative. Setting $u_k = t_k + v_k \in I\!R \oplus V$ and $\vartheta = 1/2 - 1/(p+1) > 0$, we get

$$\vartheta\|v_k\|_V^2 - \|t_k + v_k\| \leq M\,. \tag{18}$$

We now want to show that t_k and v_k are bounded. Without restrictions we may assume that $\|t_k\| \geq c > 0$ for all k. Then (18) implies

$$\vartheta\frac{\|v_k\|_V^2}{|t_k|} - \left(1 + \frac{\|v_k\|_V^2}{|t_k|^2}\right)^{1/2} \leq M/|t_k|\,,$$

or

$$\vartheta\frac{\|v_k\|_V^2}{|t_k|} - \frac{\|v_k\|_V}{|t_k|} \leq M_1\,. \tag{19}$$

Now suppose that t_k is not bounded. If $\|v_k\|_V \to \infty$, we obtain from (19)

$$\|v_k\|_V \left(\vartheta \frac{\|v_k\|_V}{|t_k|} - \frac{1}{|t_k|}\right) \leq M_1$$

and since $1/|t_k| \to \infty$, we see that $\|v_k\|_V/|t_k| \to 0$. That is, to every $\eta > 0$ there is a k_η such that $\|v_k\|_V \leq \eta |t_k|$ holds for $k \geq k_\eta$.

On the other hand, if $\|v_k\|_V$ was bounded, the inequality $\|v_k\|_V \leq \eta |t_k|$ is trivially satisfied for k large enough.

In both cases we can apply Lemma 3 to obtain

$$J(u_k) \geq \frac{1}{2}\|v_k\|_V^2 - \frac{|t|^{p+1}}{2}\frac{\alpha}{p+1}.$$

Since $\alpha < 0$, we get $J(u_k) \to \infty$ which is a contradiction. Therefore, t_k must be bounded. Now from (18) it is easy to see that $\|v_k\|_V$ must also be bounded if t_k is. This shows that every Palais-Smale sequence for J is bounded.

Now by standard arguments, using the compact embedding, resp. trace $V \hookrightarrow L^{p+1}(\Omega)$, $V \to L^{q+1}(\Gamma)$, we conclude that u_k has a convergent subsequence, thus (i) is proved.

To show that (ii) holds, fix $\rho > 0$ and let η be as in Lemma 3. Choose $u = t + v \in H^1(\Omega)$ with $\|u\| = \rho$, i. e. $t^2 + \|v\|_V^2 = \rho^2$.

Assume first that $\|v\|_V \leq \eta\,|t|$. Then $\rho^2 \leq t^2(1+\eta^2)$ or $t^2 \geq \rho^2(1+\eta^2)^{-1}$. Setting $\alpha = (1/2)\int_\Omega a(x)\,dx$. From Lemma 3 and (6) we get

$$\begin{aligned} J(u) &\geq \int_\Omega \frac{1}{2}|\nabla v|^2 - \frac{1}{p+1}a(x)|t+v|^{p+1}dx \\ &\geq \frac{-\alpha}{2}\frac{|t|^{p+1}}{p+1}. \end{aligned}$$

From (5) we see that $-\alpha > 0$ and find

$$J(u) \geq \frac{-\alpha}{2(p+1)}\left(\frac{\rho^2}{1+\eta^2}\right)^{(p+1)/2} > 0\,.$$

Now we consider the case $\|v\|_V \geq \eta\,|t|$. We have $\|u\|^2 \leq t^2 + \|v\|_V^2 \leq (\frac{1}{\eta^2}+1)\|v\|_V^2$. From the Sobolev embedding, resp. trace theorem we obtain

$$\begin{aligned} \left|\int_\Omega a(x)|u|^{p+1}dx\right| &\leq C_1\|u\|^{p+1} \leq C_1(\frac{1}{\eta^2}+1)^{1/2}\|v\|_V^{p+1}, \\ \left|\int_\Gamma b(x)|u|^{q+1}dx\right| &\leq C_2\|u\|^{q+1} \leq C_2(\frac{1}{\eta^2}+1)^{1/2}\|v\|_V^{q+1} \end{aligned}$$

with constants $C_1, C_2 > 0$. For v small enough, we infer

$$J(u) \geq \frac{1}{2}\|v\|_V^2 - \frac{C_1}{p+1}\|v\|_V^{p+1} - \frac{C_2}{q+1}\|v\|_V^{q+1} \geq \frac{1}{4}\|v\|_V^2 .$$

Since $\|u\| = \rho$ and $\|v\|_V \geq \eta\|u\|(1+\eta^2)^{-1/2}$, we obtain

$$J(u) \geq \frac{1}{4}\frac{\eta^2\rho^2}{1+\eta^2} > 0$$

Hence the choice $\delta = \min\left\{ \frac{1}{4}\frac{\eta^2\rho^2}{1+\eta^2}, \frac{-\alpha}{2(p+1)}\left(\frac{\rho^2}{1+\eta^2}\right)^{(p+1)/2} \right\}$ proves (ii).

To prove (iii), we note that by (5) there is an open subset $U \subset \Omega$, where $a(x) > 0$. Now choose a positive function $w \in H^1(\Omega)$ with compact support in U and observe that $J(\lambda w) \to -\infty$ as $\lambda \to +\infty$. For λ large enough, we infer that $e = \lambda w$ satisfies (iii).

Now the following variant of the mountain-pass lemma from [4] (cited in [3]) shows the existence of a positive solution.

Theorem 4 *Let X be a Banach space and $J \in C^1(X)$ a functional satisfying the conditions (i)–(iii) of the mountain-pass lemma. Assume that $\pi : X \to X$ is a continuous function satisfying $J(\pi(u)) \leq J(u)$ for every $u \in X$, $\pi(0) = 0$, $\pi(e) = e$ (with e defined in (iii)). Then, there exists a critical point $u \in \overline{\pi(X)}$ and $J(u) \geq \delta$.*

Setting $\pi(u) = |u|$, we obtain a positive solution, which is strictly greater than 0 by the maximum principle. Thus, Theorem 2 is proved.

References

[1] S. Alma, G. Tarantello. *On semilinear elliptic equations with indefinite nonlinearities.* Calc. Var. Partial Differ. Eq. 1, 439–475 (1993)

[2] A. Ambrosetti, P. H. Rabinowitz. *Dual variational methods in critical point theory and applications.* J. Funct. Anal.14, 349 – 381 (1973)

[3] H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg. *Variational methods for indefinite superlinear homogeneous elliptic problems.* NoDEA 2, 553 – 572 (1995)

[4] H. Brezis, L. Nirenberg. *Nonlinear Functional Analysis.* (in preparation)

[5] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations.* Reg. Conf. Ser. Math. 65 (1986)