Optimal design of experiments for multivariate response in two-factor linear models*

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Abstract—The design of experiments aims at optimizing certain charcteristics of the statistical procedures to be used, in dependence on the settings for the experimental conditions chosen. In particular, certain functionals of the (asymptotic) covariance matrix of the least squares estimator are to be minimized. For multivariate observations designing experiments is substantially more complicated than in univariate settings. If there are additionally many factors of influence which may interact in different ways, the situation becomes even worse. However, by reduction principles, we are able to generate optimal designs for broad classes of multivariate observations in multi-factor settings which will be illustrated here in the case of two factors of influence.

1. INTRODUCTION

A multivariate experimental situation is characterized by the fact, that for each experimental conditions $t \in T$ the observation $X(t)$ is multivariate, i. e. r-dimensional real valued vectors $X(t) = (X^{(1)}(t), ..., X^{(r)}(t))$ are observed instead of real valued random variables with unknown mean response $\mu(t) = E(X(t)) = (E(X^{(1)}(t)), ..., E(X^{(r)}(t))) \in \mathbb{R}^r$ and correlation structure $cov(X(t)) = \Sigma(t)$.

Here, we consider multivariate observations $X(t) = (X^{(1)}(t), ..., X^{(r)}(t))$ in which the mean response $\mu(t) = (\mu_1(t), ..., \mu_r(t))$ can be linearly parametrized for each component $\mu_\varrho(t) = E(X^{(\varrho)}(t))$, $\varrho = 1, ..., r$, in dependence on the setting $t \in T$ for the influential factors. More precisely,

$$X^{(\varrho)}(t) = a_\varrho(t)'\beta_\varrho + Z^{(\varrho)}(t), \quad t \in T, \tag{1.1}$$

where $a_\varrho : T \to \mathbb{R}^{p_\varrho}$ and $\beta_\varrho \in \mathbb{R}^{p_\varrho}$ are the known regression functions resp. the unknown parameters for the ϱth component. $Z(t) = (Z^{(1)}(t), ..., Z^{(r)}(t))$ is the random error associated with the observation $X(t)$, $E(Z(t)) = \mathbf{0}$. Hence, a multivariate observation $X(t)$ is described by the linear model

$$X(t) = a(t)\beta + Z(t), \quad t \in T, \tag{1.2}$$

or, equivalently, by its mean response

$$E(X(t)) = a(t)\beta, \quad t \in T, \tag{1.3}$$

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where $\beta = (\beta_1', \beta_2', ..., \beta_r')' \in \mathbb{R}^p$, $p = \sum_{\varrho=1}^r p_\varrho$, is the vector of all unknown parameters and the matrix $a(t) = \mathrm{diag}(a_\varrho(t)')_{\varrho=1,\ldots,r}$ of regression functions is block diagonal.

A design d of size N is given as an N-dimensional vector $(t_1, ..., t_N)'$ which contains the adjustments of the factors of influence for N different experiments $X_n(t_n) = a(t_n)\beta + Z_n(t_n)$, $n = 1, ..., N$.

With $X_d^{(\varrho)} = (X_1^{(\varrho)}(t_1), ..., X_N^{(\varrho)}(t_N))'$ and $Z_d^{(\varrho)} = (Z_1^{(\varrho)}(t_1), ..., Z_N^{(\varrho)}(t_N))'$ being the vectors of observations and errors, respectively, the observations for the ϱth component can be written in the usual matrix notation

$$X_d^{(\varrho)} = A_d^{(\varrho)}\beta_\varrho + Z_d^{(\varrho)} \tag{1.4}$$

where $A_d^{(\varrho)} = (a_\varrho(t_1), a_\varrho(t_2), ..., a_\varrho(t_N))'$ is the design matrix for the ϱth component. In accordance with the common notations in multivariate analysis we vectorize the model and arrange its components successively. i. e. $X_d = (X_d^{(1)\prime}, X_d^{(2)\prime}, ..., X_d^{(r)\prime})'$ and $Z_d = (Z_d^{(1)\prime}, Z_d^{(2)\prime}, ..., Z_d^{(r)\prime})'$. Hence, we obtain the vectorized linear model equation for multivariate response

$$X_d = A_d\beta + Z_d \tag{1.5}$$

in matrix form, where the design matrix $A_d = \mathrm{diag}(A_d^{(\varrho)})_{\varrho=1,\ldots,r}$ is block diagonal.

Next, we assume a homogeneous error structure $\Sigma(t) = \Sigma$ for which Σ is positive definite to avoid degenerated cases, and uncorrelated observations, $cov(Z_m(t_m), Z_n(t_n)) = \mathbf{0}$ for $m \neq n$. This results in a covariance structure $cov(Z_d) = \Sigma \otimes \mathbf{E}_N$ for the vectorized errors Z_d, where $\mathbf{E}_N$ is the $N \times N$ identity matrix and $\otimes$ denotes the Kronecker product.

In the linear model (1.5) the general (weighted) least squares estimator $\widehat{\beta}_d = (A_d'(\Sigma \otimes \mathbf{E}_N)^{-1}A_d)^{-1}A_d'(\Sigma \otimes \mathbf{E}_N)^{-1}X_d$ is the best linear unbiased estimator of the unknown parameters β. We will consider only such designs for which β is linearly identifiable (estimable), i. e. for which A_d has full column rank. The quality of the estimator $\widehat{\beta}_d$ is based on the design d and, hence, the quality of the design d itself will be measured by the covariance matrix $cov(\widehat{\beta}_d) = (A_d'(\Sigma \otimes \mathbf{E}_N)^{-1}A_d)^{-1}$.

The information matrix $\mathbf{I}(d) = A_d'(\Sigma \otimes \mathbf{E}_N)^{-1}A_d$ of a design d of size N is defined as the inverse of the corresponding covariance matrix $cov(\widehat{\beta}_d)$. By rearranging terms it can be seen that the information matrix is the sum of the informations contained in each single setting, $\mathbf{I}(d) = \sum_{n=1}^N a(t_n)'\Sigma^{-1}a(t_n)$.

If there are only few different settings and, hence, many replications in an elementary design d it is more convenient to write

$$d = \begin{pmatrix} t_1 & t_2 & \cdots & t_J \\ N_1 & N_2 & \cdots & N_J \end{pmatrix} \tag{1.6}$$

where the settings t_j are mutually different and the N_j are the numbers of replications at the settings t_j, $\sum_{j=1}^J N_j = N$. Hence, we obtain for the information matrix $\mathbf{I}(d) = \sum_{j=1}^J N_j a(t_j)'\Sigma^{-1}a(t_j)$. As the sample size N is fixed, a design d is determined by its relative frequencies $w_j = \frac{N_j}{N}$ at the actual settings. Thus, d may be identified with a dicrete measure δ_d on a finite support, which assigns weights w_j to its support points t_j.

To solve the integer optimization problem for N_j is often a difficult task. Therefore it is reasonable to embed the optimization problem in a continuous setting and to drop the requirement of w_j to be an integer multiple of $\frac{1}{N}$. Hence, as a (generalized) design

δ we define any measure on T, normalized to one, which is concentrated on a finite number of supporting points. In accordance with hte above considerations we define the information matrix

$$\mathbf{I}(\delta) = \int a(t)'\Sigma^{-1}a(t)\,\delta(\mathrm{d}t), \tag{1.7}$$

such that, for any design d of sample size N, $\mathbf{I}(\delta_d) = \frac{1}{N}\mathbf{I}(d)$ is a normalized version of the information matrix $\mathbf{I}(d)$.

Besides this traditional extension of the design problem to general designs, in order to embed the optimization problem into a richer set which is convex, a more substantial reason for considering general designs is evident. Because the supposition of a prespecified number of units with prespecified variation is sometimes rather artificial a more general idea should be associated with the concept of general designs (see Kurotschka, 1988). Assume that we may observe at certain design points $t_1, ..., t_J$ with intensity proportional to $w_j > 0$, $\sum_{j=1}^{J} w_j = 1$, i. e. $cov(X_j(t_j)) = w_j^{-1}\Sigma$, $j = 1, ..., J$, then in this general multivariate linear model the best linear unbiased estimator is given by the weighted least squares estimator $\widehat{\beta}_\delta = (A_\delta'(\Sigma^{-1} \otimes W_\delta)A_\delta)^{-1}A_\delta'(\Sigma^{-1} \otimes W_\delta)X_\delta$ where $X_\delta = (X_\delta^{(\varrho)})_{\varrho=1,\ldots,r}$ with $X_\delta^{(\varrho)} = (X_j^{(\varrho)}(t_j))_{j=1,\ldots,J}$ are the vectorized observations, $A_\delta = \mathrm{diag}(A_\delta^{(\varrho)})_{\varrho=1,\ldots,r}$ with $A_\delta^{(\varrho)} = (a_\varrho(t_1), ..., a_\varrho(t_J))'$ is the design matrix associated with the supporting points t_j of the design δ and the diagonal matrix $W_\delta = \mathrm{diag}(w_j)_{j=1,\ldots,J}$ is the intensity matrix of the observations. In this setting the covariance structure of the observations obviously becomes $cov(X_\delta) = \Sigma \otimes W_\delta^{-1}$.

We notice that $\mathbf{I}(\delta) = \sum_{j=1}^{J} w_j a(t_j)'\Sigma^{-1}a(t_j) = A_\delta'(\Sigma^{-1} \otimes W_\delta)A_\delta = cov(\widehat{\beta}_\delta)$. Hence, a design δ which assigns weights w_j to the supporting points t_j, $j = 1, ..., J$, has to be understood as an instruction which demands that J different observations are to be made at the design points t_j with intensity proportional to w_j, $j = 1, ..., J$.

The natural optimality criterion of minizing the covariance $cov(\widehat{\beta}_\delta)$ in the positive difinite sense is too stringent in general settings. As a compromise different matrix means can be considered which are commonly used in both multivariate analysis and optimum design theory. In the following we will be concerned with the D-criterion of generalized variance which aims at minizing the determinant of the covariance or, equivalently, at maximizing the determinant $\det \mathbf{I}(\delta, \Sigma)$ of the information matrix. For general statements on generalized designs in multivariate settings we refer to Fedorov (1972). More recently, Krafft and Schaefer (1992) and Bischoff (1993) investigated models in which the complexity increases from component to component. Their results lead to weighted optimality criteria as proposed by Läuter (1974) for model uncertainty in univariate settings. An extensive survey of situations in which a single design is simultaneously optimal for all components will be given in Kurotschka and Schwabe (1996).

In most experiments more than one factor of influence is active and the design problems become substantially more complicated. For univariate observations techniques have been developed in Schwabe (1996) to reduce the design problems for multi-factor experiments to their single-factor counterparts which can be solved by standard methods of optimum design theory. In the present treatise we exhibit that those ideas can be extended to situations with multivariate responses. In particular, the optimality of product designs is established, and the marginals of them satisfy certain optimality conditions within the associated marginal models in which only a single factor is involved.

For notational ease we confine to the case of two factors of influence,

$$E(X(t_1,t_2)) = a(t_1,t_2)\beta, \quad t_1 \in T_1,\ t_2 \in T_2. \tag{1.8}$$

The related higher dimensional problems can be treated by combining several different factors to a new "meta-factor".

In the following sections different interaction structures are investigated. Section 2 deals with Kronecker product type structures in which all influences related to the single factors factorize. The practically more realistic case of additive modelling without interactions is treated in Section 3. In Section 4 invariance structures are introduced for situations in which, at least, one factor of influence is qualitative, i. e. it acts on a finite number of levels. Finally, the concept of conditioning is exposed for intra-class models with multivariate response and some possible extensions are indicated.

2. KRONECKER PRODUCT TYPE STRUCTURES

We consider the class of two-factor models

$$E(X(t_1,t_2)) = (a_{(1)}(t_1) \otimes a_{(2)}(t_2))\beta, \quad t_1 \in T_1,\ t_2 \in T_2, \tag{2.1}$$

in which the regression function factorizes into components corresponding to the single factors of influence, $a_{(k)} : T_k \to I\!R^{p_{(k)} \times r_k}$. Here $\otimes$ denotes the usual Kronecker product. Additionally, we assume that the covariance structure $\Sigma = \Sigma_1 \otimes \Sigma_2$ factorizes appropriately into $r_k \times r_k$ covariance matrices Σ_k. The associated marginal models are given by

$$E(X_{(k)}(t_k)) = a_{(k)}(t_k)\beta_{(k)}, \quad t_k \in T_k, \tag{2.2}$$

with covariance structure Σ_k. Note that $r = r_1 r_2$ for the dimensions of the responses and $p = p_{(1)}p_{(2)}$ for the dimensions of parameters.

THEOREM 2.1. *The product design* $\delta^* = \delta_1^* \otimes \delta_2^*$ *is D-optimal in the Kronecker product type two-factor model* (2.1) *with covariance structure* $\Sigma = \Sigma_1 \otimes \Sigma_2$, *if the designs* δ_k^* *are D-optimal in the marginal models* (2.2) *with covariances* Σ_k, $k = 1, 2$, *respectively.*

Proof. Let $\mathbf{I}_k(\delta_k, \Sigma_k)$ be the information matrix in the kth martinal model. Then $\mathrm{tr}(\Sigma^{-1}a(t_1,t_2)\mathbf{I}(\delta^*,\Sigma)^{-1}a(t_1,t_2)') = \prod_{k=1}^{2} \mathrm{tr}(\Sigma_k^{-1}a_{(k)}(t_k)\mathbf{I}_k(\delta_k^*,\Sigma_k)^{-1}a_{(k)}(t_k)')$ and the result follows from the multivariate version of the Kiefer-Wolfowitz equivalence theorem (Fedorov, 1972, p. 212).

Example 2.1. Assume that both marginal models have a constant and a linear component $a_{(k),1}(t_k) = 1$ and $a_{(k),2}(t_k) = (1, t_k)'$ each. Then the Kronecker product type two-factor model has four components, which are either constant, linear in one of the factors, or linear in both of the factors with an interaction term, $a_1(t_1,t_2) = 1$, $a_2(t_1,t_2) = (1,t_2)'$, $a_3(t_1,t_2) = (1,t_1)'$ and $a_4(t_1,t_2) = (1,t_1,t_2,t_1t_2)'$. For T_k a compact interval the D-optimal marginal designs δ_k^* assign equal weights $\frac{1}{2}$ to both endpoints. Hence, the D-optimal design $\delta^* = \delta_1^* \otimes \delta_2^*$ assigns equal weights $\frac{1}{4}$ to the four corners of the rectangle $T_1 \times T_2$.

Remark 2.1. If $a_2 = \mathbf{E}_{r_2}$, i. e. the second factor is not present, then the Kronecker product generates a model with r_2 replicates of the first marginal. Hence, by Theorem 2.1 a design δ_1^* which is D-optimal for the first marginal model is also D-optimal for the replicated model, if Σ factorizes. This establishes the results of Chang (1994).

3. ADDITIVE MODELS

In the class of additive models without interactions between the factors less is required for the model structure. The additive components

$$E(X_\varrho(t_1,t_2)) = \beta_{\varrho,0} + f_{\varrho,1}(t_1)'\beta_{\varrho,1} + f_{\varrho,2}(t_2)'\beta_{\varrho,2}, \quad t_1 \in T_1,\ t_2 \in T_2 \tag{3.1}$$

can be combined to a multivariate additive model

$$E(X(t_1,t_2)) = \beta_0 + f_1(t_1)\beta_1 + f_2(t_2)\beta_2, \quad t_1 \in T_1,\ t_2 \in T_2 \tag{3.2}$$

with covariance structure Σ, where $f_k(t_k) = \operatorname{diag}(f_{\varrho,k}(t_k)')_{\varrho=1,\dots r}$. The associated marginal models are given by

$$E(X_{(k)}(t_k)) = \beta_0 + f_k(t_k)\beta_k, \quad t_k \in T_k. \tag{3.3}$$

LEMMA 3.1. *Every design δ is dominated by the product of its marginals δ_1 and δ_2, i. e.* $\det \mathbf{I}(\delta,\Sigma) \le \det \mathbf{I}(\delta_1 \otimes \delta_2, \Sigma)$.

Proof. As in the univariate case we can center the regression functions according to $\widetilde{f}_k = f_k - \int f_k \,\mathrm{d}\delta_k$ without changing the determinant of the information matrices. Hence,

$$\begin{aligned} \det \mathbf{I}(\delta,\Sigma) &= \det \begin{pmatrix} \Sigma^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \int \widetilde{f}_1'\Sigma^{-1}\widetilde{f}_1 \,\mathrm{d}\delta_1 & \int \widetilde{f}_1'\Sigma^{-1}\widetilde{f}_2 \,\mathrm{d}\delta \\ \mathbf{0} & \int \widetilde{f}_2'\Sigma^{-1}\widetilde{f}_1 \,\mathrm{d}\delta & \int \widetilde{f}_2'\Sigma^{-1}\widetilde{f}_2 \,\mathrm{d}\delta_2 \end{pmatrix} \\ &\ge \det \Sigma^{-1} \det \int \widetilde{f}_1'\Sigma^{-1}\widetilde{f}_1 \,\mathrm{d}\delta_1 \det \int \widetilde{f}_2'\Sigma^{-1}\widetilde{f}_2 \,\mathrm{d}\delta_2, \end{aligned} \tag{3.4}$$

with equality for product designs $\delta = \delta_1 \otimes \delta_2$ because $\int \widetilde{f}_1'\Sigma^{-1}\widetilde{f}_2 \,\mathrm{d}(\delta_1 \otimes \delta_2) = \mathbf{0}$ for the off-diagonal elements.

It remains to find the best product design.

THEOREM 3.2. *The product design $\delta^* = \delta_1^* \otimes \delta_2^*$ is D-optimal in the additive model* (3.2) *with covariance Σ, if the designs δ_k^* are D-optimal in the marginal models* (3.3) *with covariances Σ, $k = 1, 2$, respectively.*

Proof. Denote by $\mathbf{I}_k(\delta_k,\Sigma)$ the information matrices in the marginal models. Then, by (3.4), $\det \mathbf{I}(\delta_1 \otimes \delta_2, \Sigma) = \det \Sigma \det \mathbf{I}_1(\delta_1,\Sigma) \det \mathbf{I}_2(\delta_2,\Sigma)$ which attains its maximum for the D-optimal marginals δ_1^* and δ_2^*.

Remark 3.1. As in the case of univariate observations the presence of a constant term in each component is required for the result of Theorem 3.1 to hold.

Example 3.1. In an additive model in which the components are given by their regression functions $a_1(t_1,t_2) = (1, t_1, t_1^2, t_2)'$ and $a_2(t_1,t_2) = (1, t_1, t_2, t_2^2)'$, respectively, the marginals contain a linear and a quadratic regression each. Hence, in view of the results of Krafft and Schaefer (1992), the D-optimal marginal designs δ_k^* assign weights $\frac{3}{8}$ to both endpoints and weight $\frac{1}{4}$ to the midpoint of the interval T_k. By Theorem 3.1 the product design $\delta_1^* \otimes \delta_2^*$ is D-optimal which assigns weights to the four corners, to the four midpoints of the edges and to the center of the rectangle $T_1 \times T_2$ in the proportion 9 to 6 to 4.

Example 3.2. In an additive model with components $E(X_1(i,u)) = \alpha_{1,i} + \beta_1 u$ and $E(X_2(i,u)) = \alpha_{2,i} + \beta_{2,1}u + \beta_{2,2}u^2$, $i = 1, ..., I$, $u \in U$, the first factor is qualitative and its influence is described by a one-way layout. The D-optimal marginal design δ_1^* is uniform on the levels $1, ..., I$. The second marginal model contains a linear and a quadratic regression and the D-optimal design is δ_2^* as given in Example 3.1 (with $T_2 = U$). By Theorem 3.1 the product design $\delta_1^* \otimes \delta_2^*$ is D-optimal which is concentrated on $3I$ design points with weights $\frac{3}{8I}$ and $\frac{1}{4I}$, respectively.

Remark 3.2. The result of Theorem 3.1 remains true even if some of the components $f_{\varrho,k}$ of the marginal regression functions are missing. For example, if in case $r = 2$ the components $E(X_k(t_1, t_2)) = \beta_{k,0} + f_k(t_k)'\beta_k$ depend on a single factor each, then the product design $\delta^* = \delta_1^* \otimes \delta_2^*$ is D-optimal for which the marginals δ_k^* are D-optimal in the univariate single-factor models given by the components.

4. INVARIANCE

In this section we consider the class of symmetric multivariate linear models in which one of the factors of influence is qualitative, i. e. it acts on a finite number of levels, and in which the model equations are not affected by permutations of those levels of the qualitative factor.

$$E(X(i,u)) = f_0(u)\beta_0 + f_1(u)\beta_i, \quad i = 1, ..., I, \ u \in U, \tag{4.1}$$

with covariance structure Σ. In the particular case of an additive model we have seen in Example 3.2 that a D-optimal design can be obtained as a product with a uniform marginal for the qualitative factor. This result will be extended to general models (4.1). While the first marginal model is a one-way layout, we have to consider both the complete marginal model

$$E(X_{(2)}(u)) = f_0(u)\beta_0 + f_1(u)\beta_1, \quad u \in U, \tag{4.2}$$

and the reduced marginal model

$$E(X_{(2,1)}(u)) = f_1(u)\beta_1, \quad u \in U, \tag{4.3}$$

for the second factor of influence. Denote by $\mathbf{I}_2(\delta_2, \Sigma)$ and $\mathbf{I}_{2,1}(\delta_2, \Sigma) = \int f_1'\Sigma^{-1}f_1 \, \mathrm{d}\delta_2$ the information matrices in these two marginal models, respectively.

THEOREM 4.1. *The product design* $\delta^* = \delta_1^* \otimes \delta_2^*$ *is D-optimal in the symmetric model* (4.1) *with covariance* Σ, *if the design* δ_1^* *is uniform on* $\{1, ..., I\}$ *and the design* δ_2^* *maximizes* $\det \mathbf{I}_2(\delta_2, \Sigma) \det \mathbf{I}_{2,1}(\delta_2, \Sigma)^{I-1}$.

Proof. The matrix a of regression functions of the symmetric model can be written as $a(i,u) = (f_0(u), \mathbf{e}_i' \otimes f_1(u))$ where $\mathbf{e}_i$ is the ith I-dimensional unit vector. Every permutation of the qualitative levels $\{1, ..., I\}$ induces a linear transformation on a and, hence, every design δ is dominated by it s symmetrization $\delta_1^* \otimes \delta_2$, i. e. $\det \mathbf{I}(\delta, \Sigma) \leq \det \mathbf{I}(\delta_1^* \otimes \delta_2, \Sigma)$, where δ_2 is the second marginal of δ (for details see Schwabe, 1996, Sections 3.2 and 6.2). It remains to find the best δ_2. ¿From the representation

$$\mathbf{I}(\delta_1^* \otimes \delta_2, \Sigma) = \begin{pmatrix} \int f_0'\Sigma^{-1}f_0 \, \mathrm{d}\delta_2 & \frac{1}{I}\mathbf{1}_I' \otimes \int f_0'\Sigma^{-1}f_1 \, \mathrm{d}\delta_2 \\ \frac{1}{I}\mathbf{1}_I \otimes \int f_1'\Sigma^{-1}f_0 \, \mathrm{d}\delta_2 & \frac{1}{I}\mathbf{E}_I \otimes \int f_1'\Sigma^{-1}f_1 \, \mathrm{d}\delta_2 \end{pmatrix} \tag{4.4}$$

of the information matrix we obtain $\det \mathbf{I}(\delta_1^* \otimes \delta_2, \Sigma) = I^{-Ip_{(2,1)}} \left(\det \int f_1' \Sigma^{-1} f_1 \, \mathrm{d}\delta_2\right)^I$ $\det \left(\int f_0' \Sigma^{-1} f_0 \, \mathrm{d}\delta_2 - \int f_0' \Sigma^{-1} f_1 \, \mathrm{d}\delta_2 (\int f_1' \Sigma^{-1} f_1 \, \mathrm{d}\delta_2)^{-1} \int f_1' \Sigma^{-1} f_0 \, \mathrm{d}\delta_2\right)$ by the determinant formula for partitioned matrices, where $p_{(2,1)}$ is the dimension in the reduced second marginal model. By formally letting $I = 1$ in the above arguments we obtain the related representation of $\det \mathbf{I}_2(\delta_2, \Sigma)$ which establishes the optimization problem for δ_2 stated in the theorem.

Remark 4.1. The optimization problem of Theorem 4.1 is another application of the weighted optimality criterion proposed by Läuter (1974).

Remark 4.2. For $f_1 = \mathbf{E}_I$ we recover the optimality result of Theorem 3.1 for an additive model with, at least, one qualitative factor. Also in the present setting some components of f_0 or f_1 may be missing.

5. INTRA-CLASS MODELS

If f_0 is completely omitted from the model in the previous section, then there are no common effects and each group $i = 1, \dots, I$ can be treated separately. If we allow additionally that the modelling may differ from group to group we get the general class of multivariate intra-class models

$$E(X(i,u)) = f_i(u)\beta_i, \quad i = 1, \dots, I, \; u \in U, \tag{5.1}$$

with covariance structure Σ. In contrast to the previous sections we have to consider conditional models

$$E(X_{(2,i)}(u)) = f_i(u)\beta_{(i)}, \quad u \in U, \tag{5.2}$$

for every level i of the qualitative factor. Using Kurotschka's (1984) conditional approach we obtain

THEOREM 5.1. *The δ_1^*-mixture $\delta^* = \delta_1^* \otimes \delta_2^*$ of the kernel δ_2^* is D-optimal in the intra-class model* (5.1) *with covariance Σ, if*

(i) *the conditional desgin $\delta_2^*(i,\cdot)$ is D-optimal in the ith marginal model (5.2) with covariance Σ, for every $i = 1, \dots, I$, and*

(ii) *the design δ_1^* assigns weights $p_{(2,i)} / \sum_{j=1}^I p_{(2,j)}$ to the levels $i = 1, \dots, I$ which are proportional to the dimension $p_{(2,i)}$ of $\beta_{(i)}$ in the ith conditional model.*

Proof. The result is straightforward from the block diagonal structure of the information matrix $\mathbf{I}(\delta_1 \otimes \delta_2) = \mathrm{diag}\left(\delta_1(i) \int f_i' \Sigma^{-1} f_i \, \mathrm{d}\delta_2(i,\cdot)\right)_{i=1,\dots,I}$ for every δ_1-mixture of a kernel δ_2.

Remark 5.1. Due to the conditional approach in Theorem 5.1 the common design region U and the common cavariance structure Σ may be replaced by a conditional region U_i and a conditional covariance structure Σ_i, respectively.

6. ADDITIONAL REMARKS

Remark 6.1. The results on additive and Kronecker product type models (Section 2 and 3) can immediately be extended to an arbitrary number K of factors of influence. Moreover, generalizations to intermediate models with complete M-factor interactions are possible if the covariance $\Sigma = \Sigma_1 \otimes \dots \otimes \Sigma_K$ factorizes appropriately.

Remark 6.2. The considerations of Sections 2, 4 and 5 are also suitable for other optimality criteria like the commonly used A-criterion of minimizing the trace and the E-criterion of minimizing the largest eigenvalue of the covariance matrix. Special emphasis may also be laid on particular linear aspects of the parameters within and between the components like contrasts and mean effects adjusted for blocks.

Remark 6.3. Usually there is no knowledge available on the actual form of the covariance Σ in advance. Hence, the weighted least squares estimator $\widehat{\beta}_\delta$ can not be calculated in general and different alternatives have to be considered. One common method is to estimate the covariance Σ by a random matrix $\widehat{\Sigma}$. Then a $D_{\widehat{\Sigma}}$-optimal design should aim at minimzing the determinant of the covariance matrix for the resulting randomly weighted least squares estimator in which Σ is replaced by $\widehat{\Sigma}$. Asymptotically this will coincide with the D-optimality considered above under mild regularity conditions.

Alternatively, the ordinary least squares estimator may be used which results in the covariance matrix $(\int a'a \, \mathrm{d}\delta)^{-1} \int a'\Sigma a \, \mathrm{d}\delta (\int a'a \, \mathrm{d}\delta)^{-1}$. Although the problem of obtaining optimal designs in this situation looks more difficult it can be seen that in many situations this optimization is equivalent to the usual D-optimality.

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