FRACTIONAL OSCILLATIONS AND MITTAG-LEFFLER FUNCTIONS

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Abstract

The fractional oscillation equation is obtained from the classical equation for linear oscillations by replacing the second-order time derivative by a fractional derivative of order α with $1 < \alpha < 2$. Using the method of the Laplace transform, it is shown that the fundamental solutions can be expressed in terms of Mittag-Leffler functions, and exhibit a finite number of damped oscillations with an algebraic decay. For completeness we also discuss both the cases $0 < \alpha < 1$ (fractional relaxation) and $2 < \alpha \le 3$ (growing oscillations), showing the key role of the Mittag-Leffler functions.

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1. Introduction

For real $\alpha > 0$ (later only for $0 < \alpha \le 3$) we consider the fractional differential equation

$$D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0^+) \right) = -u(t) + q(t)\,, \quad t > 0\,, \tag{1.1}$$

where $q(t)$ is a given continuous function. Here m is an integer uniquely defined by $m-1 < \alpha \le m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k\,,\ k = 0,1,2,\ldots,m-1$. Implicit in the form of (1.1) is our desire to obtain solutions $u(t)$ for which the $u^{(k)}(t)$ are continuous for $t \ge 0,\ k = 0,1,\ldots,m-1$.

The purpose of this paper is to evaluate the solutions of (1.1) in analytical form, providing a detailed discussion of the main properties of interest in mechanics. In our analysis we point out the key role of the Mittag-Leffler functions, improving the results recently obtained by Gorenflo and Rutman [1] and by Mainardi [2], who had considered some mathematical aspects of (1.1).

A numerical treatment of (1.1) has been recently given by Blank [3], who has provided reliable results by means of the collocation approximation with polynomial splines. The plots of our analytical solutions confirm the numerical results in [3].

Several applications of fractional differential equations in physics and engineering have also appeared, see *e.g.* [4-15].

Before starting our analysis of (1.1), we have to explain the meaning of the operator D^α of fractional differentiation. Hereafter, for the sake of completeness, we provide the essentials of Fractional Calculus according to the Riemann-Liouville approach. Let us first define the operator of fractional integration J^β, $\beta > 0$,

$$J^\beta\, u(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\, u(\tau)\, d\tau\,, \quad t > 0\,. \tag{1.2}$$

Then, taking the positive integer m with $m-1 < \alpha \le m$, we define, with $D = \frac{d}{dt}$,

$$D^\alpha\, u(t) := D^m\, J^{m-\alpha}\, u(t)\,, \quad t > 0\,. \tag{1.3}$$

We also define $J^0 = D^0 = I$, where I denotes the Identity operator. We refer to the recent books on Fractional Calculus [16-18] for general treatment and discussion of the operators of fractional integration and derivation. Here we limit ourselves to report the simple, but relevant results for the power function, with $t > 0$,

$$J^p\, t^q = \frac{\Gamma(q+1)}{\Gamma(q+p+1)}\, t^{q+p}\,, \quad D^p\, t^q = \frac{\Gamma(q+1)}{\Gamma(q-p+1)}\, t^{q-p}\,, \quad p \ge 0\,,\ q > -1\,. \tag{1.4}$$

We also observe that an alternative definition of fractional derivative, which is due to Caputo [4], is

$$\widetilde{D}^\alpha\, u(t) := J^{m-\alpha}\, D^m\, u(t)\,. \tag{1.5}$$

We note in general that

$$D^m\, J^{m-\alpha}\, u(t) \ne J^{m-\alpha}\, D^m\, u(t)\,. \tag{1.6}$$

In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m-1 < \alpha < m$, and $t > 0$

$$D^\alpha\, u(t) = \widetilde{D}^\alpha\, u(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}\, u^{(k)}(0^+)\,, \tag{1.7}$$

and therefore, recalling the fractional derivative of the power function (1.4),

$$D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = \widetilde{D}^\alpha\, u(t)\,. \tag{1.8}$$

We also note that while

$$\alpha \to m^\mp\,, \quad D^\alpha\, u(t) \to D^m\, u(t)\,; \tag{1.9a}$$

we have

$$\begin{cases} \alpha \to m^-\,, & \widetilde{D}^\alpha\, u(t) \to D^m\, u(t)\,; \\ \alpha \to m^+\,, & \widetilde{D}^\alpha\, u(t) \to D^m\, u(t) - u^{(m)}(0^+)\,. \end{cases} \tag{1.9b}$$

Therefore in (1.1) we have substantially adopted Caputo's definition for the fractional derivative, which incorporates the initial values of the function and of its integer derivatives of lower order. This approach appears suitable in the applications, where the initial conditions are explicitly expressed in terms of the conventional integer derivatives. Observe, however, that in mathematical treatments of fractional differential equations, the initial conditions are usually given as limiting values of fractional integrals; see *e.g.* [17-18].

The equation (1.1) with the given initial conditions can be solved by elementary functions if α is an integer. For completeness of presentation we write down the corresponding solutions. In the case $\alpha = 1$ we have the equation of *relaxation*

$$u'(t) = -u(t) + q(t)\,, \tag{1.10}$$

whose solution, under the initial condition $u(0) = c_0$, is

$$u(t) = c_0\, \mathrm{e}^{-t} + \int_0^t q(t-\tau)\, \mathrm{e}^{-\tau}\, d\tau\,. \tag{1.11}$$

In the case $\alpha = 2$ we have the equation of *oscillation*

$$u''(t) = -u(t) + q(t)\,, \tag{1.12}$$

whose solution, under the initial conditions $u(0) = c_0$ and $u'(0) = c_1$, is

$$u(t) = c_0\, \cos t + c_1\, \sin t + \int_0^t q(t-\tau)\, \sin\tau\, d\tau\,. \tag{1.13}$$

In the case $\alpha = 3$ we have the equation

$$u'''(t) = -u(t) + q(t)\,, \tag{1.14}$$

whose solution, under the initial conditions $u(0) = c_0$, $u'(0) = c_1$, $u''(0) = c_2$, is

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + c_2\, u_2(t) - \int_0^t q(t-\tau)\, u_0'(\tau)\, d\tau\,. \tag{1.15}$$

Here

$$\begin{cases} u_0(t) = \dfrac{1}{3}\,\mathrm{e}^{-t} + \dfrac{2}{3}\,\mathrm{e}^{t/2}\,\cos\dfrac{\sqrt{3}}{2}\,t\,, \\ u_1(t) = J\,u_0(t) = -\dfrac{1}{3}\,\mathrm{e}^{-t} + \dfrac{2}{3}\,\mathrm{e}^{t/2}\,\cos\left(\dfrac{\sqrt{3}}{2}\,t - \dfrac{\pi}{3}\right)\,, \\ u_2(t) = J\,u_1(t) = J^2\,u_0(t) = \dfrac{1}{3}\,\mathrm{e}^{-t} + \dfrac{2}{3}\,\mathrm{e}^{t/2}\,\cos\left(\dfrac{\sqrt{3}}{2}\,t - 2\dfrac{\pi}{3}\right)\,, \end{cases} \tag{1.16}$$

with the operator $J = J^1$ defined according to $J\,u(t) = \int_0^t u(\tau)\,d\tau$.

We note that in the cases $\alpha = 1$ and $\alpha = 2$ we can use a notation in analogy with the case $\alpha = 3$, namely we write $u_0(t) = \mathrm{e}^{-t}$ and $u_0(t) = \cos t$, $u_1(t) = J\,u_0(t) = \sin t = \cos(t - \pi/2)$, respectively.

Furthermore, we note that as for $\alpha = 3$ [see (1.15)], also for $\alpha = 1$ and $\alpha = 2$ [see (1.11) and (1.13), respectively], the free term $q(t)$ appears convoluted with the function

$$u_\delta(t) := -u_0'(t)\,. \tag{1.17}$$

This function represents the so called *impulse response solution* for the corresponding equation [(1.10), (1.12), (1.14)], namely the particular solution of the inhomogeneous equation with all $c_k \equiv 0$, $k = 0, 1, \ldots, m-1$, $m = 1, 2, 3$ and with $q(t) = \delta(t)$. Here $\delta(t)$ denotes the Dirac delta distribution.

R e m a r k : The more general equation

$$D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\, u^{(k)}(0^+) \right) = -\rho^\alpha\, u(t) + q(t)\,, \quad \rho > 0\,, \quad t > 0\,, \tag{1.18}$$

can be reduced to (1.1) by a change of scale $t \to t/\rho$. We prefer, for ease of notation, to discuss the "dimensionless" form (1.1).

2. Formal solutions of the fractional differential equation

We prefer to write (1.1) with the prescribed initial conditions as an equivalent (fractional) integral equation and then treat the integral equation by the Laplace transform method (see *e.g.* [19] for this method). For this purpose we note that

$$J^\alpha \widetilde{D}^\alpha u(t) = J^\alpha J^{m-\alpha} D^m u(t) = J^m D^m u(t) = u(t) - \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} . \tag{2.1}$$

Applying the operator J^α to both sides of (1.1) we obtain

$$u(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} - J^\alpha u(t) + J^\alpha q(t) , \tag{2.2}$$

Using the sign $\div$ for the juxtaposition of a function depending on t with its Laplace transform depending on s:

$$u(t) \div U(s) , \quad q(t) \div Q(s) ,$$

we have

$$U(s) = \sum_{k=0}^{m-1} \frac{c_k}{s^{k+1}} - \frac{1}{s^\alpha} U(s) + \frac{1}{s^\alpha} Q(s) ,$$

hence

$$U(s) = \sum_{k=0}^{m-1} c_k \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1} Q(s) . \tag{2.3}$$

Introducing the functions

$$e_\alpha(t) \div \frac{s^{\alpha-1}}{s^\alpha + 1} , \tag{2.4}$$

and

$$u_k(t) := J^k e_\alpha(t) \div \frac{s^{\alpha-k-1}}{s^\alpha + 1} , \tag{2.5}$$

we find, from inversion of the Laplace transforms in (2.3),

$$u(t) = \sum_{k=0}^{m-1} c_k u_k(t) - \int_0^t q(t-\tau) u_0'(\tau) d\tau . \tag{2.6}$$

For finding the last term in the R.H.S of (2.6) we have used the well-known rule for the Laplace transform of the derivative noting that $u_0(0) = e_\alpha(0) = 1$ and

$$\frac{1}{s^\alpha + 1} = - \left(s \frac{s^{\alpha-1}}{s^\alpha + 1} - 1 \right) \div -u_0'(t) . \tag{2.7}$$

The formula (2.6) encompasses the solutions (1.11), (1.13) and (1.15) found for $\alpha = 1\,,\,2\,,\,3\,,$ respectively.

When α is not integer, namely for $m-1 < \alpha < m\,,$ we note that $m-1$ represents the integer part of α (usually denoted by $[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. The m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \dots, m-1$ represent those particular solutions which satisfy the following m initial conditions

$$u_k^{(h)}(0^+) = \delta_{h\,k}\,, \quad h = 0, 1, \dots, m-1\,, \tag{2.8}$$

and therefore they are referred to as the *fundamental solutions* of the fractional equation (1.1). In the next section we are going to compute the *fundamental solutions* for $0 < \alpha < 3$ ($\alpha \neq 1, 2$). For completeness of presentation we also exhibit the *impulse response solution*, $u_\delta(t)$.

R e m a r k : The reader is invited to verify that the solution (2.6) has continuous derivatives $u^{(k)}(t)$ for $k = 0, 1, 2, \dots, m-1\,,$ which fulfill the m initial conditions $u^{(k)}(0^+) = c_k$. In fact, looking back at (2.2), one must recognize the smoothing power of the operator J^α. However, the so called *impulse response solution* of our equation (1.1), $u_\delta(t)$, is expected to be not so regular like the ordinary solution (2.6). In fact, from (2.3) and (2.6-7), one obtains

$$u_\delta(t) = -u_0'(t) \div \frac{1}{s^\alpha + 1}\,, \tag{2.9}$$

and therefore, using the limiting theorem for Laplace transforms, one can recognize that, being $m-1 < \alpha < m\,,$

$$u_\delta^{(k)}(0^+) = 0\,,\; k = 0, 1, \dots, m-2\,; \quad u_\delta^{(m-1)}(0^+) = \infty\,. \tag{2.10}$$

3. Explicit solutions of the fractional differential equation

In order to obtain the inversion of the Laplace transform in (2.4) we need to recall the so-called Mittag-Leffler function $E_\alpha(z)$ defined in the whole complex plane as

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}\,, \quad z \in \mathbb{C}\,, \tag{3.1}$$

This function, introduced by Mittag-Leffler at the beginning of this century, turns out to be an entire function of order $\rho = 1/\alpha$, on which the reader may get information from Erdélyi [20] and Dzherbashyan [21].

The Mittag-Leffler function is connected to the Laplace integral through the equation

$$\int_0^\infty \mathrm{e}^{-x}\, E_\alpha\left(x^\alpha\, z\right)\, dx = \frac{1}{1-z}\,, \quad \alpha > 0\,. \tag{3.2}$$

The integral was evaluated by Mittag-Leffler who showed that the region of convergence of (3.2) contains the unit circle and is bounded by the line $\mathrm{Re}\, z^{1/\alpha} = 1$. Putting in (3.2) $x = st$ and $x^\alpha\, z = \lambda t^\alpha$ with $t \geq 0$, and $\lambda \in \mathbb{C}$ we get the following Laplace transform pair

$$E_\alpha\left(\lambda\, t^\alpha\right) \div \frac{s^{\alpha-1}}{s^\alpha - \lambda}\,. \tag{3.3}$$

As a consequence we obtain from (2.4) and (3.3)

$$e_\alpha(t) = E_\alpha(-t^\alpha)\,. \tag{3.4}$$

Formally we can obtain the function $e_\alpha(t)$ in form of a series in powers of t^α by term-wise Laplace inversion of the series

$$\frac{s^{\alpha-1}}{s^\alpha+1} = \sum_{n=0}^{\infty}(-1)^n\,\frac{1}{s^{\alpha n+1}}\,, \quad |s^\alpha| > 1\,,$$

as

$$e_\alpha(t) = \sum_{n=0}^{\infty}(-1)^n\,\frac{t^{\alpha n}}{\Gamma(\alpha n+1)} = E_\alpha(-t^\alpha)\,, \quad t \geq 0\,. \tag{3.5}$$

We here prefer to derive the relevant properties of our functions $e_\alpha(t)$, directly from their representation as a Laplace inversion integral

$$e_\alpha(t) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\,st}\,\frac{s^{\alpha-1}}{s^\alpha+1}\, ds\,, \tag{3.6}$$

in detail for $0 < \alpha \leq 3$, without detour on the general theory of Mittag-Leffler functions in the complex plane. In (3.6) Br denotes the Bromwich path, *i.e.* a line $Re\, s = \sigma$ with a value $\sigma \geq 1$.

The essential step consists in decomposing $e_\alpha(t)$ into two parts according to $e_\alpha(t) = f_\alpha(t) + g_\alpha(t)$, as indicated below. The part f_α, present for α not integer, turns out to be a *completely monotonic* function, which tends to zero as t tends to infinity; the other part turns out to be of *oscillatory* character if $1 < \alpha < 3$, but identically vanishing if $0 < \alpha < 1$. For transparency reasons, we separately discuss the cases:

$$\text{(a)} \quad 0 < \alpha < 1\,, \qquad \text{(b)} \quad 1 < \alpha < 2\,, \qquad \text{(c)} \quad 2 < \alpha < 3\,, \tag{3.7}$$

recalling that for $\alpha = 1, 2, 3$ we know $e_\alpha(t)$ as elementary function, namely

$$e_1(t) = \mathrm{e}^{-t}\,, \quad e_2(t) = \cos t\,, \quad e_3(t) = \frac{1}{3}\,\mathrm{e}^{-t} + \frac{2}{3}\,\mathrm{e}^{t/2}\,\cos\frac{\sqrt{3}}{2}\,t\,. \tag{3.8}$$

For α not integer the power function s^α is uniquely defined as $s^\alpha = |s|^\alpha\,\mathrm{e}^{i\arg s}$, with $-\pi < \arg s < \pi$, that is in the complex s-plane cut along the negative real axis.

In order to obtain the desired decomposition of e_α we bend the Bromwich path of integration Br into the equivalent Hankel path $Ha(1^+)$, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|s| = 1$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis. One obtains in all cases (a), (b), (c),

$$e_\alpha(t) = f_\alpha(t) + g_\alpha(t)\,, \quad t \ge 0\,, \tag{3.9}$$

with

$$f_\alpha(t) := \frac{1}{2\pi i}\int_{Ha(\epsilon)} \mathrm{e}^{\,st}\,\frac{s^{\alpha-1}}{s^\alpha + 1}\,ds\,, \tag{3.10}$$

where now the Hankel path $Ha(\epsilon)$ denotes a loop constituted by a small circle $|s| = \epsilon$ with $\epsilon \to 0$ and by the two borders of the cut negative real axis, and

$$g_\alpha(t) := \sum_h \mathrm{e}^{\,s'_h\,t}\,Res\left[\frac{s^{\alpha-1}}{s^\alpha + 1}\right]_{s'_h}\,, \tag{3.11}$$

where s'_h are the relevant poles of $s^{\alpha-1}/(s^\alpha + 1)$. In fact the poles turn out to be $s_h = \exp\left[i(2h+1)\pi/\alpha\right]$ with unitary modulus; they are all simple but relevant are only those situated in the main Riemann sheet, *i.e.* the poles s'_h for which $-\pi < \arg s'_h < \pi$.

If $0 < \alpha < 1$, there are no such poles, since for all integer h we have $|\arg s_h| = |2h+1|\,\pi/\alpha > \pi$; as a consequence,

$$g_\alpha(t) \equiv 0\,, \quad \text{hence} \quad e_\alpha(t) = f_\alpha(t)\,, \quad \text{if} \quad 0 < \alpha < 1\,. \tag{3.12}$$

If $1 < \alpha < 3$, then there exist precisely two relevant poles, namely $s_0' = \exp(i\pi/\alpha)$ and $s_{-1}' = \exp(-i\pi/\alpha) = \overline{s_0}'$, and one obtains

$$g_\alpha(t) = \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right], \quad \text{if} \quad 1 < \alpha < 3\,. \tag{3.13}$$

We note that this function exhibits oscillations with circular frequency $\omega(\alpha) = \sin(\pi/\alpha)$ and with an amplitude which exponentially decays or grows correspondingly to the case (b) $(1 < \alpha < 2)$ or (c) $(2 < \alpha < 3)$ with the rate $\lambda(\alpha) = |\cos(\pi/\alpha)|$. Of course, in the classical case $\alpha = 2$, we recover the sinusoidal behaviour with unitary frequency, $\omega(2) = 1$.

R e m a r k : If and only if α is an odd integer there is a pole, namely -1, on the negative real axis on which the cut is no longer present.

It is now an exercise in complex analysis to show that the contribution from the Hankel path $Ha(\epsilon)$ as $\epsilon \to 0$ is provided by

$$f_\alpha(t) := \int_0^\infty \mathrm{e}^{-rt}\,K_\alpha(r)\,dr\,, \tag{3.14}$$

with

$$K_\alpha(r) = \frac{1}{\pi}\,\frac{r^{\alpha-1}\,\sin(\alpha\pi)}{r^{2\alpha} + 2\,r^\alpha\,\cos(\alpha\pi) + 1}\,. \tag{3.15}$$

This function $K_\alpha(r)$ vanishes identically if α is an integer, it is positive for all r if $0 < \alpha < 1$ or $2 < \alpha < 3$, negative for all r if $1 < \alpha < 2$. In fact in (3.15) the denominator is, for α not integer, always positive being $> (r^\alpha - 1)^2 \ge 0$. Hence $f_\alpha(t)$ has the aforementioned property of complete monotonicity: it is decreasing towards zero in cases (a) and (c), increasing towards zero in case (b). We also note that, in order to satisfy the initial condition $e_\alpha(0^+) = 1$, we find $\int_0^\infty K_\alpha(r)\,dr = 1$ if $0 < \alpha < 1$, $\int_0^\infty K_\alpha(r)\,dr = 1 - 2/\alpha$ if $1 < \alpha < 3$.

In order to compile all the *fundamental solutions* for $0 < \alpha < 3$ with $\alpha \ne 1, 2$ [cases (a), (b), (c)], we also need to compute the associated functions $J^k e_\alpha(t)$ for $k = 1, 2$, precisely $J^1\,e_\alpha(t)$ if $1 < \alpha < 3$, and $J^2\,e_\alpha(t)$ if $2 < \alpha < 3$. This can be done in direct analogy to the computation of the functions $e_\alpha(t)$, the Laplace transform of $J^k e_\alpha(t)$ being given by (2.5). We obtain

$$J^k e_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt}\,K_{\alpha,k}(r)\,dr + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right) - k\frac{\pi}{\alpha}\right]. \tag{3.16}$$

Here

$$K_{\alpha,k}(r) := (-1)^k\,r^{-k}\,K_\alpha(r) = \frac{(-1)^k}{\pi}\,\frac{r^{\alpha-1-k}\,\sin(\alpha\pi)}{r^{2\alpha} + 2\,r^\alpha\,\cos(\alpha\pi) + 1}\,, \tag{3.17}$$

with $K_\alpha(r) = K_{\alpha,0}(r)$, can be referred to as the *spectral function* of the corresponding solution $J^k\, e_\alpha(t)$. This function turns out to be absolutely integrable (locally summable) in $\mathbb{R}^+$ for any permissible integer $k \ge 0$, with

$$\int_0^\infty K_{\alpha,k}(r)\, dr = \begin{cases} 1\,, & \text{if} \quad 0<\alpha<1\,,\ k=0\,,\\ 1-\dfrac{2}{\alpha}\,, & \text{if} \quad 1<\alpha<3\,,\ k=0\,,\\ -\dfrac{2}{\alpha}\,\cos\left(k\dfrac{\pi}{\alpha}\right)\,, & \text{if} \quad 1<\alpha<3\,,\ 1\le k\le m-1=[\alpha]\,, \end{cases} \tag{3.18}$$

in order to satisfy the initial conditions $e_\alpha(0^+) = 1$ and $J^k e_\alpha(t) \to 0$ as $t \to 0^+$, where $1 \le k \le m-1 < \alpha \le m$.

For the *impulse response solution* $u_\delta(t) = -u_0'(t) = -De_\alpha(t)$ we similarly obtain

$$D\, e_\alpha(t) = \int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)+\frac{\pi}{\alpha}\right]\,. \tag{3.19}$$

In Fig 1 we show the plots of the basic *spectral functions* $K_\alpha(r)$ for some values of α in the intervals (a) $0<\alpha<1$, (b) $1<\alpha<2$, (c) $2<\alpha<3$.

We resume the *fundamental solutions* $u_k(t)$ [including $u_\delta(t)$] as follows:

(a) $0<\alpha<1$,

$$u(t) = c_0\, u_0(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{3.20a}$$

where

$$\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,0}(r)\, dr\,,\\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr\,, \end{cases} \tag{3.21a}$$

with

$$\begin{cases} u_0(0^+) = 1\,,\\ u_\delta(0^+) = \infty\,; \end{cases} \tag{3.22a}$$

(b) $1<\alpha<2$,

$$u(t) = c_0\, u_0(t) + c_1\, u_1(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{3.20b}$$

where

$$\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,0}(r)\, dr + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right]\,,\\ u_1(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,1}(r)\, dr + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)-\frac{\pi}{\alpha}\right]\,,\\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr - \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)+\frac{\pi}{\alpha}\right]\,, \end{cases} \tag{3.21b}$$

with

$$
\begin{cases} u_0(0^+) = 1\,,\ u_0'(0^+) = 0\,, \\ u_1(0^+) = 0\,,\ u_1'(0^+) = 1\,, \\ u_\delta(0^+) = 0\,,\ u_\delta'(0^+) = +\infty\,; \end{cases} \tag{3.22b}
$$

(c) $2 < \alpha < 3\,,$

$$
u(t) = c_0\, u_0(t) + c_1\, u_1(t) + c_2\, u_2(t) + \int_0^t q(t-\tau)\, u_\delta(\tau)\, d\tau\,, \tag{3.20c}
$$

where

$$
\begin{cases} u_0(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,0}(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right]\,, \\ u_1(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,1}(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right) - \frac{\pi}{\alpha}\right]\,, \\ u_2(t) = \displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,2}(r)\, dr + \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right) - 2\frac{\pi}{\alpha}\right]\,, \\ u_\delta(t) = -\displaystyle\int_0^\infty \mathrm{e}^{-rt}\, K_{\alpha,-1}(r)\, dr - \frac{2}{\alpha}\, \mathrm{e}^{t\,\cos(\pi/\alpha)}\, \cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right) + \frac{\pi}{\alpha}\right]\,, \end{cases} \tag{3.21c}
$$

with

$$
\begin{cases} u_0(0^+) = 1\,,\ u_0'(0^+) = 0\,,\ u_0''(0^+) = 0\,, \\ u_1(0^+) = 0\,,\ u_1'(0^+) = 1\,,\ u_1''(0^+) = 0\,, \\ u_2(0^+) = 0\,,\ u_2'(0^+) = 0\,,\ u_2''(0^+) = 1\,, \\ u_\delta(0^+) = 0\,,\ u_\delta'(0^+) = 0\,,\ u_\delta''(0^+) = +\infty\,. \end{cases} \tag{3.22c}
$$

We recall that in [3] plots of the solution $u_0(t)$, which we agree to refer as the *basic fundamental solution*, are reported for some values of α. In Fig. 2 we quote the plots of this solution for the following cases : (a) $\alpha = 0.25\,, 0.50\,, 0.75\,, 1\,,$ and (b) $\alpha = 1.25\,, 1.50\,, 1.75\,, 2\,,$ obtained from the first formula in (3.21a) and (3.21b), respectively. We have verified that our present results confirm those obtained in [3] by a numerical treatment and those obtained in [2] by an analytical treatment, valid when α is a rational number.

For plotting we have not considered the case (c) since the corresponding solutions exhibit an exponential growth, and therefore are unstable. For the cases (a) (b) with α not integer, *i.e.* for the cases of *fractional relaxation* and *fractional oscillation*, we note an algebraic decay as $t \to \infty$, that will be investigated in the next section.

In Fig. 3 we show some plots of the *basic fundamental solution* $u_0(t) = e_\alpha(t)$ for $\alpha = 1.25\,, 1.50\,, 1.75\,.$ Here the algebraic decay of the fractional oscillation can be recognized and compared with the two contributions provided by f_α (monotonic behaviour) and $g_\alpha(t)$ (exponentially damped oscillation).

4. The decay of the fractional relaxations and fractional oscillations

We are going to describe the asymptotic behaviour as $t \to \infty$ of the fundamental solutions for the cases (a) and (b), namely of $u_0(t) = e_\alpha(t)$ and for $0 < \alpha < 1$, and $u_0(t) = e_\alpha(t)$, $u_1(t) = J^1 e_\alpha(t)$ for $1 < \alpha < 2$. For completeness we consider the asymptotic behaviour of the corresponding impulse response solution $u_\delta(t)$.

Furthermore, for the case (b), we try to get some information about the position and the number of the zeros of the fundamental solution u_0, with special regard to the limiting cases $\alpha \to 1^+$ and $\alpha \to 2^-$.

Let us start with the asymptotic behaviour of $u_0(t)$. To this purpose we first derive an asymptotic series for the function $f_\alpha(t)$, valid for $t \to \infty$. Using the identity

$$\frac{1}{s^\alpha + 1} = 1 - s^\alpha + s^{2\alpha} - s^{3\alpha} + \ldots + (-1)^{N-1} s^{(N-1)\alpha} + (-1)^N \frac{s^{N\alpha}}{s^\alpha + 1}, \qquad (4.1)$$

in formula (3.10) and the Hankel representation of the reciprocal Gamma function, we (formally) obtain the following asymptotic expansion (for α non integer)

$$f_\alpha(t) = \sum_{n=1}^{N} (-1)^{n-1} \frac{t^{-n\alpha}}{\Gamma(1 - n\alpha)} + O\left(t^{-(N+1)\alpha}\right), \quad \text{as} \quad t \to \infty. \qquad (4.2)$$

The validity of this asymptotic expansion can be established rigorously using the generalized Watson lemma (see Bleistein and Handelsman [22]). We note that this asymptotic expansion coincides with that for $u_0(t) = e_\alpha(t)$, having assumed $0 < \alpha < 2$ $(\alpha \neq 1)$. In fact the contribution of $g_\alpha(t)$ is identically zero if $0 < \alpha < 1$ and exponentially small as $t \to \infty$ if $1 < \alpha < 2$. The asymptotic expansion of the solutions $u_1(t)$ and $u_\delta(t)$ are obtained from (4.2) integrating or differentiating term by term with respect to t.

In particular, taking the leading term in (4.2), we obtain the following asymptotic representations as $t \to \infty$,

$$\begin{cases} u_0(t) \sim \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}, & (a) \quad (b) \\ u_\delta(t) \sim -\dfrac{t^{-\alpha-1}}{\Gamma(-\alpha)}, & (a) \quad (b) \\ u_1(t) \sim \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}, & (b) \end{cases} \qquad (4.3)$$

that point out the algebraic decay of the fundamental and impulse response solutions.

R e m a r k : The asymptotic series (4.2) can also be derived from the existing formula in literature for $E_\alpha(z)$ when z is real negative and $0 < \alpha < 2$ $(\alpha \neq 1)$, see *e.g.* Erdélyi [20].

Now we find it interesting to carry out some considerations about the zeros of $e_\alpha(t)$ in the case (b), namely about the solutions of the following equation

$$e_\alpha(t) = f_\alpha(t) + \frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)}\,\cos\left[t\,\sin\left(\frac{\pi}{\alpha}\right)\right] = 0\,. \tag{4.4}$$

Of course, in the case (a) $e_\alpha(t)$ does not exhibit any zero, while in the case (c) it exhibits infinitely many zeros since the oscillating part $g_\alpha(t)$ dominates by exponential increase of amplitude the decaying part $f_\alpha(t)$.

We first note that the function $e_\alpha(t)$ exhibits an *odd* number of zeros, in that $e_\alpha(0) = 1$, and, for sufficiently large t, $e_\alpha(t)$ turns out to be permanently negative, as shown in (4.3) by the sign of $\Gamma(1-\alpha)$. The smallest zero lies in the first positivity interval of $\cos[t\sin(\pi/\alpha)]$, hence in the interval $0 < t < \pi/[2\,\sin(\pi/\alpha)]$; all other zeros can only lie in the succeeding positivity intervals of $\cos[t\,\sin(\pi/\alpha)]$, in each of these two zeros are present as long as

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)} \geq |f_\alpha(t)|\,. \tag{4.5}$$

When t is sufficiently large the zeros are expected to be found approximately from the equation

$$\frac{2}{\alpha}\,\mathrm{e}^{t\,\cos(\pi/\alpha)} \approx \frac{t^{-\alpha}}{|\Gamma(1-\alpha)|}\,, \tag{4.6}$$

obtained from (4.4) neglecting the oscillation factor of $g_\alpha(t)$ [see (3.13)] and taking the first term in the asymptotic expansion of $f_\alpha(t)$ [see (4.2-3)]. As we shall see, such an approximate equation turns out to be useful when $\alpha \to 1^+$ and $\alpha \to 2^-$.

For $\alpha \to 1^+$, only one zero is present, which is expected to be very far from the origin in view of the large period of the function $\cos[t\,\sin(\pi/\alpha)]$. In fact, since there is no zero for $\alpha = 1$, and by increasing α more and more zeros arise, we are sure that only one zero exists for α sufficiently close to 1. Putting now $\alpha = 1+\epsilon$ the asymptotic position T_* of this zero can be found from the relation (4.6) in the limit $\epsilon \to 0^+$. Assuming in this limit the first-order approximation, we get

$$\cos(\pi/\alpha) = \cos[\pi/(1+\epsilon)] \sim \cos[\pi(1-\epsilon)] \sim -1\,,$$

and

$$\Gamma(1-\alpha) = \Gamma(-\epsilon) \sim -1/\epsilon\,.$$

Therefore

$$\mathrm{e}^{-T_*} \sim \frac{\epsilon(1+\epsilon)}{2}\,T_*^{-1-\epsilon}\,,$$

from which

$$T_* \sim \log\left(\frac{2}{\epsilon}\right) + \log T_*\,. \tag{4.7}$$

Neglecting $\log T_*$ with respect to T_* we obtain

$$T_* \sim \log\left(\frac{2}{\epsilon}\right), \tag{4.8a}$$

which shows that T_* tends to infinity slower than $1/\epsilon$, as $\epsilon \to 0$. Reversing this asymptotic expression we obtain

$$\epsilon \sim 2\,\mathrm{e}^{-T_*}, \tag{4.8b}$$

which shows that ϵ tends to zero exponentially as $T_* \to \infty$.

For $\alpha \to 2^-$, there is an increasing number of zeros up to infinity since $e_2(t) = \cos t$ has infinitely many zeros [in $t_n^* = (n+1/2)\pi$, $n = 0, 1, \ldots$]. We now are going to give an asymptotic estimate for the largest zero T_* (and for the total number N_* of zeros) when $\alpha = 2 - \delta$ in the limit $\delta \to 0^+$. Assuming in this limit the first-order approximation, we get

$$\cos(\pi/\alpha) = \cos[\pi/(2-\delta)] \sim \cos[(\pi/2)(1+\delta/2)] = \sin(\pi\delta/4) \sim \pi\delta/4,$$

and

$$\Gamma(1-\alpha) = \Gamma(-1+\delta) \sim -1/\delta.$$

Therefore

$$\mathrm{e}^{-\pi\delta T_*/4} \sim \frac{(2-\delta)\delta}{2}\, T_*^{-2+\delta},$$

from which

$$\frac{\pi\delta T_*}{4} \sim \log\left(\frac{1}{\delta}\right) + 2\log T_*. \tag{4.9}$$

Since both the two terms in the R.H.S. of (4.9) diverge, $\delta\, T_* \to \infty$, so that T_* tends to infinity faster than $1/\delta$ as $\delta \to 0$, or, equivalently, that δ tends to 0 slower than $1/T_*$ as $T_* \to \infty$. As a matter of fact both terms turn out to be of the same order as we recognize putting in the L.H.S of (4.9) either $T_* \sim (A/\delta)\log\,(1/\delta)$ or $\delta \sim B\log\,(T_*)/T_*$, where a and B are positive constants to be determined. We easily obtain $A = B = 12/\pi$, namely the following equivalent asymptotic expressions

$$T_* \sim \frac{12}{\pi\,\delta}\log\left(\frac{1}{\delta}\right), \tag{4.10a}$$

$$\delta \sim \frac{12}{\pi}\,\frac{\log T_*}{T_*}. \tag{4.10b}$$

Now, for $\delta \to 0^+$ the length of the positivity intervals of $g_\alpha(t)$ tends to π and, as long as $t \le T_*$, there are two zeros in each positivity interval. Hence, in the limit $\delta \to 0^+$, there is in average one zero per interval of length π, so we expect that $N_* \sim T_*/\pi$.

R e m a r k : For the above considerations we got inspiration from an interesting paper by Wiman [23] who, after having treated the Mittag-Leffler function in the complex plane, considered the position of the zeros of the function on the negative real axis (without providing any detail). While our expression of T_* for $\alpha = 1+\epsilon$ is in agreement with that by Wiman (possibly, unless of a numerical factor!), that of T_* for $\alpha = 2-\delta$ is completely different. In fact, using our notation, Wiman's expression reads

$$\alpha = 2-\delta\,, \qquad T_* \sim \frac{4}{\pi\,\delta}\log\left(\frac{1}{\delta}\right)\,. \tag{4.11}$$

The discrepancy between Wiman's and our results is (presumably) due to the fact that Wiman (arbitrarily) neglected $\log T_*$ with respect to $\log{(1/\delta)}$ in (4.9).

The comparison with the numerical results has proven the validity of our analysis, as shown in the plots of Figures 4 and 5, where we show the position of the unique zero when $1 < \alpha \le 1.4$ and the position of the last zero as $1.8 \le \alpha < 2$, respectively.

Furthermore, for completeness we find it interesting to analyse the the phenomenon of the transition of the (odd) number of zeros as $1.4 \le \alpha \le 1.8$. For this purpose, in Table I we report the intervals of amplitute $\Delta\alpha = 0.01$ where these transitions occur, and the location T_* (evaluated within a relative error of 0.1%) of the largest zeros found at the two extreme values of the above intervals.

We recognize that the transition from 1 to 3 zeros occurs as $1.40 \le \alpha \le 1.41$, that one from 3 to 5 zeros occurs as $1.56 \le \alpha \le 1.57$, and so on. The last transition in the considered range of α is from 15 to 17 zeros, and it just occurs as $1.79 \le \alpha \le 1.80$.

N_*	α	T_*
$1 \div 3$	$1.40 \div 1.41$	$1.730 \div 5.726$
$3 \div 5$	$1.56 \div 1.57$	$8.366 \div 13.48$
$5 \div 7$	$1.64 \div 1.65$	$14.61 \div 20.00$
$7 \div 9$	$1.69 \div 1.70$	$20.80 \div 26.33$
$9 \div 11$	$1.72 \div 1.73$	$27.03 \div 32.83$
$11 \div 13$	$1.75 \div 1.76$	$33.11 \div 38.81$
$13 \div 15$	$1.78 \div 1.79$	$39.49 \div 45.51$
$15 \div 17$	$1.79 \div 1.80$	$45.51 \div 51.46$

Table I

N_* = number of zeros, α = fractional order, T_* location of the largest zero.

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