The Daugavet equation for operators not fixing a copy of $C(S)$

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Abstract. We prove the norm identity $\|Id + T\| = 1 + \|T\|$, which is known as the Daugavet equation, for operators T on $C(S)$ not fixing a copy of $C(S)$, where S is a compact metric space without isolated points.

1. Introduction

An operator $T: X \to X$ on a Banach space is said to satisfy the *Daugavet equation* if

$$\|Id + T\| = 1 + \|T\|; \tag{1.1}$$

this terminology is derived from Daugavet's theorem that a compact operator on $C[0, 1]$ satisfies (1.1). Many authors have established the Daugavet equation for various classes of operators, e.g., the weakly compact ones, on various spaces; we refer to [1], [2], [4], [7], [10], [11], [13] and the references in these papers for more information.

The most far-reaching result for operators on $L^1[0, 1]$ is due to Plichko and Popov [7, Th. 9.2 and Th. 9.8] who prove that an operator on $L^1[0, 1]$ which does not fix a copy of $L^1[0, 1]$ satisfies the Daugavet equation. (As usual, $T: X \to X$ fixes a copy of a Banach space E if there is a subspace $F \subset X$ isomorphic to E such that $T|_F$ is an (into-) isomorphism.) In this paper we shall establish the corresponding result for operators on spaces of continuous functions.

Theorem 1.1 *Let (S, d) be a compact metric space without isolated points. If $T: C(S) \to C(S)$ does not fix a copy of $C(S)$, then T satisfies the Daugavet equation.*

In the next section we shall provide a direct proof of this theorem which is measure theoretic in spirit, and in Section 3 we shall give another argument which relies on a deep result due to Rosenthal [8]. Though this proof is much shorter it is less revealing than the one in Section 2; therefore we found it worthwhile to present both arguments.

Either of our approaches depends on the analysis of the representing kernel of an operator $T: C(S) \to C(S)$; this is the family of Borel measures on S defined by $\mu_s = T^*\delta_s$, $s \in S$. In [10] (see also [11]) a necessary and sufficient condition on the kernel of T was established in order that T satisfy the Daugavet equation. Here we record a special case which suffices for our needs.

Lemma 1.2 *If the representing kernel $(\mu_s)_{s\in S}$ of a bounded linear operator $T: C(S) \to C(S)$ satisfies*

$$\inf_{s\in U} |\mu_s(\{s\})| = 0 \tag{1.2}$$

for all nonvoid open subsets U of S, then T satisfies the Daugavet equation.

Our results are valid for real and complex spaces; we denote the scalar field by $\mathbb{K}$.

2. Proof of Theorem 1.1

We fix a diffuse probability measure λ on S whose support is S. To show the existence of such a measure, consider a countable basis of open sets $O_n \subset S$. Since each O_n is perfect, we know from a theorem of Bessaga and Pełczyński (see [6, p. 52]) that there are diffuse probability measures λ_n with $\operatorname{supp} \lambda_n \subset O_n$; it remains to define $\lambda = \sum 2^{-n}\lambda_n$.

Now let $T: C(S) \to C(S)$ be a bounded linear operator and $(\mu_s)_{s\in S}$ its representing kernel. It is enough to show that T fixes a copy of $C(S)$ if (1.2) of Lemma 1.2 fails for some open set U. Thus we assume, for some open $U \subset S$ and $\alpha > 0$, that

$$|\mu_s(\{s\})| \geq \alpha \qquad \forall s \in U. \tag{2.1}$$

We shall also assume without loss of generality that $\|T\| = 1$.

We shall decompose the μ_s into their atomic and diffuse parts. It is instrumental for our argument that this can be accomplished in a measurable fashion, as was proved by Kalton [5, Th. 2.10]. More precisely we can write

$$\mu_s = \sum_{n=1}^{\infty} a_n(s)\delta_{\sigma_n(s)} + \nu_s$$

where

- each $a_n\colon S \to \mathbb{K}$ is measurable for the completion Σ_λ of the Borel sets of S with respect to λ,
- each $\sigma_n\colon S \to S$ is Σ_λ-Borel-measurable,
- each ν_s is diffuse, and $s \mapsto |\nu_s|$ is Σ_λ-Borel-measurable (we are considering the weak* Borel sets of the unit ball of $C(S)^*$),
- $|a_1(s)| \geq |a_2(s)| \geq \ldots$ for all $s \in S$,
- $\sigma_n(s) \neq \sigma_m(s)$ for all $s \in S$ whenever $n \neq m$,
- $\sum_{n=1}^{\infty} |a_n(s)| \leq 1$ for all $s \in S$.

Put $\beta = \lambda(U) > 0$. Applying the Egorov and Lusin theorems, we may find a compact subset $S_1 \subset S$ with $\lambda(S_1) \geq 1 - \beta/2$ such that

- each $a_n|_{S_1}$ is continuous on S_1,
- each $\sigma_n|_{S_1}$ is continuous on S_1,
- $\sum |a_n(s)|$ converges uniformly on S_1,
- $\lim\limits_{n\to\infty} \sup\limits_{s \in S_1} |\nu_s|(B(t, 1/n)) \to 0$ for all $t \in S$; (2.2)

here $B(t,r)$ denotes the closed ball with centre t and radius r. To prove (2.2) fix a countable dense subset Q of S. Let $\mathbf{U}_n$ be the countable set of all finite covers of S by means of open balls of radius $1/n$ with centres in Q. Then

$$\varphi_n(s) = \inf_{\mathcal{C} \in \mathbf{U}_n} \sup_{B \in \mathcal{C}} |\nu_s|(B)$$

defines a sequence of Σ_λ-measurable functions, and we have $\varphi_n \to 0$ pointwise, since the ν_s are diffuse and S is compact. By Egorov's theorem (φ_n) tends to 0 uniformly on a large subset of S. This easily implies our claim.

After discarding relatively isolated points of S_1, if necessary, we come up with a perfect subset S_1 with the above properties; note that $U \cap S_1 \neq \emptyset$ since this intersection has positive λ-measure. Next we pick $N \in \mathbb{N}$ such that

$$\sum_{n>N} |a_n(s)| \leq \alpha/3 \qquad \forall s \in S_1. \tag{2.3}$$

Let $s \in U \cap S_1$. Then

$$\alpha \leq |\mu_s(\{s\})| = \left| \sum_{n=1}^{\infty} a_n(s) \delta_{\sigma_n(s)}(\{s\}) \right|.$$

Consequently $\sigma_k(s) = s$ for some k; in addition, we must have

$$|a_k(s)| \geq \alpha \tag{2.4}$$

for this k so that $k \leq N$. This shows that

$$U \cap S_1 = \bigcup_{l=1}^{N} \{s \in U \cap S_1 \colon \sigma_l(s) = s\}.$$

Each of these N sets is relatively closed in $U \cap S_1$, so one of them, with index k say, contains a relatively interior point s_0. Therefore, there exists some $\delta > 0$ with the following properties:

For $B_l = \sigma_l(S_1 \cap B(s_0, \delta))$, $l = 1, \ldots, N$, we have

(a) $B_k = S_1 \cap B(s_0, \delta) \subset S_1 \cap U$, and $\sigma_k(s) = s$ for all $s \in B_k$,

(b) $B_l \cap B(s_0, \delta) = \emptyset$ for $l \neq k$ (since $s_0 = \sigma_k(s_0) \neq \sigma_l(s_0)$ and these functions are continuous on S_1),

(c) $|\nu_s|(B(s_0, \delta)) \leq \alpha/3$ for all $s \in S_1$ (see (2.2)).

Now let $V = S_1 \cap B(s_0, \delta/2)$. This is a compact perfect subset of S and thus $C(V)$ is isomorphic to $C(S)$ by Milutin's theorem [12, Th. III.D.19]. Also let $W = \{s \in S \colon d(s, s_0) \geq \delta\}$. Again, this is a compact subset of S, $V \cap W = \emptyset$, and we have from (a) and (b) above

(a′) $\sigma_k(s) = s$ for all $s \in V$,

(b′) $B_l \subset W$ for all $l = 1, \ldots, N$, $l \neq k$.

By the Borsuk-Dugundji theorem (see e.g. [12, Cor. III.D.17]) there exists an isometric linear extension operator $L \colon C(V \cup W) \to C(S)$. Let $E = \{f \in C(V \cup W) \colon f|_W = 0\}$ and $F = L(E)$; thus $F \cong E \cong C(V)$, and F is isomorphic to $C(S)$.

We finally prove that $T|_F$ is an isomorphism. In fact, let us show that

$$\|Tf\| \geq \frac{\alpha}{3} \|f\| \qquad \forall f \in F.$$

Suppose $f \in F$ with $\|f\| = 1$. Then $\|f|_V\| = 1$ and $f|_W = 0$. Pick $s \in V$ with $|f(s)| = 1$. Then we have (recall that $V \subset S_1 \cap U$)

$$\begin{aligned} \|Tf\| &\geq |Tf(s)| \\ &= \left| \sum_{n=1}^{\infty} a_n(s) f(\sigma_n(s)) + \int_S f \, d\nu_s \right| \\ &\geq |a_k(s)| - \sum_{n>N} |a_n(s)| - \int_S |f| \, d|\nu_s| \\ &\qquad \text{(by (a$'$) and (b$'$))} \\ &\geq \alpha - \frac{\alpha}{3} - |\nu_s|(B(s_0, \delta)) \\ &\qquad \text{(by (2.4), (2.3) and since } f|_W = 0) \\ &\geq \frac{\alpha}{3} \qquad \text{(by (c)).} \end{aligned}$$

This completes the proof of the theorem. □

3. Operators whose adjoints have separable ranges

In this section we discuss another, formally shorter, approach to Theorem 1.1 via the following proposition.

Proposition 3.1 *Suppose S is a compact Hausdorff space without isolated points. If $T: C(S) \to C(S)$ is a bounded linear operator such that* $\operatorname{ran}(T^*)$ *is separable, then T satisfies the Daugavet equation.*

Proof. Let $(\mu_s)_{s \in S}$ be the representing kernel of T, and let $\{\mu_{s_n}: n \in \mathbb{N}\}$ be dense in $\{\mu_s: s \in S\}$. We define $\mu = \sum 2^{-n} |\mu_{s_n}|$; then all the μ_s are absolutely continuous with respect to μ. Hence $\mu_s(\{s\}) \neq 0$ only if $|\mu|(\{s\}) \neq 0$. Therefore, $\{s \in S: \mu_s(\{s\}) \neq 0\}$ is countable, and, since S is perfect, its complement is dense by Baire's theorem. This implies that the condition of Lemma 1.2 is satisfied, and the proposition is proved. □

Proposition 3.1 obviously covers the case of compact operators, operators factoring through a space with a separable dual, and, in the case of metric S, weakly compact operators; for the latter observe that $T^{**}(C(S)^{**}) \subset C(S)$ so that T^{**} and, consequently, T^* have separable ranges. But even for non-metrizable perfect S the Daugavet equation for weakly compact operators on $C(S)$ can be derived from the argument of Proposition 3.1; one only has to recall that by a theorem of Bartle, Dunford and Schwartz [3, p. 306] also in this case all the μ_s are absolutely continuous with respect to some finite measure μ. This seems to yield the easiest proof of the Daugavet equation for these classes of operators.

If S is a metric space then Proposition 3.1 clearly follows from Theorem 1.1. On the other hand, Rosenthal [8] (see also [9]) has shown that an operator $T: C(S) \to C(S)$, S an uncountable compact metric space, with $\operatorname{ran}(T^*)$ nonseparable fixes a copy of $C(S)$. Therefore Proposition 3.1 implies Theorem 1.1 so that both these results are in fact equivalent. Moreover we note that in this setting T does not fix a copy of $C(S)$ if and only if T does not fix a copy of ℓ^1 if and only if T is weakly sequentially precompact, i.e., if (f_n) is a bounded sequence in $C(S)$, then (Tf_n) admits a weak Cauchy subsequence.

We finally extend Proposition 3.1 to the class of nicely embedded Banach spaces introduced in [11] so that we obtain a unified approach to several results of that paper.

Let S be a Hausdorff topological space, and let $C^b(S)$ be the sup-normed Banach space of all bounded continuous scalar-valued functions. The functional $f \mapsto f(s)$ on $C^b(S)$ is denoted by δ_s. We say that a linear map $J: X \to C^b(S)$ on a Banach space X is a *nice embedding* and that X is *nicely embedded* into $C^b(S)$ if J is an isometry such that for all $s \in S$ the following properties hold:

(N1) *For* $p_s := J^*(\delta_s) \in X^*$ *we have* $\|p_s\| = 1$.
(N2) $\operatorname{lin}\{p_s\}$ *is an* L*-summand in* X^*.

The latter condition means that there are projections Π_s from X^* onto $\operatorname{lin}\{p_s\}$ such that

$$\|x^*\| = \|\Pi_s(x^*)\| + \|x^* - \Pi_s(x^*)\| \qquad \forall x^* \in X^*.$$

We will also need the equivalence relation

$$s \sim t \ \text{ if and only if } \ \Pi_s = \Pi_t$$

on S. Then s and t are equivalent if and only if p_s and p_t are linearly dependent, which implies by (N1) that $p_t = \lambda p_s$ for some scalar of modulus 1. The equivalence classes of this relation are obviously closed.

We will consider the following nondiscreteness condition.

(N3) *None of the equivalence classes* $Q_s = \{t \in S: s \sim t\}$ *contains an interior point.*

If the set $\{p_s\colon s \in S\}$ is linearly independent, this simply means:

(N3′) *S does not contain an isolated point.*

By (N2), the p_s are linearly independent as soon as they are pairwise linearly independent.

Proposition 3.2 *Let S be a Baire topological Hausdorff space and suppose that X is nicely embedded into $C^b(S)$ so that additionally* (N3) *holds. If T: $X \to X$ is a bounded linear operator with* $\operatorname{ran}(T^*)$ *separable, then T satisfies the Daugavet equation.*

Proof. We stick to the above notation and put $q_s = T^*(p_s)$. By [11, Prop. 2.1] it is sufficient to prove that

$$S' = \{t \in S\colon\ \Pi_t(q_s) = 0\ \forall s\}$$

is dense in S. If $\{q_{s_n}\colon n \in \mathbb{N}\}$ denotes a dense subset of $\{q_s\colon s \in S\}$, then $t \notin S'$ if and only if $\Pi_t(q_{s_n}) \neq 0$ for some n. This implies that $S \setminus S'$ consists of at most countably many equivalence classes for the equivalence relation $\sim$ (cf. [11, Lemma 2.3]), and by (N3) and Baire's theorem S' must be dense. $\square$

It is proved in [11] that the following classes of Banach spaces satisfy the assumptions of Proposition 3.2:

- X is a function algebra whose Choquet boundary does not contain isolated points,
- X is a real L^1-predual space so that the set $\operatorname{ex} B_{X^*}$ of extreme functionals does not contain isolated points (for the weak* topology),
- X is a complex L^1-predual space for which the quotient space $\operatorname{ex} B_{X^*}/{\sim}$ does not contain isolated points, where $\sim$ means linear dependence,
- X is a space of type C_Λ for certain subsets Λ of abelian discrete groups.

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