Approximation of accumulation points of sequences of continuous functions on Polish spaces by subsequences

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Abstract. It is known that accumulation points of a pointwise bounded sequence $(f_n)_n$ of continuous functions on a Polish space are pointwise limits of subsequences if all accumulation points are Borel.

Here we present a quantitative variant of this result: For $\epsilon_0 \geq 0$ and any accumulation point f there is a subsequence $(f_{n_k})_k$ with $\limsup_k |f(x) - f_{n_k}(x)| \leq \epsilon_0$ for all x provided that every accumulation point is $\epsilon_0/2$-close to a Borel function.

We will indicate the rôle of Martin's axiom if one wants to decide whether there is a best possible ϵ_0 with the above property.

keywords: Polish space, Martin's axiom

1. Introduction

Let $(T, \mathcal{T})$ be a Polish space, i.e., $\mathcal{T}$ is generated by a metric with respect to which T is separable and complete. Further, let $(f_n)_{n\in\mathbb{N}}$ be a pointwise bounded sequence of continuous real-valued functions on T and f an accumulation point with respect to pointwise convergence of the (f_n). Several authors have considered the problem under which conditions f is in fact the limit of a subsequence: This is true if every accumulation point of the (f_n) is Borel measurable; see [7], [4].

One of the many interesting consequences of this fact is the possibility to characterize the Banach spaces X which contain isomorphic copies of l^1; see chapter III in [5] and the references cited there.

In [1], [2] and [3] the author has investigated quantitative aspects of such questions: In Rosenthal's l^1-theorem one can relate the Banach-Mazur distance to l^1 of subspaces generated by subsequences of a given sequence with the convergence properties (with respect to the weak topology) of subsequences. Here we aim at treating a quantitative variant of the abovementioned question concerning accumulation points: Given $\epsilon_0 \geq 0$, is it true that there is a subsequence (f_{n_k}) such that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ for every x?

It will be shown that this holds provided that every accumulation point of the (f_n) is $(\epsilon_0/2)$-close to a Borel function. Our methods of proof are similar to those in chapter III of [5].

The relation between ϵ_0-approximations for all $\epsilon_0 > 0$ and the case $\epsilon_0 = 0$ is also discussed. We will indicate how Martin's axiom comes into play when such questions are to be investigated. Under the assumption of this axiom the following will be shown: Let T be a set with $\mathrm{card}(T) < 2^{\aleph_0}$, and (f_n) a sequence of real-valued functions on T. If for every $\epsilon_0 > 0$ there is a subsequence (f_{n_k}) with $\limsup_k |f_{n_k}(x)| \leq \epsilon_0$ for every x, then (f_n) contains a subsequence which converges pointwise to zero.

The paper is organized as follows. *Section 2* contains preliminary material on non-Borel functions and on the length of descending chains of closed sets in a topological space. In *section 3* we derive the main result on the approximation of accumulation points by subsequences. Finally, in *section 4*, we discuss how approximations in the discrete case lead to Martin's axiom.

2. Preliminaries

Whereas several techniques are known to show that a certain function is a Borel function there seem to exist only few methods to prove that a given f is *not* Borel. Our first lemma provides an abundance of such functions, it will play a crucial rôle in section 3.

Let M be the set $\{0,1\}^{\mathbb{N}}$ together with the product topology. M can be identified with the Cantor set, we will prefer, however, to regard M as the collection of all subsets of $\mathbb{N}$. For $n \in \mathbb{N}$, let $h_n : M \longrightarrow \{0,1\}$ be the natural projection onto the n'th coordinate; according to the respective interpretation the h_n correspond to the Rademacher functions on the Cantor set or the Dirac measures on the subsets of the integers.

Consider now the functions h in the closure of the family (h_n) (closure with respect to pointwise convergence). It is an easy exercise to show that $h \mapsto h^{-1}(1)$ defines a bijection between these h and the ultrafilters on $\mathbb{N}$, where the h which are not in $\{h_n \mid n \in \mathbb{N}\}$, i. e., the accumulation points, correspond to the free ultrafilters.

Lemma 2.1 *Let h be an accumulation point of the (h_n). Then h is not a Borel function. In fact there is not even a Borel function g such that $\|g - h\|_\infty < 0.5$.*

Proof. At first we prove that h is not Borel: We assume the contrary and it will be shown that this leads to a contradiction.

We provide M with the product measure P with respect to the uniform distribution on the factors, and we regard M as a topological group. Since every h_n is a group homomorphism and satisfies $h_n(x) + h_n(1-x) = 1$ for $x \in M$ the same properties hold for the pointwise accumulation point h.

By assumption h is Borel so that $A := h^{-1}(1)$ is measurable. Since $h(x) + h(1-x) = 1$, the set M is the disjoint union of A and $1 - A$, and therefore $P(A) = 0.5$ (note that P is translation invariant).

On the other hand it is straightforward to show that A is a tail event with respect to the functions $h_1, h_2, \ldots$. Hence, by the zero-one-law, $P(A) = 0$ or $P(A) = 1$, and this proves the first part of the claim.

Finally suppose that there exists a Borel function g such that $\|h - g\|_\infty < 0.5$. Then $\{g < 0.5\} = \{h = 0\}$ and $\{g > 0.5\} = \{h = 1\}$ would be Borel sets in contrast to that what has been shown before. □

In the second half of this section we deal with constructions where one repeatedly chooses proper closed subsets: After how many steps will such a construction terminate?

Definition 2.2 Let $(T, \mathcal{T})$ be a topological space and $\mathcal{A}$ the collection of closed subsets. A function $\Phi : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *subset assignment* if $\Phi(A)$ is a proper subset of A for all nonempty A, and $\Phi(\emptyset) = \emptyset$.

Proposition 2.3 *Let Φ be a subset assignment on a topological space $(T, \mathcal{T})$. Further, let $\mathcal{V}$ be a collection of open sets which separates points from closed sets: For every closed A and every $x \notin A$ there is $V \in \mathcal{V}$ with $x \in V$ but $V \cap A = \emptyset$.*

Then there is a collection $\mathcal{S}$ of subsets of T with $\mathrm{card}(\mathcal{S}) \leq \mathrm{card}(\mathcal{V})$ *such that T is the union of the $S \in \mathcal{S}$ and each of these S is of the form $A \setminus \Phi(A)$.*

In particular there is a countable $\mathcal{S}$ if T is a first countability space.

Proof. Call a collection $\mathcal{B}$ of closed sets a Φ-system if the following hold:

(i) $T \in \mathcal{B}$; (ii) if $A, B \in \mathcal{B}$ then $A \subset B$ or $B \subset A$, and if A is a proper subset of B then $A \subset \Phi(B)$; (iii) with B also $\Phi(B)$ is in $\mathcal{B}$; (iv) for every x which is not in $\bigcap\{B \mid B \in \mathcal{B}\}$ there exists $B \in \mathcal{B}$ with $x \in B \setminus \Phi(B)$.

For example, $\{T, \Phi(T), \Phi^2(T), \ldots\}$ is a Φ-system, and a routine calculation shows that the collection of all Φ-systems is inductively ordered with respect to inclusion.

Let $\mathcal{B}_0$ be a maximal Φ-system. Necessarily $B_0 := \bigcap\{B \mid B \in \mathcal{B}_0\} = \emptyset$ since otherwise $\mathcal{B}_0 \cup \{B_0, \Phi(B_0), \ldots\}$ would be a Φ-system strictly larger than $\mathcal{B}_0$. By property (iv) T is the union over the $B \setminus \Phi(B)$, $B \in \mathcal{B}_0$. Also $\mathcal{B}_0$ has at most card($\mathcal{V}$) elements: Choose, for $B \in \mathcal{B}_0$ with $B \neq \emptyset$, a V_B in $\mathcal{V}$ such that $B \cap V_B \neq \emptyset = \Phi(B) \cap V_B$; $B \mapsto V_B$ is clearly one-to-one. □

3. Accumulation points and approximations by subsequences

As before let $(T, \mathcal{T})$ be a topological space and (f_n) a pointwise bounded sequence of real-valued continuous functions on T. Then, by compactness, there are plenty of accumulation points f of the (f_n) with respect to pointwise convergence. Fix any of these f, we will investigate how closely f can be approximated by subsequences of (f_n).

More precisely we are interested in the numbers $\epsilon_0 \geq 0$ such that there is a subsequence (f_{n_k}) for which $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ (all $x \in T$).

Lemma 3.1 *Let $(f_n), f$ be as above and choose any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $f_n(x) \xrightarrow{\mathcal{U}} f(x)$ for all x. Further suppose that $(T, \mathcal{T})$ is a complete metric space. Let $a < b$ be real numbers such that $U_V := \{n | \sup f_n|_V > b, \inf f_n|_V < a\}$ belongs to $\mathcal{U}$ for any open nonvoid V. Then (f_n) admits an accumulation point h such that $\|g-h\|_\infty \geq (b-a)/2$ for every Borel function g on T.*

Proof. We will suppose that $[a, b] = [0, 1]$. We start with a nonvoid open V such that diam(V), the diameter of V, is ≤ 1. Fix any $n_1 \in U_V$ and choose open $V_0, V_1 \subset V$ with $\text{diam}(V_0) \leq 1/2, \text{diam}(V_1) \leq 1/2$, $f_{n_1}|_{V_0} < 0$, $f_{n_1}|_{V_1} > 1$.

Now select an $n_2 \in U_{V_0} \cap U_{V_1}$ with $n_2 > n_1$, open V_{00}, V_{01} in V_0 as well as open $V_{10}, V_{11} \subset V_1$ such that the diameters of the V_{ij} are $\leq 1/2^2$ and $f_{n_1}|_{V_{j0}} < 0, f_{n_1}|_{V_{j1}} > 1$ for $j = 0, 1$. It should now be clear how to continue: We construct functions $f_{n_1}, f_{n_2}, \ldots$ and nonvoid open subsets $V_{j_1 \ldots j_r}$ for $r \in \mathbb{N}$, $j_1, \ldots, j_r \in \mathbb{N}$ such that $\text{diam}(V_{j_1,\ldots,j_r}) \leq 1/2^r$, and f_{n_r} is < 0 (resp. > 1) on $V_{j_1 \ldots j_{r-1} 0}$ (resp. $V_{j_1 \ldots j_{r-1} 1}$).

By construction and the completeness assumption on T there is, for any $(j_1, j_2, \ldots) \in \{0,1\}^{\mathbb{N}}$, precisely one point $x_{j_1 j_2 \ldots}$ in $\bigcap_r V^-_{j_1 \ldots j_r}$, and this gives rise to a mapping $\phi : T \longrightarrow \{0,1\}^{\mathbb{N}}$. It is routine to show that ϕ is continuous.

Now let h be any accumulation point of the (f_{n_k}), we will show that h has the claimed properties. To this end note that the $f_{n_k} \circ \phi$ behave like the functions h_k from lemma 2.1. More precisely, let τ on $\mathbb{R}$ be defined by $\tau(x) := \min\{\max\{x, 0\}, 1\}$; then $\tau \circ f_{n_k} \circ \phi = h_k$. By the continuity of τ, the function $\tau \circ h \circ \phi$ is an accumulation point of the (h_k). Now let g be a Borel function on T and suppose that $\|g - h\|_\infty < 0.5$. Then, since τ is a contraction, $\|\tau \circ g \circ \phi - \tau \circ h \circ \phi\|_\infty < 0.5$. But $\tau \circ g \circ \phi$ is Borel, and this contradicts lemma 2.1. □

Theorem 3.2 *Let $(T, \mathcal{T})$ be a Polish space and (f_n) a pointwise bounded sequence of continuous real-valued functions on T. Suppose that $\epsilon_0 \geq 0$ is such that every accumulation point of the (f_n) is $\epsilon_0/2$-close to a Borel function.*

Then for every accumulation point f there is a subsequence (f_{n_k}) such that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ *for $x \in T$.*

Proof. Fix f and choose an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $f_n(x) \xrightarrow{\mathcal{U}} f(x)$ for all x. For rational $a < b$ with $b - a > \epsilon_0$ and closed subsets $A \subset T$ define $\Phi_{a,b}(A)$ to be the collection of all $x \in A$ such that $\{n | \sup f_n|_{V\cap A} > b, \inf f_n|_{V\cap A} < a\}$ belongs to $\mathcal{U}$ for all open sets V with $x \in V$. Clearly $\Phi_{a,b}(A)$ is a closed subset of A, and it is – for nonempty A – a proper subset by lemma 3.1. Thus $\Phi_{a,b}$ is a subset assignment, and since T is Polish we get from proposition 2.3 a countable cover $\mathcal{S}^*_{a,b}$ of T with each member being of the form $A \setminus \Phi_{a,b}(A)$.

Fix a countable base $\mathcal{T}^*$ of the topology. By the definition of $\Phi_{a,b}$ there is, for A closed and $x \in A \setminus \Phi_{a,b}(A)$, a $V \in \mathcal{T}^*$ such that $x \in V$ and $\{n \mid \sup f_n|_{V\cap A} > b, \inf f_n|_{V\cap A} < a\}$ does *not* belong to $\mathcal{U}$.

Next wellknown properties of ultrafilters apply: If $M \subset \mathbb{N}$ and $M \notin \mathcal{U}$, then $\mathbb{N} \setminus M \in \mathcal{U}$, and $M \cup N \in \mathcal{U}$ yields $M \in \mathcal{U}$ or $N \in \mathcal{U}$. In our situation this gives $\{n \mid \sup f_n|_{V\cap A} \leq b\} \in \mathcal{U}$ or $\{n \mid f_n|_{V\cap A} \geq a\} \in \mathcal{U}$.

We are now going to construct (f_{n_k}) such that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ for $x \in T$. To this end we consider the families $\mathcal{S}_{a,b} := \{S^* \cap V \mid S^* \in \mathcal{S}^*_{a,b},\ V \in \mathcal{T}^*\}$ for rational $a < b$ such that $b - a > \epsilon_0$. We also define a family $\mathcal{U}_{a,b}$ of subsets of $\mathcal{U}$ by $U \in \mathcal{U}_{a,b}$ iff there exists $S \in \mathcal{S}_{a,b}$ such that $U = U_{S,\geq a} := \{n | \sup f_n|_S \geq a\}$ or $U = U_{S,\leq b} := \{n | \sup f_n|_S \leq b\}$. Then $\mathcal{U}^* := \bigcup \mathcal{U}_{a,b}$, where a, b run over all admissible values, is a countable subset of $\mathcal{U}$. Enumerate $\mathcal{U}^*$ as $\{U_1, U_2, \ldots\}$ and choose $n_1 < n_2 < \cdots$ such that $n_k \in U_1 \cap \cdots \cap U_k$.

We claim that $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$ for $x \in T$. Let x and $\epsilon > \epsilon_0$ be given. Choose rational a, b, a', b' such that $a < b < f(x) < a' < b'$,

$b-a, b-a' > \epsilon_0$, and $b'-f(x), f(x)-a < \epsilon$. By our construction we find $S \in \mathcal{S}_{a,b}$, $S' \in \mathcal{S}_{a',b'}$ with $x \in S \cap S'$, and $U_{S,\geq a}$ or $U_{S,\leq b}$ as well as $U_{S',\geq a'}$ or $U_{S',\leq b'}$ belong to $\mathcal{U}$. Since $\{n \mid b < f(x) < a'\}$ is in $\mathcal{U}$ it is not possible, however, that $U_{S',\geq a'}$ or $U_{S,\leq b}$ lie in this ultrafilter (otherwise $\emptyset \in \mathcal{U}$ would follow). Hence $U_{S,\geq a}, U_{S',\leq b'} \in \mathcal{U}^*$ so that, for sufficiently large k, $n_k \in U_{S,\geq a} \cap U_{S,\leq b'}$ which implies $|f_{n_k}(x) - f(x)| \leq \epsilon$. □

4. How Martin's axiom comes into play

Theorem 3.2 can be rephrased as follows:

Let $(T, \mathcal{T})$ and (f_n) be as in that theorem and denote, for $\epsilon_0 \geq 0$, by $(\epsilon_0)_{\text{Borel}}$ and $(\epsilon_0)_{\text{appr.}}$ the properties

$(\epsilon_0)_{\text{Borel}}$: Every accumulation point of the (f_n) is ϵ_0-close to a Borel function;

$(\epsilon_0)_{\text{appr.}}$: For every accumulation point of the (f_n) there is a subsequence (f_{n_k}) with $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon_0$, $x \in T$.

Then $(\epsilon_0/2)_{\text{Borel}}$ implies $(\epsilon_0)_{\text{appr.}}$.

Since the collection of all Borel functions is closed with respect to the supremum norm it is clear that $(\epsilon)_{\text{Borel}}$, all $\epsilon > 0$, yields $(0)_{\text{Borel}}$ and hence $(0)_{\text{appr.}}$, i.e., every accumulation point is the limit of a subsequence.

One might suspect that this last assertion could be established by a different argument: Given an f such that for $\epsilon > 0$ there is a subsequence (f_{n_k}) with $\limsup_k |f_{n_k}(x) - f(x)| \leq \epsilon$, $x \in T$, then there also should be a subsequence with pointwise limit f. We will concentrate on the set theoretical variant of this problem (which corresponds to the special case of discrete topological spaces). Also we may and will assume that the function to be approximated is the zero function.

Proposition 4.1 *Let T be a set and (f_n) a sequence of real-valued functions on T such that for every $\epsilon > 0$ there is a subsequence (f_{n_k}) with* $\limsup |f_{n_k}(x)| \leq \epsilon$.

(i) *If T is countable then there is a subsequence of (f_n) which converges pointwise to zero.*

(ii) *If* $\text{card}(T) \geq 2^{\aleph_0}$ *then this is not necessarily true.*

Proof. (i) is obvious. For the proof of (ii) it suffices to treat the case $T = \mathbb{N}^{\mathbb{N}}$. We define, for integers m and n, a function $h_{m,n} : T \longrightarrow [0,1]$ by $h_{m,n}(\phi) := 1$ if $n \leq \phi(m)$ and $:= 1/m$ otherwise. Let $f_1, f_2, \ldots$ be any enumeration of the family $(h_{n,m})$. Then (f_n) has subsequences with arbitrarily small lim sup, but there is no subsequence converging to zero: Let (h_{m_k,n_k}) be any subsequence; if the m_k are bounded, by m' say, then $(h_{m_k,n_k}) \geq 1/m'$, and if they are unbounded we can choose a ϕ_0 such that $\phi_0(m_k) > n_k$ infinitely often so that $(h_{m_k,n_k})(\phi_0)$ does not converge to zero. $\square$

The proposition leaves open the case of sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$. Under the assumption of the continuum hypothesis (CH) there are no such sets, but what about set theories without CH ? We refer the reader to [6] where a number of interesting set theoretical axioms and their implications for various branches of mathematics are discussed. For our problem Martin's axiom (MA) will be of importance. The precise meaning of MA is rather technical, fortunately it suffices for us to know

Theorem 4.2 *Under the assumption of* MA *the following is true:*

Whenever M is a set with cardinality less than $2^{\aleph_0}$ and $(C_x)_{x\in M}$ is a collection of subsets of $\mathbb{N}$ such that all finite intersections $C_{x_1} \cap \cdots \cap C_{x_r}$ are infinite, then there exists an infinite $I \subset \mathbb{N}$ for which all $I \setminus C_x$ are finite.

Proof. Let κ be the cardinality of M. Then, with the notation of [6], MA(κ) holds which implies, by theorem 11C of [6], that P(κ^+). But $\kappa < \kappa^+$, and our result follows from [6, 11B(c)]. $\square$

It should be noted that MA is "nearly equivalent" with the statement of the theorem. The precise meaning of this can be found in [6, Bell's theorem, 14C].

Theorem 4.3 *Under the assumption of Martin's axiom every set T with* card$(T) < 2^{\aleph_0}$ *has the following property: If the (f_n) are as in proposition 4.1, then there is a subsequence which converges pointwise to zero.*

Proof. Set $S := T \times \mathbb{N}$. For $(x,m) \in S$ define $C_{x,m} := \{k \mid f_k(x) < 1/m\}$. The assumption implies that finite intersections of such sets are infinite, and since card$(S) < 2^{\aleph_0}$ we may apply MA. Theorem 4.2 provides $I = \{n_1, n_2, \ldots\}$ such that all $C_{x,m} \setminus I$ are finite. It is then clear that (f_{n_k}) converges pointwise to zero. $\square$

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