PRECOMPACTNESS AND COMPACTNESS IN THE NATURAL FUNCTION SPACES OF THE CATEGORY OF SEMIUNIFORM CONVERGENCE SPACES

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Abstract. Semiuniform convergence spaces form a common generalization of (symmetric) limit spaces (and thus of symmetric topological spaces) as well as of uniform limit spaces (and thus of uniform spaces) with many convenient properties such as cartesian closedness, hereditariness and the fact that products of quotients are quotients. Here necessary and sufficient conditions for precompactness in the natural function spaces of the category of semiuniform convergence spaces are studied. The results are applied to characterize compactness in these function spaces and to derive the classical Ascoli theorem.

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1. Introduction

Semiuniform convergence spaces have been studied by the author (cf. [17], [18], [19] and [20]) as an alternative concept of space in topology. They form a strong topological universe, i.e. a cartesian closed and hereditary topological category such that products of quotients are quotients. They are the suitable framework for studying continuity, Cauchy continuity and uniform continuity as well as convergence structures in function spaces, namely simple convergence, continuous convergence and uniform convergence. Furthermore, the localization of compactness (when considered in the realm of semiuniform convergence spaces) leads to a cartesian closed topological category and, in contrast to the situation for topological spaces, the locally compact spaces are exactly the compactly generated spaces (even without the Hausdorff axiom!). Similarly, the localization of precompactness leads even to a topological universe (= cartesian closed and hereditary topological category) and the locally precompact spaces are exactly the precompactly generated spaces. Obviously, they form a generalization of locally compact spaces.

Here it turns out that locally precompactness of the domain of the natural function spaces (= power objects) of the category of semiuniform convergence spaces is needed in order to characterize precompact subsets of these function spaces. The results of Wyler [23] who characterized precompactness in the power objects of the category of uniform limit spaces (= uniform convergence spaces) are partially enlarged to semiuniform convergence spaces. Furthermore, the results are applied to characterize compact subsets of the above mentioned function spaces and to derive the *classical Ascoli theorem.*

In the last three decades many authors have worked in the field of generalizations of the Ascoli theorem. Among them are (in chronological order): Cook and Fischer [6], Poppe [14], Dubuc [7], Kneis [12], Gähler [9], Gray [11], Wyler [23], Bentley and Herrlich [3]. The termilogy of this article corresponds to [1] and [15].

Conventions. 1) A filter on a set X is not allowed to contain the empty set ϕ.
2) Subcategories are always assumed to be full and isomorphism–closed.

2. Some basic definitions and results

2.1 DEFINITIONS. 1) A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that the following are satisfied:

$UC_1)$ $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in A\}$,

$UC_2)$ $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \supset \mathcal{F}$,

$UC_3)$ $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$,

where $F^{-1} = \{(y,x) : (x,y) \in F\}$.

2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *diagonal* provided that the filter (Δ_X) generated by the diagonal Δ_X of $X \times X$ belongs to $\mathcal{J}_X$.

3) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *semiuniform limit space* provided that the following is satisfied:

$UC_4)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$.

4) A semiuniform limit space $(X, \mathcal{J}_X)$ is called a *uniform limit space* provided that the following is satisfied:

$UC_5)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$ (whenever $\mathcal{F} \circ \mathcal{G}$ exists, i.e. $F \circ G = \{(x,y) : \exists z \in X \text{ with } (x,z) \in G \text{ and } (z,y) \in F\} \neq \phi$ for every $F \in \mathcal{F}, G \in \mathcal{G}$), where $\mathcal{F} \circ \mathcal{G}$ is the filter generated by the filter base $\{F \circ G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$.

5) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{J}_X) \subset \mathcal{J}_Y$.

6) The category of semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**, whereas its full subcategories of semiuniform limit spaces and uniform limit spaces are denoted by **SULim** and **ULim** respectively.

2.2 REMARKS. 1) The category **Unif** of uniform spaces (and uniformly continuous maps) in the sense of A. Weil [22] can be nicely embedded into the category **SUConv**, namely **Unif** is isomorphic to the full subcategory **PrULim** of **ULim** whose object class consists of all *principal uniform limit spaces* (a uniform limit space $(X, \mathcal{J}_X)$ is called a principal uniform limit space provided that there is a filter $\mathcal{F}$ on $X \times X$ such that $\mathcal{J}_X = [\mathcal{F}]$, where $[\mathcal{F}] = \{\mathcal{G} : \mathcal{G}$ is a filter on $X \times X$ with $\mathcal{G} \supset \mathcal{F}\})$. Since **PrULim** is a bireflective subcategory of **ULim** (cf. [15;4.2.8.1]) and **ULim** is a bireflective subcategory of **SUConv** (cf. [17;2.2]), **Unif** can be embedded into **SUConv** as a bireflective subcategory. Obviously, a principal uniform limit space (= uniform space) is diagonal.

2) **SUConv** is a topological category, in which initial and final structures can be easily described (cf. e.g. [18;2.2.2]).

2.3 DEFINITIONS. 1) A *Kent convergence space* is a pair (X, q), where X is a set and $q \subset F(X) \times X$ (provided $F(X)$ denotes the set of all filters on X), such that the following are satisfied:

C_1) $(\dot{x}, x) \in q$ for each $x \in X$,

C_2) $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{G} \supset \mathcal{F}$,

C_3) $(\mathcal{F} \cap \dot{x}, x) \in q$, whenever $(\mathcal{F}, x) \in q$.

If (X, q) is a Kent convergence space, then we say that $\mathcal{F} \in F(X)$ *converges* to $x \in X$ provided that $(\mathcal{F}, x) \in q$.

2) A Kent convergence space (X, q) is called *symmetric* provided that the following is satisfied:

$$(\mathcal{F}, x) \in q \quad \text{and} \quad y \in \bigcap_{F \in \mathcal{F}} F \quad \text{imply} \quad (\mathcal{F}, y) \in q\ .$$

3) A map $f : (X, q) \to (X', q')$ between Kent convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.

2.4 REMARKS. 1) a) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying (symmetric) Kent convergence space* $(X, q_{\gamma_{\mathcal{J}_X}})$ defined as follows:

$$(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}} \Longleftrightarrow \mathcal{F} \cap \dot{x} \in \gamma_{\mathcal{J}_X}$$

where $\gamma_{\mathcal{J}_X} = \{\mathcal{H} \in F(X) : \mathcal{H} \times \mathcal{H} \in \mathcal{J}_X\}$.

b) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *convergence space* provided that

$$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \text{there are some } x \in X \text{ and } \mathcal{G} \in F(X) \text{ with } (\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}} \text{ such that } \mathcal{F} \supset \mathcal{G} \times \mathcal{G}\},$$

in other words: A convergence space is a semiuniform convergence space which is generated by all convergent filters. The category **Conv** of convergence spaces (and uniformly continuous maps) is isomorphic to the category $\mathbf{KConv}_s$ of symmetric Kent convergence spaces (and continuous maps); it is a bicoreflective subcategory of **SUConv** (cf. [18;3.2, 3.3 and 3.4]) and thus closed under formation of quotients and coproducts (formed in **SUConv**).

2) If (X, q) is a Kent convergence space, then for each subset A of X we define the *closure* $\overline{A}$ of A by

$$\overline{A} = \{x \in X : \text{there is some } \mathcal{G} \in F(X) \text{ with } (\mathcal{G}, x) \in q \text{ and } A \in \mathcal{G}\}$$

3) **Conv** is closed under formation of closed subspaces and products (formed in **SUConv**) [18;3.13].

2.5 Definition. A semiuniform convergence spaces $(X, \mathcal{J}_X)$ is called *regular* (resp. *t–regular*) provided that for each $\mathcal{F} \in \mathcal{J}_X$ the subfilter generated by the filter base $\{\overline{F} : F \in \mathcal{F}\}$ (resp. $\{F \in \mathcal{F} : F = \overline{F}\}$) belongs to $\mathcal{J}_X$, where $\overline{F}$ denotes the closure of F with respect to the underlying Kent convergence space of the product space $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$.

2.6 Remark. Every topological R_0–space $(X, \mathcal{X})$, i.e. every topological space $(X, \mathcal{X})$ such that $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$ for each $(x, y) \in X \times X$, defines a symmetric Kent convergence space $(X, q_{\mathcal{X}})$ as follows:

$$(\mathcal{F}, x) \in q_{\mathcal{X}} \quad \text{iff} \quad \mathcal{F} \supset \mathcal{U}(x) \ ,$$

where $\mathcal{U}(x)$ denotes the neighborhood filter of x with respect to $\mathcal{X}$. Thus it may be considered to be a convergence space in the sense of 2.4 1)b). Having this explanation in mind, we obtain:

a) *Every regular topological space (in the usual sense) is t–regular.*

b) *Every uniform space (= principal uniform limit space) is t–regular.*

c) *Every t–regular semiuniform convergence space is regular.*

2.7 Definitions. 1) A *filtermerotopic space* (shortly: a *filter space*) is a pair (X, γ), where X is a set and $\gamma \subset F(X)$ such that the following are satisfied:

(1) $\dot{x} \in \gamma$ for each $x \in X$,

(2) $\mathcal{G} \in \gamma$ whenever $\mathcal{F} \in \gamma$ and $\mathcal{F} \subset \mathcal{G}$.

2) A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that the filter $f(\mathcal{F})$ generated by $\{f[F] : F \in \mathcal{F}\}$ belongs to γ' for each $\mathcal{F} \in \gamma$.

3) The category of filter spaces (and Cauchy continuous maps) is denoted by **Fil**.

2.8 Remarks. 1) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying filter space* $(X, \gamma_{\mathcal{J}_X})$ defined as follows

$$\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(X) : \mathcal{F} \times \mathcal{F} \in \mathcal{J}_X\} \ .$$

The elements of $\gamma_{\mathcal{J}_X}$ are called *$\mathcal{J}_X$–Cauchy filters.*

2) If a semiuniform convergence space $(X, \mathcal{J}_X)$ is called **Fil**–*determined* provided that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \text{there is some } \mathcal{G} \in \gamma_{\mathcal{J}_X} \text{ with } \mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$ (i.e. a semiuniform convergence space $(X, \mathcal{J}_X)$ is **Fil**–determined iff it is 'generated' by all $\mathcal{J}_X$–Cauchy filters), then the category **Fil–D–SUConv** of all **Fil**–determined semiuniform

convergence spaces (and uniformly continuous maps) is isomorphic to **Fil** and a bireflective and bicoreflective subcategory of **SUConv** (cf. [18;2.4]).

3) The category **Prox** of proximity spaces (and δ–maps) in the sense of Efremovic [18] can be embedded into **Fil** as a bireflective subcategory (cf. [17;2.4.1)] and [15]), whereas the category **Conv** can be embedded into **Fil** as a bicoreflective subcategory (cf. [18;2.8, 2.11.3) and 3.2]).

2.9 Proposition. *Let $(X, \mathcal{J}_X)$ be a semiuniform limit space. A $\mathcal{J}_X$–Cauchy filter $\mathcal{F}$ on X converges to $x \in X$ in $(X, q_{\gamma_{\mathcal{J}_X}})$ iff $\mathcal{F} \times \dot{x} \in \mathcal{J}_X$.*

Proof. $'' \Longrightarrow''$. Obvious.

$'' \Longleftarrow''$. $\mathcal{F} \times \dot{x} \in \mathcal{J}_X$ implies $(\mathcal{F} \times \dot{x})^{-1} = \dot{x} \times \mathcal{F} \in \mathcal{J}_X$. Furthermore, $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$, since $\mathcal{F}$ is a $\mathcal{J}_X$–Cauchy filter. By $UC_1), \dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$. Then it follows from $UC_4)$ that

$$(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) = (\mathcal{F} \times \mathcal{F}) \cap (\mathcal{F} \times \dot{x}) \cap (\dot{x} \times \mathcal{F}) \cap (\dot{x} \times \dot{x}) \in \mathcal{J}_X \ ,$$

i.e. $\mathcal{F}$ converges to x in $(X, q_{\gamma_{\mathcal{J}_X}})$.

2.10 Definitions. A semiuniform convergence $(X, \mathcal{J}_X)$ is called

1) *precompact* (or totally bounded) provided that each ultrafilter $\mathcal{U}$ on X is a $\mathcal{J}_X$–Cauchy filter,
2) *compact* provided that each ultrafilter $\mathcal{U}$ on X converges in $(X, q_{\gamma_{\mathcal{J}_X}})$,
3) *complete* provided that each $\mathcal{J}_X$–Cauchy filter converges in $(X, q_{\gamma_{\mathcal{J}_X}})$,
4) *locally precompact* (resp. *locally compact*) provided that each $\mathcal{F} \in \mathcal{J}_X$ contains a precompact (resp. compact) subset of the product space $(X, \mathcal{J}_X) \times (X, \mathcal{J}_X)$, where *a subset* of a semiuniform convergence space *is called precompact* (*resp. compact*) provided that it is precompact (resp. compact) as a subspace.

2.11 Proposition. (cf. [18;3.7]). *Every convergence space is complete.*

2.12. The concepts defined under 2.10. are related to each other by the following implication scheme (cf. [20]).

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\text{precompact}} \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift left, "\text{complete}"] & \boxed{\text{locally precompact}} \arrow[l, Rightarrow, shift left, "\text{diagonal}"] \arrow[d, Rightarrow, shift left, "t\text{–regular and complete}"] \\
\boxed{\text{compact}} \arrow[u, Rightarrow, shift left] \arrow[r, Rightarrow, shift left] & \boxed{\text{locally compact}} \arrow[l, Rightarrow, shift left, "\text{diagonal}"] \arrow[u, Rightarrow, shift left]
\end{tikzcd}$$

2.13 Theorem (cf. [19;3.12] and [20;3.15]). *A semiuniform convergence space* $(X, \mathcal{J}_X)$ *is locally compact (resp. locally precompact) iff it is compactly generated (resp. precompactly generated), i.e.* $\mathcal{J}_X$ *is the final* **SUConv**–*structure with respect to the family* $(j_i : (K_i, \mathcal{J}_{K_i}) \to (X, \mathcal{J}_X))$ *of the inclusions of all compact (resp. precompact) subspaces of* $(X, \mathcal{J}_X)$.

2.14 Definitions. 1) A topological category $\mathcal{C}$, i.e. a concrete category over the category **Set** of sets (and maps) which is initially complete and small–fibred such that all constant maps between its objects are morphisms, is called *hereditary* provided that in $\mathcal{C}$ final (epi–) sinks are hereditary, i.e. if $(f_i : A_i \to A)_{i \in I}$ is a final (epi–) sink in $\mathcal{C}, B$ a subspace (= extremal subject) of A, B_i as subspace of A_i with underlying set $f_i^{-1}[B]$ and $g_i : B_i \to B$ is the corresponding restriction of f_i, then $(g_i : B_i \to B)_{i \in I}$ is a final (epi–) sink in $\mathcal{C}$ too.

2) A cartesian closed topological category $\mathcal{C}$ is called a *topological universe* provided that $\mathcal{C}$ is hereditary.

3) A topological universe $\mathcal{C}$ is called *strong* provided that in $\mathcal{C}$ products of quotients are quotients.

2.15 Theorem (cf. [17]). **SUConv** *is a strong topological universe.*

2.16 Corollaries. 1) **SULim** *and* **Fil** *are strong topological universes* (cf. [17]).

2) *The category* **LPC**–**SUConv** *of locally precompact semiuniform convergence spaces is a topological universe* (cf. [20]).

3) *The category* **Conv** *is a cartesian closed topological category in which quotient maps are productive* (cf. [18]).

4) **ULim** *as well as the category* **LC**–**SUConv** of locally compact semiuniform convergence spaces are cartesian closed topological categories (cf. [2], [13] and [19]).

2.17 Remark. The power objects (= *natural function spaces*) in the category **SUConv** are defined as follows:

Let $\mathbf{X} = (X, \mathcal{J}_X)$ and $\mathbf{Y} = (Y, \mathcal{J}_X)$ be semiuniform convergence spaces. Then the power–object $\mathbf{Y}^{\mathbf{X}}$ is the set $[\mathbf{X}, \mathbf{Y}]$ of all uniformly continuous maps from $\mathbf{X}$ into $\mathbf{Y}$ endowed with the **SUConv**–structure $\mathcal{J}_{X,Y} = \{\phi \in F([\mathbf{X}, \mathbf{Y}] \times [\mathbf{X}, \mathbf{Y}]) : \phi(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X\}$, where $\phi(\mathcal{F})$ is the filter generated by $\{A(F) : A \in \phi, F \in \mathcal{F}\}$ with $A(F) = \{(f(a), g(b)) : (f, g) \in A, (a, b) \in F\}$. $\mathcal{J}_{X,Y}$ is also called the *uniformly continuous* **SUConv**–*structure.*

2.18 Proposition. *Let* $\mathbf{X} = (X, \mathcal{J}_X)$ *be a semiuniform convergence space and* $\mathbf{Y} = (Y, \mathcal{J}_Y)$ *a regular semiuniform limit space. Then* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ *is a complete semiuniform limit space provided that* $\mathbf{Y}$ *is complete.*

Proof. By [2;1.5.1], $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is a semiuniform limit space provided that $\mathbf{Y}$ is a semiuniform limit space. Since 2.9 is valid, the proof of 2.18 parallels [13;theorem 3.9.]

2.19 Definition (cf. [16;2.5]). Let $\mathcal{E} \subset |\mathbf{SUConv}|$. Then a semiuniform convergence space $(X, \mathcal{J}_X)$ is called *totally $\mathcal{E}$–separated* provided that for each $x \in X$ the $\mathcal{E}$–quasicomponent K_x of X containing x is a singleton, i.e. for any two distinct elements $x, y \in X$ there exists an object $(E, \mathcal{J}_E) \in \mathcal{E}$ and a uniformly continuous map $f : (X, \mathcal{J}_X) \to (E, \mathcal{J}_E)$ such that $f(x) \neq f(y)$. The class of all totally $\mathcal{E}$–separated semiuniform convergence spaces is denoted by $Q\mathcal{E}$.

2.20 EXAMPLES. 1) Let $\mathcal{E}$ be the class of all T_0–,T_1– or T_2–spaces respectively, then $\mathcal{E} = Q\mathcal{E}$ (Note: A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a

a) *T_0–space* provided that for each pair $(x, y) \in X \times X, (\dot{x}, y) \in q_{\gamma_{\mathcal{J}_X}}$ and $(\dot{y}, x) \in q_{\gamma_{\mathcal{J}_X}}$ imply $x = y$,

b) *T_1–space* provided that for each pair $(x, y) \in X \times X, (\dot{x}, y) \in q_{\gamma_{\mathcal{J}_X}}$ implies $x = y$,

c) *T_2–space* provided that each filter on X converges to at most one element of X in $(X, q_{\gamma_{\mathcal{J}_X}})$).

2) If $D_H\mathcal{E}$ denotes the class of all *totally $\mathcal{E}$–disconnected semiuniform convergence spaces*, then $QD_H\mathcal{E} = D_H\mathcal{E}$ (cf. [18;6.30] and [15;1.3.13]).

2.21 PROPOSITION. *Let $\mathbf{X} = (X, \mathcal{J}_X)$ and $\mathbf{Y} = (Y, \mathcal{J}_Y)$ be semiuniform convergence spaces. Then $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is totally $\mathcal{E}$–separated provided that $\mathbf{Y}$ is totally $\mathcal{E}$–separated.*

PROOF. Let $f_0 \in [\mathbf{X}, \mathbf{Y}]$ and let Q_{f_0} be the $\mathcal{E}$–quasicomponent of $[\mathbf{X}, \mathbf{Y}]$ containing f_0. For each $x \in X$ define

$$h_x : [\mathbf{X}, \mathbf{Y}] \to X \times [\mathbf{X}, \mathbf{Y}]$$

by $h_x(f) = (x, f)$. Obviously, h_x is uniformly continuous (provided that $X \times [\mathbf{X}, \mathbf{Y}]$ is endowed with the product structure with respect to **SUConv**). Since Q_{f_0} is $\mathcal{E}$–connected with respect to $[\mathbf{X}, \mathbf{Y}], h_x[Q_{f_0}] = \{x\} \times Q_{f_0}$ is $\mathcal{E}$–connected with respect to $X \times [\mathbf{X}, \mathbf{Y}]$ (cf. [15;1.3.7(3)(b)]), and, since the evaluation map $e_{X,Y} : \mathbf{X} \times \mathbf{Y}^{\mathbf{X}} \to \mathbf{Y}$ is uniformly continuous, $e_{X,Y}[\{x\} \times Q_{f_0}] = \{f(x) : f \in Q_{f_0}\}$ is $\mathcal{E}$–connected with respect to Y. Since $\mathbf{Y} \in Q\mathcal{E}$, it follows that $f(x) = f_0(x)$ for each $f \in Q_{f_0}$ and for each $x \in X$, i.e. Q_{f_0} is a singleton. Thus $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is totally $\mathcal{E}$–separated.

2.22 REMARK. Obviously, *the above proposition is valid in every cartesian closed topological category.*

3. Precompactness and compactness in the power objects of SUConv

3.1 DEFINITION. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *semi–pseudouniform* provided the following is satisfied:

(Ps) $\mathcal{F} \in \mathcal{J}_X$ whenever $\mathcal{U} \in \mathcal{J}_X$ for each ultrafilter $\mathcal{U} \supset \mathcal{F}$.

3.2 REMARKS. 1) Every semi–pseudouniform space $(X, \mathcal{J}_X)$ is a semiuniform limit space (cf. [2;1.1.5]).

2) The category **SUnif** of semiuniform spaces (and uniformly continuous maps) in the sense of Čech [5] can be embedded into the category **SPsU** of semi–pseudouniform spaces (and uniformly continuous maps) as a bireflective subcategory (cf. [2;1.4.10]).

3) **SPsU** is a bireflective subcategory of **SULim** (cf. [2;1.3.2. and 1.3.6.a)] and **SULim** is a bireflective subcategory of **SUConv** (cf. [2;1.3.4] or [17;2.2.2)b)]).

3.3 PROPOSITION. *Let $(X,\mathcal{J}_X)$ be a semi–pseudouniform space and $A \subset X$ a precompact subset. Then the filter (Δ_A) on $X \times X$ generated by the diagonal Δ_A in $A \times A$ belongs to $\mathcal{J}_X$.*

PROOF. Obviously, A is a precompact subset of X iff each ultrafilter $\mathcal{U}$ on X containing A is a $\mathcal{J}_X$–Cauchy filter. Let $\mathcal{W}$ be an ultrafilter on $X \times X$ finer than (Δ_A). Then there is an ultrafilter $\mathcal{U}$ on X containing A such that $\Delta_\mathcal{U} = \mathcal{W}$, where $\Delta_\mathcal{U}$ is the filter on $X \times X$ generated by the filter $\Delta'_\mathcal{U} = \{\Delta_U : U \in \mathcal{U}\}$ on Δ_X, namely $\mathcal{U} = \{F \subset X : \Delta_F \in \mathcal{W}\}$ (note: since $\mathcal{U}$ is an ultrafilter, $\Delta_\mathcal{U} \subset \mathcal{W}$ is an ultrafilter too, because $\Delta'_\mathcal{U}$ is an ultrafilter). Since $\mathcal{U} \times \mathcal{U} \subset \Delta_\mathcal{U}$ and A is a precompact subset of $X, \Delta_\mathcal{U} = \mathcal{W}$ belongs to $\mathcal{J}_X$. Thus, since $(X,\mathcal{J}_X)$ is semi–pseudouniform, (Δ_A) belongs to $\mathcal{J}_X$.

3.4 COROLLARY. *Every precompact semi-pseudouniform space is diagonal.*

3.5 PROPOSITION (cf. [2;1.5.1.d)] or [13;3.11]). *Let $\mathbf{X} = (X,\mathcal{J}_X)$ and $\mathbf{Y} = (Y, J_Y)$ be semiuniform convergence spaces. Then $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$ is a semi–pseudouniform space provided that $\mathbf{Y}$ is a semi-pseudouniform space. The inverse implication is valid provided that $X \neq \phi$.*

3.6 DEFINITON. Let $\mathbf{X} = (X,\mathcal{J}_X)$ and $\mathbf{Y} = (Y,\mathcal{J}_Y)$ be semiuniform convergence spaces and $M \subset [\mathbf{X},\mathbf{Y}]$. Then M is called *uniformly equicontinuous* provided that $(\Delta_M) \in \mathcal{J}_{X,Y}$, i.e. for each $\mathcal{F} \in \mathcal{J}_X$ the filter $(\Delta_M)(\mathcal{F})$ generated by $\{\Delta_M(F) : F \in \mathcal{F}\}$ with $\Delta_M(F) = \{(f(a), f(b)) : f \in M, (a,b) \in F\}$ belongs to $\mathcal{J}_Y$.

3.7 REMARKS. 1) If $\mathbf{X}$ and $\mathbf{Y}$ are principal uniform limit spaces (= uniform spaces), then “uniformly equicontinuous“ in the sense of 3.6 means “*uniformly equicontinuous*“ *in the usual sense.*

2) Let $(X,\mathcal{X})$ be a topological R_0–space (= symmetric topological space) and $\mathbf{X} = (X,\mathcal{J}_{\gamma_{q_\mathcal{X}}})$ its corresponding semiuniform convergence space, i.e. $\mathcal{J}_{\gamma_{q_\mathcal{X}}} = \{\mathcal{F} \in F(X \times X) : \exists x \in X$ such that $\mathcal{F} \supset \mathcal{U}(x) \times \mathcal{U}(x)\}$, where $\mathcal{U}(x)$ denotes the neighborhood filter of x with respect to the topology $\mathcal{X}$. Furthermore, let $\mathbf{Y}$ be a principal uniform limit space (=uniform space). Then “uniformly equicontinuous“ means “*equicontinuous*″ *in the usual sense.*

3.8 DEFINITION. Let Δ–**SUConv** be the category of diagonal semiuniform convergence spaces (and uniformly continuous maps). If X is a set and $(Y,\mathcal{J}_Y) \in |\boldsymbol{\Delta}$–**SUConv**$|$, then the $\boldsymbol{\Delta}$–**SUConv**–structure

$$\mathcal{J}^u_{Y^X} = \{\phi \in F(Y^X \times Y^X) : \phi((\Delta_X)) \in \mathcal{J}_Y\}$$

is called the *structure of uniform convergence* on Y^X.

3.9 REMARK. If the set X in 3.8 is endowed with the discrete $\boldsymbol{\Delta}$–**SUConv**–structure $[(\Delta_X)] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset (\Delta_X)\}$, then $[(X,[(\Delta_X)]),(Y,\mathcal{J}_Y)]_{\boldsymbol{\Delta}\text{–}\mathbf{SUConv}} = [(X,[(\Delta_X)]),$

$(Y, \mathcal{J}_Y)]_{\mathbf{SUConv}} = Y^X$ and the uniformly continuous **SUConv**–structure $\mathcal{J}_{X,Y}$ (cf. 2.17) coincides with the structure of uniform convergence on Y^X.

3.10 PROPOSITION. *Let* $\mathbf{X} = (X, \mathcal{J}_X)$ *and* $\mathbf{Y} = (Y, \mathcal{J}_X)$ *be diagonal semiuniform convergence spaces,* $\mathcal{J}_{X,Y}$ *the uniformly continuous* **SUConv**–*structure on* $[\mathbf{X}, \mathbf{Y}]_{\mathbf{SUConv}}$ *and* $\mathcal{J}^u_{X,Y}$ *the structure induced on* $[\mathbf{X}, \mathbf{Y}]_{\mathbf{SUConv}}$ *by* $\mathcal{J}^u_{Y^X}$ *. Then*

$$1_{[\mathbf{X},\mathbf{Y}]} : ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y}) \to ([\mathbf{X}, \mathbf{Y}], \mathcal{J}^u_{X,Y})$$

is uniformly continuous, i.e. $\mathcal{J}_{X,Y}$ *is finer than* $J^u_{X,Y}$.

PROOF. Since $\mathbf{X}$ is diagonal, $(\Delta_X) \in \mathcal{J}_X$ and, by definition of $\mathcal{J}_{X,Y}$, $\phi((\Delta_X)) \in \mathcal{J}_Y$ for each $\phi \in \mathcal{J}_{X,Y}$. If $i : [\mathbf{X}, \mathbf{Y}] \to Y^X$ denotes the inclusion map, then, since $\phi \subset (i \times i)(\phi)$, it follows that $(i \times i)(\phi)((\Delta_X)) \in \mathcal{J}_Y$, i.e. $(i \times i)(\phi) \in \mathcal{J}^u_{Y^X}$. Thus $\phi \in \mathcal{J}^u_{X,Y}$ for each $\phi \in \mathcal{J}_{X,Y}$.

3.11 REMARK. It is easily verified that $\mathbf{\Delta}$–**SUConv** is a bireflective subcategory of **SUConv** and therefore initial structures in $\mathbf{\Delta}$–**SUConv** are formed as in **SUConv**. Consequently, $\mathcal{J}^u_{X,Y}$ is also a $\mathbf{\Delta}$–**SUConv**–structure. On the other hand, $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ in 3.10 is generally not diagonal, e.g. if $\mathbb{R}_u$ denotes the set $\mathbb{R}$ of real numbers endowed with the usual uniformity, then $([\mathbb{R}_u, \mathbb{R}_u], \mathcal{J}_{\mathbb{R},\mathbb{R}})$ is a uniform limit space, which is not diagonal (cf. [10;2.2.]). Thus, $\mathcal{J}_{X,Y} \neq \mathcal{J}^u_{X,Y}$ in general.

3.12 THEOREM (*Necessary conditions for precompactness in* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$).
Let $\mathbf{X} = (X, \mathcal{J}_X)$ *be a semiuniform convergence space,* $\mathbf{Y} = (Y, \mathcal{J}_Y)$ *a semi-pseudouniform space and* M *a precompact subset of* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$*. Then the following are satisfied:*

(1) M *is uniformly equicontinuous,*
(2) $M(x) = \{f(x) : f \in M\} \subset Y$ *is precompact for each* $x \in X$.

PROOF. Since $\mathbf{Y}$ is semi-pseudouniform, it follows from 3.5 that $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is semi-pseudouniform. Thus, by 3.3, M is uniformly equicontinuous. For each $x \in X$, the restriction map $e_{X,Y}|_{\{x\}\times M} : \{x\} \times M \to Y$ of the evaluation map $e_{X,Y} : X \times [\mathbf{X}, \mathbf{Y}] \to Y$ is uniformly continuous. Since $\{x\}$ and M are precompact, $M \times \{x\}$ is precompact (cf. [18;4.2]). Consequently $(e_{X,Y}|_{\{x\}\times M})[\{x\} \times M] = M(x) \subset Y$ is precompact (cf.[20;2.11]).

3.13 DEFINITION. A semi-pseudouniform space $\mathbf{Y} = (Y, \mathcal{J}_Y)$ is called a *semi-Ascoli space* provided that for every precompact uniform limit space $\mathbf{X} = (X, \mathcal{J}_X)$ and every uniformly equicontinuous set $M \subset [\mathbf{X}, \mathbf{Y}]$ the subspace structure of M with respect to the product structure of Y^X coincides with the subspace structure of M in $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$.

3.14 REMARK. Obviously, a semi-pseudouniform space $\mathbf{Y} = (Y, \mathcal{J}_Y)$ is a semi-Ascoli space iff for every precompact uniform limit space $\mathbf{X} = (X, \mathcal{J}_X)$ and every uniformly equicontinuous set $M \subset [\mathbf{X}, \mathbf{Y}]$ the subspace structure of M in $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ contains the subspace structure of M with respect to the product struture of Y^X (i.e. the structure of *simple convergence* on M).

3.15 THEOREM (*Sufficient conditions for precompactness in* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$.

Let $\mathbf{X} = (X, \mathcal{J}_X)$ *be a locally precompact uniform limit space and* $\mathbf{Y} = (Y, \mathcal{J}_Y)$ *a semi-Ascoli space. Then* $M \subset [\mathbf{X}, \mathbf{Y}]$ *is a precompact subset of* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ *provided that the following are satisfied:*

(1) M *is uniformly equicontinuous,*

(2) $M(x) = \{f(x) : f \in M\} \subset Y$ *is precompact for each* $x \in X$.

PROOF. Let N be the space $(M(\mathcal{J}_{Y^X})_M)$, where $(\mathcal{J}_{Y^X})_M$ denotes the subspace structure of M in the product space $(Y^X, \mathcal{J}_{Y^X})$. Since $M \subset \prod\limits_{x \in X} M(x) \subset Y^X, N$ is precompact (note (2) and cf. [18;4.2]). Consider first the case where $\mathbf{X}$ is precompact. Since $\mathbf{Y}$ is a semi–Ascoli space and M is uniformly equicontinuous, $N = (M, (\mathcal{J}_{X,Y})_M)$ where $(\mathcal{J}_{X,Y})_M$ denotes the subspace structure of M with respect to $\mathcal{J}_{X,Y}$. Then the restriction h of the evaluation map $e_{X,Y} : \mathbf{X} \times ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y}) \to \mathbf{Y}$ to $\mathbf{X} \times N$ is uniformly continuous. Since $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is a power object in **SUConv**, there is a unique uniformly continuous map $\overline{h} : N \to ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ such that $e_{X,Y} \circ (1_X \times \overline{h}) = h$. Then $\overline{h}(f) = f$ for every $f \in M$ and consequently $\overline{h}[N] = (M, (\mathcal{J}_{X,Y})_M)$. Since N is precompact, it follows that $(M, (\mathcal{J}_{X,Y})_M)$ is precompact (cf. [20;2.11]). Consider next any locally precompact uniform limit space $\mathbf{X}$. By 2.13, the family $(j_i : \mathbf{K}_i \to \mathbf{X})_{i \in I}$ of inclusions of all precompact subspaces $\mathbf{K}_i = (K_i, \mathcal{J}_{K_i})$ of $\mathbf{X}$ is a final epi–sink in **SUConv**. Then the source $(h_i : ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y}) \to ([\mathbf{K}_i, \mathbf{Y}], \mathcal{J}_{K_i,Y})_{i \in I}$ with $h_i(f) = f \circ j_i = f|K_i$ is an initial mono–source in **SUConv**. For each $i \in I, M_i = h_i[M]$ is uniformly equicontinuous (namely, since h_i is uniformly continuous, $h_i \times h_i((\Delta_M)) = (\Delta_{M_i}) \in \mathcal{J}_{K_i,Y}$ for each $i \in I$). Furthermore, for each $x_i \in K_i, M_i(x_i) = M(x_i)$ is precompact by assumption. Thus, using the first case of this proof, M_i is precompact for each $i \in I$. Since $(h_i|M : (M, (\mathcal{J}_{X,Y})_M) \to (M_i, (\mathcal{J}_{K_i,Y})_{M_i})_{i \in I}$ is an initial mono–source in **SUConv**, $(M, (\mathcal{J}_{X,Y})_M)$ is precompact (cf. [18;4.2]).

3.16 REMARK. 1) *Every uniform space* (=principal uniform limit space) $(X, \mathcal{J}_X)$ *is a semi–Ascoli space* (cf. [23;4.7] and note that every Ascoli–space in the sense of Wyler is a semi–Ascoli space).

2) The function space $([\mathbb{R}_u, \mathbb{R}_u], \mathcal{J}_{\mathbb{R},\mathbb{R}})$ considered under 3.11 is a *semi–Ascoli space* (since it is an Ascoli–space in the sense of Wyler by [23;4.4]), *which is not uniform* (otherwise it would be diagonal).

3.17 COROLLARY 1. (*Necessary conditions for compactness in* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$).
Let $\mathbf{X} = (X, \mathcal{J}_X)$ *be a semiuniform convergence space and* $\mathbf{Y} = (Y, \mathcal{J}_Y)$ *is a semi–pseudouniform* T_2*–space. If* M *is a compact subset of* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$*, then the following are satisfied:*

(1) M *is closed in* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$*, i.e.* $M = \overline{M}$*, where* $\overline{M}$ *denotes the closure of* M *with respect to the underlying Kent convergence space of* $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$*,*

(2) M *is uniformly equicontinuous,*

(3) $M(x) \subset Y$ *is compact for each* $x \in X$.

PROOF. (1) By 2.20 and 2.21, $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is a T_2–space. Since M is compact, M is closed (cf. [9;3.16.14.]).

(2) is obvious, since M is precompact and 3.12 is valid.

(3) cf. the corresponding proof under 3.12 and replace "precompact" by "compact".

3.18 COROLLARY 2 (*Sufficient conditions for compactness in* $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$).
Let $\mathbf{X}=(X,\mathcal{J}_X)$ be a locally precompact uniform limit space and $\mathbf{Y}=(Y,\mathcal{J}_Y)$ a regular complete semi–Ascoli space (e.g. a complete uniform space). Then $M\subset[\mathbf{X},\mathbf{Y}]$ is a compact subset of $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$ provided that the following are satisfied:

(1) M is closed in $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$,

(2) M is uniformly equicontinuous,

(3) $M(x)\subset Y$ is compact for each $x\in X$.

PROOF. By 2.18 and 3.2.1), $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$ is complete. By 3.15, M is precompact. Since M is closed, M is complete too (cf. [18;3.12]). Thus, M is compact.

3.19 COROLLARY 3. *Let $(X,\mathcal{X})$ be a compact topological T_2–space and $(Y,\mathcal{W})$ a separated uniform space (= uniform T_2–space). Further, let $C(X,Y)$ be the set of all continuous maps from $(X,\mathcal{X})$ into $(Y,\mathcal{Y}_{\mathcal{W}})$, where $\mathcal{Y}_{\mathcal{W}}$ denotes the topology on Y induced by the uniformity $\mathcal{W}$ for Y. If $C_u(X,Y)$ is the set $C(X,Y)$ endowed with the uniform struture of uniform convergence (cf. also* [18;7.6.2)]*), then $M\subset C_u(X,Y)$ is relatively compact, i.e. the closure $\overline{M}$ of M with respect to the topology of uniform convergence is compact, if and only if M is equicontinuous and for each $x\in X$, the set $M(x)$ is relatively compact in $(Y,\mathcal{W})$.*

PROOF. " $\Leftarrow$ ". Let $\mathcal{U}$ be the unique uniformity inducing $\mathcal{X}$. It follows from the assumptions on $(X,\mathcal{X})$ and $(Y,\mathcal{W})$ that $C_u(X,Y)=C_c(X,Y)$, where $C_c(X,Y)$ denotes the set $C(X,Y)$ endowed with the uniformity of compact convergence (cf. [21;II 4.5]). Furthermore, equicontinuity of M means uniform equicontinuity (cf. [4;chap. 10,§2, no 1, corollaire 2]). Thus, the closure $\overline{M}$ of M in $C_u(X,Y)$ coincides with the closure of M in Y^X endowed with the product uniformity [= structure of simple convergence] (cf. [4;chap. 10,§3, no 4, corollaire]). Since $p_x[\overline{M}]=\overline{M}(x)\subset\overline{p_x[M]}=\overline{M(x)}$, where $p_x:Y^X\to Y$ denotes the x–th projection for each $x\in X$, $\overline{M}(x)$ is relatively compact and consequently precompact for each $x\in X$. Thus, by 3.15, $\overline{M}$ is precompact in $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$, where $\mathbf{X}=(X,\mathcal{J}_X)$ and $\mathbf{Y}=(Y,\mathcal{J}_Y)$ with $\mathcal{J}_X=[\mathcal{U}]$ and $\mathcal{J}_Y=[\mathcal{W}]$ respectively (note: According to [4;chap. 10,§2, no 3, proposition 6], $\overline{M}$ is uniformly equicontinuous). Hence, by 3.10, $\overline{M}$ is precompact in $([\mathbf{X},\mathbf{Y}],\mathcal{J}^u_{X,Y})=C_u(X,Y)$. Since $\overline{M}\subset\prod_{x\in X}\overline{M}(x)\subset Y^X$ and $\overline{M}(x)$ is compact and consequently complete, $\overline{M}$ is complete with respect to the structure of simple convergence [as a closed subspace of a complete product space] and thus, $\overline{M}$ is complete with respect to the structure of compact convergence (cf. [4;chap. 10, §1, no 5, corollaire 2]), i.e. as a subspace of $C_c(X,Y)=C_u(X,Y)$. Hence, $\overline{M}$ is precompact and complete, i.e. $\overline{M}$ is compact.

" $\Rightarrow$ ". Let $N=\overline{M}$. By assumption, N is compact in $C_u(X,Y)=([\mathbf{X},\mathbf{Y}],\mathcal{J}^u_{X,Y})$, where $\mathbf{X}$ and $\mathbf{Y}$ are defined as under " $\Leftarrow$ ". Then N is compact in $([\mathbf{X},\mathbf{Y}],\mathcal{J}_{X,Y})$ too, since filter convergence with respect to $\mathcal{J}^u_{X,Y}$ coincides with filter convergence with respect to $\mathcal{J}_{X,Y}$ (use [13;3.5] and note that $\mathbf{X}$ and $\mathbf{Y}$ are diagonal uniform limit spaces). Thus, by

3.17, N is uniformly equicontinuous and $p_x[N] = N(x)$ is compact in Y for each $x \in X$. It follows from $M \subset N$ that M is uniformly equicontinuous (= equicontinuous) and, since $\mathbf{Y}$ is a T_2–space, $p_x[M] = M(x)$ is relatively compact in $\mathbf{Y}$, i.e. in $(Y, \mathcal{W})$ (cf. [18;5.8]), for each $x \in X$.

3.20 Remarks. 1) Note, that for the first part of the proof of 3.19 (" $\Leftarrow$ ") the uniform space $(Y, \mathcal{W})$ need not be separated.

2) If in 3.19 the uniform space $(Y, \mathcal{W})$ is metrizable, then $C_u(X, Y)$ is metrizable (cf. [4;chap. 10, §3, no 1, proposition 1]) and from 3.19 results the *classical Ascoli theorem*:

If $(X, \mathcal{X})$ is a compact topological T_2–space and $(Y, \mathcal{W})$ a metrizable uniform space, then the following are equivalent for each $M \subset C(X, Y)$:

(1) *Each sequence (f_n) in M has a subsequence converging uniformly to some $f \in C(X, Y)$,*

(2) *M is equicontinuous and $M(x)$ is relatively compact in $(Y, \mathcal{W})$ for each $x \in X$.*

If $(Y, \mathcal{W}) = \mathbb{R}_u$, then the above conditions are equivalent to

(3) *M is equicontinuous and $M(x)$ is bounded in $\mathbb{R}$ for each $x \in X$.*

3.21 Problem. Does there exist a semi–Ascoli space which is not an Ascoli space in the sense of Wyler [23]?

References

[1] Adámek, J., H. Herrlich, and G.E. Strecker: 1990, *Abstract and Concrete Categories*, Wiley–Intersci. Publ., New York.

[2] Behling, A.: 1992, *Einbettung uniformer Räume in topologische Universen*, Thesis, Free University, Berlin.

[3] Bentley, H.L., and H. Herrlich: 1984, 'Ascoli's theorem for a class of merotopic spaces', In: Convergence Structures (Proc. Conf. Convergence, Bchyně), pp. 47-53, *Mathematical Research* **24**, Akademie Verlag, Berlin.

[4] Bourbaki, N.: 1961, *Topologie générale*, Hermann, Paris.

[5] Čech, E.: 1966, *Topological Spaces*, Revised edition by M. Frolik and M. Katětov, Interscience, London.

[6] Cook, C.H., and H.R. Fischer: 1965, 'On equicontinuity and continuous convergence', *Math. Ann.* **159**, 94–104.

[7] Dubuc, E.J.: 1979, 'Concrete quasitopoi', In: Applications of Sheaves (Proc. Durham Conf., 1977), pp. 239-254, *Lecture Notes Math.* **753**, Springer, Berlin.

[8] Efremovič, V.A.: 1952, 'Geometry of proximity', *Math. Sb.* **31** (**73**), 189–200.

[9] Gähler, W.: 1977/78, *Grundstrukturen der Analysis*, Band 1 und 2, Birkhäuser, Basel.

[10] Gazik, R.J., D.C. Kent and G.D. Richardson: 1974, 'Regular completions of uniform convergence spaces', *Bull. Austral. Math. Soc.* **11**, 413–424.

[11] Gray, J.W.: 1982, 'Ascoli's theorem for topological categories', In: Categorical Aspects of Topology and Analysis (Proc. Carleton Conf., 1981), pp 86–104, *Lecture Notes Math.* **915**, Springer, Berlin.

[12] Kneis, G.: 1977, 'Zum Satz von Arzelà–Ascoli in pseudouniformen Räumen', *Math. Nachr.* **79**, 49–54.

[13] Lee, R.S.: 1978, *Function Spaces in the Category of Uniform Convergence Spaces*, Thesis, Carnegie–Mellon University, Pittsburgh.

[14] Poppe, H.: 1974, *Compactness in General Function Spaces*, Deutscher Verlag der Wissenschaften, Berlin.

[15] Preuß, G.: 1988, *Theory of Topological Structures*, Reidel, Dordrecht.

[16] Preuß, G.: 1991, 'Point separation axioms, monotopological categories and MacNeille completions', In: Category Theory at Work (H. Herrlich and H.–E. Porst, eds.), pp.47–55, *Research and Exposition in Math.* **18**, Heldermann, Berlin.

[17] Preuß, G.: 1993, 'Cauchy spaces and generalizations', *Math. Japonica* **38**, 803–812.

[18] Preuß, G.: 1995, 'Semiuniform convergence spaces' , *Math. Japonica* **41**, 465–491.

[19] Preuß, G.: 1996, 'Local compactness in semiuniform convergence spaces', *Quaest. Math.* **19** (to appear).

[20] Preuß, G.: 199?, 'The topological universe of locally precompact semiuniform convergence spaces', Special issue of *Topology Appl.* in memory of K. Morita (to appear).

[21] Schubert, H.: 1964, *Topologie*, Teubner, Stuttgart.

[22] Weil, A.: 1937, *Sur les espaces à structures uniformes et sur la topologie générale*, Hermann, Paris.

[23] Wyler, O.: 1984, 'Ascoli theorems for cartesian closed categories', In: Categorical Topology (Proc. Conf. Toledo, Ohio 1983), pp. 599-617, *Sigma Series in Pure Mathematics* **5**, Heldermann, Berlin.