On a Semilinear Schrödinger Equation with Periodic Potential

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1 Introduction

We study the semilinear Schrödinger equation

$$(P) \qquad -\Delta u + V(x)u = f(x,u) \quad \text{in} \quad I\!R^N$$

with a periodic, continuous potential V and superquadratic, subcritical nonlinearity f, $f(x,0)=0$, with the same period in x as V. We look for nontrivial solutions of (P) having finite energy. There is a number of papers dealing with this problem. At first, the so-called definite case (when V is strictly positive) was considered. Using the Nehari variational principle, the first author [3] has proved an existence theorem for ground states, i.e. solutions having smallest energy among all nontrivial solutions. (Even the more general asymptotically almost periodic case was treated there). Later on P. Rabinowitz [4] has obtained the existence of a nontrivial solution (not necessary a ground state) under less restrictive assumptions on f. Moreover, in [1] V. Coti Zelati and P. Rabinowitz have proved the existence of infinitely many solutions, imposing some additional (perhaps, technical) conditions.

For general (indefinite) V it is known that the operator $-\Delta + V$ on $L^2(I\!R^N)$ has a purely continuous spectrum consisting of closed disjoint intervals. In recent papers, C. Troestler and M. Willem [8], and W. Kryszewski and A. Szulkin [7] have proved the existence of a nontrivial solution of (P) under the assumption that 0 is in a gap of the spectrum of $-\Delta + V$. Their proofs are based on a decomposition of $H^1(I\!R^N)$ into two infinite dimensional subspaces and a generalized linking theorem which is applied to the energy functional associated to (P). This linking theorem requires the construction of a new degree theory in order to handle the lack of compactness in this problem. Moreover, in [7] it is stated that there exists infinitely many solutions provided the nonlinearity f is odd.

In this paper we present another proof for the existence of nontrivial solutions which is based on an approximation technique with periodic functions. Such an approach was used by P. Rabinowitz [4] in the positive case. We shall show that this procedure leads to a nontrivial solution of (P) without constructing any new degree theory.

Denote by $F(x,u) = \int_0^u f(x,\xi)d\xi$ the primitive function of f. We make the following assumptions:

1^o $f : I\!R^N \times I\!R \to I\!R$ and $V : I\!R^N \to I\!R$ are continuous and 1-periodic functions in each x_i, $i = 1, \ldots, N$.

2^o $|f(x,u)| \le C(1+|u|^{p-1})$, where $2 < p < 2N(N-2)$.

3^o $f(x,u) = o(|u|)$, as $u \to 0$, uniformly in x.

4^o There exists $\gamma > 2$ such that $0 < \gamma F(x,u) \leq uf(x,u)$ for every $u \neq 0$.

5^o 0 is in a spectral gap of the operator $-\Delta + V$.

One of the main results of this paper is

Theorem 1 *Under assumptions 1^o–5^o there exists a nontrivial solution of (P).*

To prove this result, we consider a sequence of approximating problems in the next section and prove a general theorem which ensures the existence of a nontrivial limit. In Section 3 we apply the Linking Theorem of Rabinowitz to our approximating problems. This gives us a sequence of periodic solutions which leads to a solution of (P). In Section 4 we show that these solutions are asymptotically equal to a finite sum of $H^1(\mathbb{R}^N)$-solutions. In the last Section 5 we prove the existence of a ground state solution by means of Nehari's variational principle and periodic approximations. Additionally we prove that, on the cube of periods, these solutions look like a ground state when the period is large enough (see (18)).

It should be pointed out that all the results presented here have natural counterparts in the case of periodic Hamiltonian systems.

2 Approximation by periodic functions

Associated with (P) we study the approximate problem in cubes Q_k of $\mathbb{R}^N$ with edge length $k \in \mathbb{N}$:

$$(P)_k \qquad -\Delta u + V(x)u = f(x,u) \quad \text{in} \quad Q_k\,, \quad u \in E_k = H^1_{per}(Q_k)\,,$$

where $H^1_{per}(Q_k)$ denotes the space of $H^1(Q_k)$-functions which are k-periodic in x_i, $i = 1, \ldots, N$. The operator $-\Delta + V$ on $L^2_{per}(Q_k)$ has discrete spectrum with eigenvalues $\lambda_{k,1} \leq \lambda_{k,2} \leq \ldots \to \infty$ and it exists a finite

$$j(k) = \min\{i \mid \lambda_{k,i} > 0\}\,.$$

Moreover, every eigenvalue $\lambda_{k,i}$ is contained in the spectrum of $-\Delta + V$ on the whole space. This follows from the spectral decomposition theorem XIII.97 in [6]. Therefore, if $(-\alpha, \beta)$, $\alpha, \beta > 0$ denots the spectral gap around 0, assumed in 5^o, we claim that $\lambda_{k,i} \notin (-\alpha, \beta)$ for every $k, i \in \mathbb{N}$. We denote by $\phi_{k,i}$ the corresponding eigenfunctions.

Since every function $u \in E_k$ is, by periodicity, also in E_{mk} for every $m \in I\!N$, we claim that every eigenvalue of $-\Delta + V$ on $L^2_{per}(Q_k)$ is also an eigenvalue of this operator in $L^2_{per}(Q_{mk})$, $m \in I\!N$.

Now we define an orthogonal decomposition of E_k by

$$E_k = Y_k \oplus Z_k\,, \qquad \text{where} \qquad Y_k = \{\phi_{k,1}, \ldots, \phi_{k,j(k)-1}\}\,, \quad Z_k = Y_k^{\perp}\,.$$

The associated energy functional to $(P)_k$ is

$$J_k(u) = \frac{1}{2}\int_{Q_k} |\nabla u|^2 + V(x)|u|^2\,dx - \int_{Q_k} F(x,u)\,dx$$

and the quadratic part of J_k,

$$\ell_k(u) = \int_{Q_k} |\nabla u|^2 + Vu^2 dx\,,$$

is positive on Z_k and negative on Y_k. Moreover, we may define a new scalar product $(\cdot,\cdot)_k$ on E_k with corresponding norm $\|\cdot\|_k$, such that

$$\begin{aligned}\int_{Q_k} |\nabla u|^2 + V(x)|u|^2\,dx &= -\|u\|_k^2 \quad \text{for} \quad u \in Y_k\,,\\ \int_{Q_k} |\nabla u|^2 + V(x)|u|^2\,dx &= \|u\|_k^2 \quad \text{for} \quad u \in Z_k\,.\end{aligned}$$

If we denote by $P_k : E_k \to Y_k$ and $T_k : E_k \to Z_k$ the orthogonal projections, our functional becomes

$$J_k(u) = \frac{1}{2}(\|T_k u\|_k^2 - \|P_k u\|_k^2) - \int_{Q_k} F(x,u)\,dx.$$

From our assumptions it follows that J_k is C^1 on E_k and its derivative is given by the formula

$$\langle J_k'(u), v\rangle = (T_k u, v)_k - (P_k u, v)_k - \int_{Q_k} f(x,u)v\,dx\,.$$

Lemma 1 *The norm $\|\cdot\|_k$ is equivalent to the standard norm $\|\cdot\|_{H^1}$ in $H^1(Q_k)$, with constants not depending on k.*

Proof. Let $C_0 = \sup|V(x)|$ and $(-\alpha,\beta)$, $\alpha,\beta > 0$ be the spectral gap of $-\Delta+V$ around 0. For $u \in E_k$ we have

$$\|u\|_k^2 \le \int_{Q_k} |\nabla u|^2 + C_0|u|^2\,dx \le C_1\|u\|_{H^1}^2\,.$$

On the other hand, for $z \in Z_k$ and $\varepsilon > 0$

$$\begin{aligned}
\|z\|_k^2 &= \varepsilon \left(\int_{Q_k} |\nabla z|^2 + V|z|^2 dx \right) + (1-\varepsilon) \left(\int_{Q_k} |\nabla z|^2 + V|z|^2 dx \right) \\
&\geq \varepsilon \int_{Q_k} |\nabla z|^2 dx - \varepsilon C_0 \int_{Q_k} |z|^2 dx + (1-\varepsilon)\beta \int_{Q_k} |z|^2 dx \, .
\end{aligned}$$

For ε small enough this implies

$$\|z\|_k^2 \geq C_2 \|z\|_{H^1}^2 \, ,$$

where C_2 only depends on C_0, β and ε. A similar argument for $y \in Y_k$ shows that

$$\begin{aligned}
\|y\|_k^2 &\geq \varepsilon \int_{Q_k} |\nabla y|^2 dx - \varepsilon C_0 \int_{Q_k} |y|^2 dx + \alpha \int_{Q_k} |y|^2 dx \\
&\geq C_3 \|y\|_{H^1}^2
\end{aligned}$$

with C_3 depending on C_0, α and ε. □

Now we want to show that any sequence u_k of critical points of J_k has a nontrivial limit in $H^1_{loc}(I\!R^N)$ which is a solution of (P), provided the sequence $J_k(u_k)$ is bounded from above. To this end we need the following results concerning the behaviour of such sequences.

Lemma 2 *Any critical point u_k of J_k satisfies the estimate $\|u_k\|_k \leq C_k$ with a constant $C_k > 0$ only depending on the critical value.*

Proof. From $c_k = J_k(u_k)$ and $J_k'(u_k) = 0$ we obtain

$$\begin{aligned}
c_k &= J_k(u_k) - \frac{1}{2} \langle J_k'(u_k), u_k \rangle \\
&\geq (\frac{1}{2} - \frac{1}{\gamma}) \int_{Q_k} f(x, u_k) u_k dx \, . \qquad (1)
\end{aligned}$$

From Assumptions $2^o - 4^o$ we have the inequalities

$$\begin{aligned}
|f(x,u)|^2 &\leq Cuf(x,u), \quad \text{if} \quad |u| \leq 1, \qquad (2) \\
|f(x,u)|^{p'} &\leq C|u|^{(p-1)(p'-1)} |f(x,u)| = Cuf(x,u), \quad \text{if} \quad |u| > 1, \qquad (3)
\end{aligned}$$

where $p' = p/(p-1)$ is the conjugate exponent. Denote $B_k = \{x \in Q_k \mid |u_k(x)| \leq 1\}$. Then from (1)–(3) we get

$$c_k \geq C_1 \left(\int_{B_k} |f(x,u_k)|^2 dx + \int_{Q_k \setminus B_k} |f(x,u_k)|^{p'} dx \right) \qquad (4)$$

with $C_1 = (\gamma - 2)/(2\gamma C)$. Hence

$$I_1 := \int_{B_k} |f(x, u_k)|^2 dx \le c_k / C_1, \quad \text{and} \quad I_2 := \int_{Q_k \setminus B_k} |f(x, u_k)|^{p'} dx \le c_k / C_1 .$$

Now set $y_k = P_k u_k$ and $z_k = T_k u_k$. From $\langle J_k'(u_k), y_k \rangle = 0$ and Hölder's inequality, we see that

$$\begin{aligned} \|y_k\|_k^2 &= -\int_{Q_k} f(x, u_k) y_k dx \\ &\le I_1^{1/2} \|y_k\|_{L^2} + I_2^{1/p'} \|y_k\|_{L^{p'}} , \end{aligned}$$

where $|\cdot|_q$ denotes the usual L^q-norm. Then the Sobolev embedding theorem implies

$$\|y_k\|_k^2 \le C_2 \left(I_1^{1/2} + I_2^{1/p'} \right) \|y_k\|_k .$$

Clearly, the same argument works for z_k and we obtain

$$\|u\|_k \le 2C_2 \left(\left(\frac{c_k}{C_1} \right)^{1/2} + \left(\frac{c_k}{C_1} \right)^{1/p'} \right) = C_k$$

with C_1 and C_2 independent of k. □

Lemma 3 *There exists $\varepsilon_1 > 0$ not depending on k such that $\|u_k\|_k \ge \varepsilon_1$, $\|u\|_{H^1} \ge \varepsilon_1$ holds for any nontrivial critical point u_k of J_k and u of J. In addition, there exists a constant $\varepsilon_2 > 0$ not depending on k such that $J_k(u_k) \ge \varepsilon_2$, $J(u) \ge \varepsilon_2$ for any nontrivial critical point u_k of J_k and u of J.*

Proof. Let $\ell_k(u)$ be the quadratic part of J_k. Since $(-\alpha, \beta)$ is in the spectral gap, there is a constant $C_0 = C_0(\alpha, \beta) > 0$, such that

$$|\ell_k(u)| \ge C_0 \, \|u\|_{L^2(Q_k)}^2 .$$

Let $C_1 = |\min\{V(x)\}|$. Then we obtain

$$\begin{aligned} |\ell_k(u)| &= \varepsilon \, | \int_{Q_k} |\nabla u|^2 + V u^2 dx| + (1 - \varepsilon) | \int_{Q_k} |\nabla u|^2 + V u^2 dx| \\ &\ge \varepsilon \int_{Q_k} |\nabla u|^2 dx - \varepsilon C_1 \, \|u\|_{L^2}^2 + (1 - \varepsilon) C_0 \, \|u\|_{L^2}^2 \\ &\ge C_2 \|u\|_k^2 \end{aligned}$$

for ε small enough, and $C_2 > 0$ does not depend on k. Now let u_k be a nontrivial critical point of J_k. Then we claim from $\langle J_k'(u_k), u_k\rangle = 0$ and Assumption 4^o, that $\ell_k(u_k) \geq 0$. We obtain

$$\begin{aligned}
\|u\|_k^2 &\leq \frac{1}{C_2}\ell_k(u) \\
&= \frac{1}{C_2}\int_{Q_k} f(x,u_k)u_k\,dx \\
&\leq \frac{1}{C_2}(\varepsilon\|u\|_k^2 + C_\varepsilon\|u\|_k^p)\,.
\end{aligned}$$

Consequently,

$$\|u\|_k \geq \left(\frac{1-\varepsilon/C_2}{C_\varepsilon}\right)^{\frac{1}{p-2}} \tag{5}$$

and ε can be choosen such that $1-\varepsilon/C_2 > 0$. Again this estimate is independent of k and also holds for critical points of J. For the corresponding critical value we obtain (with $s > 0$ satisfying $1/s + 1/\gamma = 1/2$, γ from Assumption 4^o)

$$\begin{aligned}
J_k(u_k) &\geq \frac{1}{2}\ell_k(u_k) - \frac{1}{\gamma}\int_{Q_k} f(x,u_k)u_k\,dx \\
&= \frac{1}{s}\ell_k(u_k) + \frac{1}{\gamma}\underbrace{(\ell_k(u_k) - \int_{Q_k} f(x,u_k)u_k\,dx)}_{=0} \\
&\geq \frac{C_2}{s}\|u\|_k^2\,.
\end{aligned}$$

This shows, together with (5), that

$$J_k(u_k) \geq \frac{C_2}{s}\left(\frac{1-\varepsilon/C_2}{C_\varepsilon}\right)^{\frac{2}{p-2}}$$

with constants not depending on k. Clearly the same arguments work for J. $\square$

Lemma 4 *Let Q_n be the cube of edge length $l_n \to \infty$ centered at the origin, and $K_r(\xi)$ be the closed r-cube centered at the point ξ. Consider a sequence $u_n \in H^1_{loc}(I\!R^N)$ of l_n-periodic functions such that $\|u_n\|_{H^1(Q_n)} \leq C$. Assume that there is $r > 0$ such that*

$$\liminf_{n\to\infty}\left(\sup_\xi \int_{K_r(\xi)} |u_n|^2 dx\right) = 0.$$

Then $\|u_n\|_{L^q(Q_n)} \to 0$ for $q \in (2, 2N/(N-2))$.

Proof. Let us denote by Q_n' (resp. Q_n'') the cube of edge length $l_n + r$ (resp. $l_n + 2r$) centered at the origin. Choose a cut-of function $\chi_n \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \chi_n(x) \le 1$, $\chi_n = 1$ on Q_n', supp $\chi_n \subset Q_n''$ and $|\nabla\chi_n(x)| \le C$, $x \in \mathbb{R}^N$, where $C > 0$ does not depend on n. Evidently, such a function exists.

Now we set $v_n = \chi_n u_n$. For n being large enough it is easy to verify (using the periodicity of u_n) that

$$\sup_\xi \int_{K_r(\xi)} |v_n|^2 dx = \sup_\xi \int_{K_r(\xi)} |u_n|^2 dx$$

and v_n is bounded in $H^1(\mathbb{R}^N)$. By the well-known concentration lemma of P.-L. Lions (see [2], Lemma I.1) $v_n \to 0$ in $L^q(\mathbb{R}^N)$, $q \in (2, 2N/(N-2))$. Since $u_n = v_n$ on Q_n, we get the assertion. □

Lemma 5 *If a sequence $u_k \in E_k$ is uniformly bounded in the $H^1(Q_k)$-norm and satisfies $J_k'(u_k) \to 0$, then the following alternative holds: Either*

(i) $\|u_k\|_k \to 0$ as $k \to \infty$; or

(ii) there exist numbers $r, \eta > 0$ and a sequence of points $\xi_k \in \mathbb{R}^N$, such that

$$\liminf_{k\to\infty} \left(\int_{K_r(\xi_k)} |u_k|^2 dx \right) \ge \eta\,.$$

Proof. Assume that (ii) does not hold. Then we know from Lemma 4 that $\|u_k\|_{L^q(Q_k)} \to 0$ for $2 < q < 2N/(N-2)$. Set $y_k = P_k u_k$, $z_k = T_k u_k$. From $\langle J_k'(u_k), z_k\rangle \le \varepsilon_k \|z_k\|_k$ with $\varepsilon_k \to 0$, Assumptions $2^o, 3^o$ and Hölder's inequality, we get

$$\begin{aligned} \|z_k\|_k^2 &\le \int_{Q_k} f(x, u_k) z_k\, dx + \varepsilon_k \|z_k\|_k \\ &\le \varepsilon \|u_k\|_{L^2} \|z_k\|_{L^2} + C_\varepsilon \|u_k\|_{L^p}^{p-1} \|z_k\|_{L^p} + \varepsilon_k \|z_k\|_k\,. \end{aligned}$$

This implies that $\|z_k\|_k \to 0$. In the same way it follows that $\|y_k\|_k \to 0$. □

Theorem 2 *Let Assumptions 1^o–5^o be satisfied and let $u_k \in E_k$ be a sequence of nontrivial critical points of J_k such that $c_k = J_k(u_k)$ is bounded from above with a constant not depending on k. Then there exists a nontrivial solution $u \in H^1(\mathbb{R}^N)$ of (P).*

Proof. Since $\|u_k\|_k$ is uniformly bounded (by Lemma 2) and $J_k(u_k) \ge \varepsilon_2 > 0$ with ε_2 not depending on k (Lemma 3), case (i) of Lemma 5 is not possible. From (ii) we obtain (at least for a subsequence)

$$\|u_k\|_{L^2(K_r(\xi_k))}^2 \ge \eta/2\,.$$

Thus, we find a sequence of integer vectors $b_k \in \mathbb{Z}^N$ and a number $s > 0$, such that the sequence $\tilde{u}_k$, defined by $\tilde{u}_k(x) = u_k(x + b_k)$, satisfies

$$\|\tilde{u}_k\|^2_{L^2(K_s(0))} \geq \eta/2\,. \tag{6}$$

Since V and f are invariant under translations by b_k, we have $J_k(\tilde{u}_k) = J_k(u_k)$ and $\|J_k'(\tilde{u}_k)\| = \|J_k'(u_k)\|$. Lemma 2 implies that the sequence $\tilde{u}_k$ is uniformly bounded in $\|\cdot\|_k$ and consequently converges weakly in $H^1_{loc}(\mathbb{R}^N)$ to a function $u \in H^1(\mathbb{R}^N)$. Then, for a subsequence, we have $f(x, \tilde{u}_k) \to f(x, u)$ in $L^1_{loc}(\mathbb{R}^N)$ and we obtain for any test function $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}
\langle J'(u), \varphi\rangle &= \int_{\mathbb{R}^N} \nabla u \nabla\varphi + V u\varphi - f(x,u)\varphi\, dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^N} \nabla \tilde{u}_k \nabla\varphi + V \tilde{u}_k\varphi - f(x,\tilde{u}_k)\varphi\, dx \\
&= 0\,.
\end{aligned}$$

Therefore, u is a weak solution of (P) and is not trivial in view of (6). $\square$

3 Existence of periodic critical points

In this section we construct a sequence of critical points which satisfies the assumptions of Theorem 2.

Lemma 6 *There exist real numbers $\delta, r > 0$ not depending on k, such that*

$$\inf_{u\in N_k} J_k(u) \geq \delta\,, \qquad \textit{where} \qquad N_k = \{z \in Z_k \mid \|z\|_k = r\}\,.$$

Proof. For $z \in Z_k$ we have

$$J_k(z) = \frac{1}{2}\|z\|_k^2 - \int_{Q_k} F(x,z)\, dx$$

and from our growth conditions $2^o, 3^o$ we claim that for every $\varepsilon > 0$ there is a constant C_ε, such that $|F(x,u)| \leq \varepsilon|u|^2 + C_\varepsilon|u|^p$. From the Sobolev embedding theorem we obtain a constant C, not depending on k, such that

$$\int_{Q_k} F(x,z)\, dx \leq C\left(\varepsilon\|z\|_k^2 + C_\varepsilon\|z\|_k^p\right)$$

holds. Consequently, for ε and $\|z\|_k$ small enough, we obtain the assertion. $\square$

Now for each k we fix a function $z_k \in Z_k$ with $\|z_k\|_k = 1$. For $\rho > 0$ we define the sets

$$M_k = \{y + tz_k \mid \|y + tz_k\|_k < \rho,\ t \geq 0,\ y \in Y_k\}.$$

Lemma 7 *There exists a $\rho > r$ not depending on k, such that*

$$\sup_{u \in \partial M_k} J_k(u) = 0\,.$$

Proof. Remark that

$$\partial M_k = \{y + tz_k \mid (\|y + tz_k\|_k = \rho,\ t \geq 0) \ \text{ or } \ (\|y + tz_k\|_k \leq \rho,\ t = 0)\}\ .$$

Since $J_k(0) = 0$, we have to show $\sup_{u \in \partial M_k} J_k(u) \leq 0$. For $y + tz_k \in M_k$ we have

$$J_k(y + tz_k) = \frac{1}{2}t^2 - \frac{1}{2}\|y\|_k^2 - \int_{Q_k} F(x, y + tz_k)\,dx\,.$$

From our Assumptions $2^o - 4^o$ we claim that for every $\varepsilon > 0$ there is a constant C_ε, such that $F(x,u) \geq -\varepsilon|u|^2 + C_\varepsilon |u|^\gamma$ where $2 < \gamma < 2N/(N-2)$. It follows that

$$\int_{Q_k} F(x, y + tz_k)\,dx \geq -\varepsilon\|y\|_{L^2}^2 - \varepsilon t^2 \|z_k\|_{L^2}^2 + C_\varepsilon \|y + tz_k\|_{L^\gamma}^\gamma\,.$$

From the Sobolev embedding theorem we obtain with a constant C not depending on k

$$J_k(y + tz_k) \leq -\frac{1}{2}\|y\|_k^2 + \varepsilon C\,\|y\|_k^2 + \frac{1}{2}t^2 + \varepsilon C t^2 - C_\varepsilon \|y + tz_k\|_{L^\gamma}^\gamma\,. \tag{7}$$

Now $X_k = Y_k \oplus \mathbb{R} z_k$ is continuously embedded in $L^\gamma(Q_k)$ and there exists a continuous projection $\pi_k : X_k \to \mathbb{R} z_k$ with norm $\|\pi_k\| = C_k$, i. e. $\|tz_k\|_{L^\gamma} \leq C_k \|y + tz_k\|_{L^\gamma}$. Since z_k is not trivial, we clearly have $C_k \geq 1$ for every k. Choosing ε in (7) so small that $\varepsilon C = 1/4$, we get

$$J_k(y + tz_k) \leq -\frac{1}{4}\|y\|_k^2 + \frac{3}{4}t^2 - C_\varepsilon C t^\gamma\,,$$

where we have used again the Sobolev embedding. Note that the right hand side of this estimate does not depend on k. Consequently, if $t = 0$, we have $J_k(y) \leq 0$ and for $\|y + tz_k\|_k \to \infty$ we have $J_k(y + tz_k) \to -\infty$. This shows that we can choose ρ independent of k such that Lemma 7 holds. $\square$

Lemma 8 *For the set of deformations* $\Gamma_k = \{h \in C^0(E_k, E_k) \mid h_{|\partial M_k} = id\}$ *the numbers*

$$c_k = \inf_{h\in\Gamma_k} \sup_{u\in M_k} J(h(u))$$

are critical values of J_k *and satisfy* $0 < \delta \leq c_k \leq M < \infty$.

Proof. Since Y_k is finite-dimensional and N_k and ∂M_k link, the first assertion follows from the Linking Theorem 4.6 of P. Rabinowitz [5]. This theorem also shows that $c_k \geq \delta$, where $\delta > 0$ is independent of k by Lemma 2.

Since the constant ρ in the definition of M_k is also independent of k, and J_k is bounded on bounded sets, we obtain (taking $h = \mathrm{id}$)

$$c_k \leq \sup_{u\in M_k} J_k(u) \leq M\,,$$

where M only depends on ρ. □

The proof of Theorem 1 now follows immediately from Lemma 8 and Theorem 2.

4 Further properties of periodic solutions

In this section we show that periodic solutions of (P) with large period look like a finite sum of finite energy solutions in the whole space, i. e. they converge to a sum of $H^1(\mathbb{R}^N)$-solutions as the period tends to ∞. We state even a little more general assertion dealing with periodic "almost solutions", namely, Theorem 3 below. It will be used, in full generality, to prove Theorem 4 of Section 5. This statement is the periodic analog of Proposition 4.2 of [7] and Proposition 2.13 of [1] (in the positive definite case).

We need the following additional assumption on f:

6^o There exist positive constants C and ε such that for all $x \in \mathbb{R}^N$

$$|f(x, u+v) - f(x,u)| \leq C|v|(1 + |u|^{p-1}) \quad \text{if} \quad |v| \leq \varepsilon\,.$$

Under this additional assumption we have

Theorem 3 *Assume* 1^o– 6^o. *Let* $u_k \in E_k$ *be a uniformly bounded sequence which satisfies* $J_k'(u_k) \to 0$ *and* $c_k = J_k(u_k) \to c > 0$. *Then there exist critical points* u^i, $i = 1, \ldots, n$ *of* J *and sequences* $b_k^i \in \mathbb{Z}^N$ *such that*

$$\|u_k - \sum_{i=1}^n u^i(\cdot + b_k^i)\|_k \to 0 \quad \textit{and} \quad \sum_{i=1}^n J(u^i) = c\,.$$

Proof. We fix the following notations. Lemma 5 shows that there exist $b_k \in \mathbb{Z}^N$ such that $\tilde{u}_k = u_k(\cdot + b_k) \to u$ weakly in H^1_{loc} and strongly in L^p_{loc} and $u \in H^1(\mathbb{R}^N)$ is a nontrivial solution of (P). Let $\chi_k \in C_0^\infty$ be a function such that $0 \leq \chi_k \leq 1$, $\chi_k = 1$ on Q_{k-1}, $\chi_k = 0$ outside of Q_k and $|\nabla \chi_k| \leq C$. Set

$$w_k = \chi_k u \quad \text{ and } \quad v_k = \tilde{u}_k - w_k$$

(w_k may be either viewed as a function in E_k or in $H^1(\mathbb{R}^N)$). Since $w_k \to u$ in $H^1(\mathbb{R}^N)$, we claim that

$$\|\tilde{u}_k - u\|_k \to 0 \quad \text{iff} \quad \|v_k\|_k \to 0. \tag{8}$$

First we state the following assertions:

(i) For any $\varepsilon > 0$ there exists $\Omega \subset \mathbb{R}^n$ and $k_\varepsilon \in \mathbb{N}$ such that for $k \geq k_\varepsilon$

$$\|w_k\|_{H^1(\mathbb{R}^N \setminus \Omega)} \leq \varepsilon, \quad \|w_k\|_{L^\infty(\mathbb{R}^N \setminus \Omega)} \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus \Omega} F(x, w_k) dx \leq \varepsilon\,.$$

(ii) $J_k'(v_k) \to 0$ as $k \to \infty$.

(iii) $J_k(v_k) \to c - J(u)$ as $k \to \infty$.

(i) The L^∞- estimate follows from [4], the others are clear.

(ii) Since $J \in C^1$, we have $J'(w_k) \to 0$ and $J_k(w_k) = J(w_k) \to J(u)$. Since every function $\varphi_k \in E_k$ can be extended (by periodicity and cut off) to a function $\tilde{\varphi}_k \in H^1(\mathbb{R}^N)$ such that $\tilde{\varphi}_k = \varphi_k$ on Q_k and $\|\tilde{\varphi}_k\|_{H^1(\mathbb{R}^N)} \leq C \|\varphi_k\|_k$ with C not depending on φ and k, we obtain

$$\langle J_k'(w_k), \varphi_k \rangle = \langle J'(w_k), \tilde{\varphi}_k \rangle \leq \varepsilon_k \|\tilde{\varphi}_k\|_{H^1(\mathbb{R}^N)} \leq \varepsilon_k C \|\varphi_k\|_k$$

for all $\varphi_k \in E_k$, where $\varepsilon_k \to 0$. This shows $J_k'(w_k) \to 0$. For v_k we have

$$\begin{aligned} \langle J_k'(v_k), \varphi_k \rangle &= \langle J_k'(\tilde{u}_k), \varphi_k \rangle - \langle J_k'(w_k), \varphi_k \rangle \\ &\quad + \int_{Q_k} [f(x, \tilde{u}_k) - f(x, w_k) - f(x, \tilde{u}_k - w_k)]\, \varphi_k \, dx\,. \end{aligned} \tag{9}$$

Since $\tilde{u}_k \to u$ locally in L^p, we get for any bounded domain $\Omega \subset \mathbb{R}^N$ and k large enough

$$\left| \int_\Omega [f(x, \tilde{u}_k) - f(x, w_k) - f(x, \tilde{u}_k - w_k)]\, \varphi_k \, dx \right| \leq \varepsilon \|\varphi_k\|_k\,.$$

Since $w_k \in H^1(\mathbb{R}^N)$, we also have

$$\left| \int_{Q_k \setminus \Omega} f(x, w_k) \varphi_k \, dx \right| \leq \varepsilon \|\varphi_k\|_k$$

for large k and appropriate Ω. Finally, from Assumption 6^o and (i) we get

$$\int_{Q_k\setminus\Omega} |f(x,v_k-w_k)-f(x,v_k)|\,|\varphi_k|\,dx \;\le\; \int_{Q_k\setminus\Omega} C(1+|v_k|^{p-1})|w_k|\,|\varphi_k|dx$$

$$\begin{aligned}
&\le\; C\|\varphi_k\|_k\left[\|w_k\|_{H^1(\mathbb{R}^N\setminus\Omega)}+\|w_k\|_{L^\infty(\mathbb{R}^N\setminus\Omega)}\|v_k\|^{p-1}_{H^1(\mathbb{R}^N\setminus\Omega)}\right]\\
&\le\; \varepsilon\|\varphi_k\|_k\,.
\end{aligned}$$

Thus (9) implies (ii).

(iii) We have

$$\begin{aligned}
J_k(\tilde u_k) \;&=\; J_k(v_k)+J_k(w_k)+\int_{Q_k}\nabla v_k\nabla w_k+Vv_kw_k\,dx\\
&\quad -\int_{Q_k}F(x,v_k+w_k)-F(x,v_k)-F(x,w_k)\,dx\,. \qquad (10)
\end{aligned}$$

For $\Omega\subset\mathbb{R}^N$ and k large enough, the first integral can be estimated by

$$\int_\Omega \nabla v_k\nabla u+Vv_ku\,dx+C\|v_k\|_k\,\|w_k\|_{H^1(\mathbb{R}^N\setminus\Omega)}\,. \qquad (11)$$

Since v_k tends weakly to 0 in H^1_{loc} and $\|v_k\|_k$ is uniformly bounded, (i) implies that (11) tends to 0 as $k\to\infty$. For the second integral we get

$$\left|\int_\Omega F(x,v_k+w_k)-F(x,v_k)-F(x,w_k)\,dx\right|\le\varepsilon$$

if k is sufficiently large. By Hölder's inequality we have

$$\left|\int_{\mathbb{R}^N\setminus\Omega}F(x,v_k+w_k)-F(x,v_k)\,dx\right|$$

$$\begin{aligned}
&\le\; \int_{Q_k\setminus\Omega}\left[|v_k|+|w_k|+C(|v_k|+|w_k|)^{p-1}\right]\,|w_k|\,dx\\
&\le\; C\left[\|v_k\|_k+\|w_k\|_k+\|v_k\|_k^{p-1}+\|w_k\|_k^{p-1}\right]\|w_k\|_{H^1(\mathbb{R}^N\setminus\Omega)}\\
&\le\; \varepsilon\,.
\end{aligned}$$

Together with the third estimate in (i) we obtain from (10)

$$c=\lim_{k\to\infty}J_k(\tilde u_k)=\lim_{k\to\infty}J_k(v_k)+J(u)$$

To complete the proof of the theorem we observe that any function $\varphi\in E_k$, $\varphi\neq 0$ satisfies

$$J_k(\varphi)-\frac{1}{2}\langle J_k'(\varphi),\varphi\rangle\ge\left(\frac12-\frac1\gamma\right)\int_{Q_k}f(x,\varphi)\varphi\,dx>0\,, \qquad (12)$$

which implies

$$J_k(v_k) \to c - J(u) \geq 0, \quad \text{i. e.} \quad J(u) \leq c\,.$$

If $J(u) = c$, we claim $\|v_k\|_k \to 0$, since otherwise we can use the same arguments as above for v_k instead of u_k and obtain a nontrivial solution v of (P) with $J(v) \leq 0$, which is not possible in view of Lemma 3. If $J(u) < c$, then Lemma 3 shows $J(u) \geq \varepsilon_2 > 0$ and we can repeat the arguments above with u_k replaced by v_k and c replaced by $c - J(u) \leq c - \varepsilon_2$. After a finite number of steps we are done. □

5 Ground states

In this section we use a different variational method to construct a nontrivial solution of (P), which in addition has minimal energy (ground state). To do this we need a more restrictive assumption on f, namely we assume that f is C^1 in the second variable and instead of 4^o we assume

$4'^o$ There exists $0 < \theta < 1$ such that $0 < y^{-1} f(x,y) \leq \theta \frac{\partial f}{\partial y}(x,y)$ for every $y \neq 0$.

This implies in particular

$$0 < F(x,y) \leq \frac{\theta}{1+\theta} f(x,y) y \tag{13}$$

for $y \neq 0$, and $\theta/(1+\theta) < 1/2$. Moreover, setting $\sigma = 1/\theta$, we obtain from $4'^o$

$$|y|^\sigma y \frac{d}{dy}\Big(|y|^{-\sigma} f(x,y)\Big) = -\sigma f(x,y) + y\,\frac{d}{dy} f(x,y) \geq 0 \quad \text{for every} \quad y \neq 0\,.$$

Now if $t > 0$ is a real number, integration of $\frac{d}{dy}(|y|^{-\sigma} f(x,y))$ over the interval $[u, tu]$ (resp. $[tu, u]$) leads to the inequalities

$$f(x,tu)u \left\{ \begin{array}{lll} \geq & t^\sigma f(x,u)u & \text{if } t \geq 1 \\ \leq & t^\sigma f(x,u)u & \text{if } t \leq 1 \end{array} \right. . \tag{14}$$

For $k = 1, 2, \ldots$ we define

$$I_k(u) = \langle J_k'(u), u \rangle = \int_{Q_k} |\nabla u|^2 + V u^2\, dx - \int_{Q_k} f(x,u) u\, dx$$

In a similar manner we define the functional I on $H^1(I\!R^N)$. We consider the minimization problems

$$m_k = \inf\{J_k(u) \mid u \in S_k\}\,, \quad \text{where} \quad S_k := \{u \in E_k \mid u \neq 0,\ I_k(u) = 0\}, \tag{15}$$

$$m = \inf\{J(u) \mid u \in S\}\,, \quad \text{where} \quad S := \{u \in H^1(I\!R^N) \mid u \neq 0,\ I(u) = 0\}. \tag{16}$$

Using the same arguments as in Lemma 3, we can show the following (note that we have only used the fact that $\langle J_k' u_k, u_k\rangle = 0$ in the proof of Lemma 3):

Lemma 9 *There exist numbers $\varepsilon_1, \varepsilon_2 > 0$, such that $\|u_k\|_k \geq \varepsilon_1$, $\|u\| \geq \varepsilon_1$, $J_k(u_k) \geq \varepsilon_2$ and $J(u) \geq \varepsilon_2$ for all $u_k \in S_k$, $u \in S$.* □

Lemma 10 *For each k, there exists a minimizer for (15) which is a nontrivial ground state solution of $(P)_k$.*

Proof. From compactness of embeddings it follws that a minimizing sequence for (15) has a convergent subsequence with nontrivial limit (in view of Lemma 9).

Since u_k is a minimizer, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$, such that $J_k'(u_k) = \lambda I'(u_k)$. Since $\langle J_k'(u_k), u_k\rangle = 0$, we claim from $4'^o$

$$\begin{aligned}
\langle I_k'(u_k), u_k\rangle &= \ell_k(u_k) - \int_{Q_k} f_y'(x, u_k) u_k^2 dx \\
&\leq \ell_k(u_k) - \frac{1}{\theta} \int_{Q_k} f(x, u_k) u_k dx \\
&= -\frac{1}{s} \int_{Q_k} f(x, u_k) u_k dx \,,
\end{aligned}$$

where $s > 0$ is such that $1 + 1/s = 1/\theta$. Since u_k is not trivial, the right hand side is strictly less than 0 and it follows that λ must be zero. Hence, u_k is a solution of $(P)_k$. □

Theorem 4 *Assume 1^o– 3^o, $4'^o$ and 5^o and let $u_k \in S_k$ be the periodic solutions of (15). Then there is a nontrivial solution u of (P), which is a minimizer for (16).*
Moreover, if f satisfies

$$|f_y(x, y)| \leq C(1 + |y|^{p-1}) \,, \tag{17}$$

then there is a sequence $b_k \in \mathbb{Z}^N$ such that, along a subsequence,

$$\|u_k - u(\cdot + b_k)\|_{H^1(Q_k)} \to 0 \,. \tag{18}$$

Proof. First of all we derive a uniform bound for u_k . To do this we fix a positive function $\eta \in C_0^\infty(Q_1)$ which satisfies $\ell_1(\eta) > 0$. Then inequality (14) shows that

$$I(t\eta) \begin{cases} \leq & t^2 \ell(\eta) - t^{\sigma+1} \int_{\mathbb{R}^N} f(x, \eta)\eta \, dx \quad \text{if } t \geq 1 \\ \geq & t^2 \ell(\eta) - t^{\sigma+1} \int_{\mathbb{R}^N} f(x, \eta)\eta \, dx \quad \text{if } t \leq 1 \,, \end{cases} \tag{19}$$

where $\sigma = 1/\theta > 1$. Hence, there exists a number $t > 0$, such that

$$\ell_1(t\eta) = \int_{Q_1} f(x, t\eta)t\eta \, dx$$

and $v_1 := t\eta$ belongs to S_1. For $k \geq 2$ we define

$$v_k(x) = \begin{cases} v_1(x) & , \quad x \in Q_1 \\ 0 & , \quad x \in Q_k \setminus Q_1 \end{cases} .$$

Clearly,

$$v_k \in S_k \quad \text{ and } \quad J_k(v_k) = J_1(v_1) = J(v_1) \quad \text{ for all } \quad k = 1, 2, \ldots$$

This shows $J_k(u_k) \leq J_1(v_1)$. Now Lemma 2 implies $\|u_k\|_k \leq C$ with a constant not depending on k. As in the proof of Theorem 2 we find a sequence $b_k \in \mathbb{Z}^N$ such that a subsequence of $\tilde{u}_k = u_k(\cdot + b_k)$ converges locally to a function $u \in H^1(\mathbb{R}^N)$, a nontrivial solution of (P).

To prove that u is a minimizer for (16) we first establish the following: For every $v \in S$ and every $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ and $v_k \in S_k$, such that

$$J_k(v_k) \leq J(v) + \varepsilon \quad \text{ for all } \quad k \geq k_\varepsilon \, . \tag{20}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$ and J and I are continuous on $H^1(\mathbb{R}^N)$, there exists a sequence $\eta_k \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp} \eta_k \subset Q_k$ with the following properties:

$$\eta_k \;\to\; v \quad \text{ in the norm } \quad \| \cdot \|_{H^1} , \tag{21}$$

$$J(\eta_k) \;\to\; J(v), \qquad I(\eta_k) \to I(v) = 0 \, . \tag{22}$$

Now inequality (19) shows that if $I(\eta_k) \leq 0$, then there exists $0 < t_k \leq 1$ such that $I(t_k\eta_k) = 0$, while if $I(\eta_k) \geq 0$, then $I(t_k\eta_k) = 0$ for some $t_k \geq 1$. Moreover we have the following estimates for t_k:

$$\text{if } \; I(\eta_k) = \delta_k \leq 0 : \qquad 1 \geq t_k \geq \left(\frac{\ell(\eta_k)}{\int_{\mathbb{R}^N} f(x, \eta_k)\eta_k \, dx} \right)^{\frac{1}{\sigma - 1}} = \left(1 + \frac{\delta_k}{\int_{\mathbb{R}^N} f(x, \eta_k)\eta_k \, dx} \right)^{\frac{1}{\sigma - 1}}$$

$$\text{if } \; I(\eta_k) = \delta_k \geq 0 : \qquad 1 \leq t_k \leq \left(\frac{\ell(\eta_k)}{\int_{\mathbb{R}^N} f(x, \eta_k)\eta_k \, dx} \right)^{\frac{1}{\sigma - 1}} = \left(1 + \frac{\delta_k}{\int_{\mathbb{R}^N} f(x, \eta_k)\eta_k \, dx} \right)^{\frac{1}{\sigma - 1}} .$$

Since $\int_{\mathbb{R}^N} f(x, v)v \, dx = \ell(v) \geq 2\varepsilon_2$ for $v \in S$ (with $\varepsilon_2 > 0$ from Lemma 3), we can choose k_0 large enough such that η_k satisfies

$$\int_{\mathbb{R}^N} f(x, \eta_k)\eta_k \, dx \geq \varepsilon_2 \quad \text{ for all } \quad k \geq k_0 \, .$$

Since $\delta_k \to 0$, we obtain from the above inequalities

$$\begin{aligned} t_k &\to 1 \qquad \text{and} \\ J(t_k\eta_k) &\to J(v) \quad \text{as} \quad k \to \infty \, . \end{aligned}$$

Now given $\varepsilon > 0$ we can choose k_ε large enough such that $v_k := t_k\eta_k \in S_k$ and $J_k(v_k) = J(v_k) \leq J(u) + \varepsilon$ for all $k \geq k_\varepsilon$, which proves (20).

In particular, (20) implies that

$$\limsup m_k \leq J(v) + \varepsilon$$

for any $v \in S$ and for any $\varepsilon > 0$; hence,

$$\limsup m_k \leq m \, .$$

Since $\theta/(1+\theta) < 1/2$, (13) implies that the function $g(x,v) := [\frac{1}{2}f(x,v)v - F(x,v)]$ is nonnegative for every v. Moreover we have

$$J_k(v_k) = \int_{Q_k} g(x, v_k)\, dx \qquad \text{for every} \quad v_k \in S_k$$

and consequently

$$m_k = J_k(\tilde{u}_k) = \int_{Q_k} g(x, \tilde{u}_k)\, dx$$

for any $k \in \mathbb{N}$. Since the integrand is nonnegative, we claim that for any bounded $\Omega \subset \mathbb{R}^N$ and k large enough

$$m_k \geq \int_\Omega g(x, \tilde{u}_k)\, dx \, .$$

Passing to the limit, we obtain

$$\liminf m_k \geq \int_\Omega g(x, u)\, dx \, .$$

Since Ω is an arbitrary domain and u is a nontrivial solution, this gives rise to the inequality

$$\liminf m_k \geq J(u) \geq m \, .$$

Therefore,

$$\lim m_k = m = J(u) \tag{23}$$

and u is a ground state.

Thus, it remains to prove (18). Note that (17) implies Assumption 6^o from Section 4. Since $J_k(u_k) \to J(u) = m$, the assertion of Theorem 3 holds with $c = m$, $n = 1$ and $u^1 = u$. □

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