THE TOMATO SALAD PROBLEM IN SPHERICAL STEREOLOGY[1]

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Abstract

In 1925 S.D. Wicksell studied the problem of determining the probability density of the sizes of spherical particles embedded in a solid opaque medium from that of the sizes of their intersections with a random plane cut. Mathematical modelling leads to a first kind Abel type integral equation. Beginning in 1958 G. Bach published a series of papers on a modified problem, replacing the cut by a slice of positive thickness bounded by parallel planes and measuring the radii of circles obtained by orthogonal projection of the cut-out parts of spheres on a plane of observation lying parallel to the slice. P.L. Goldsmith in 1967 presented a solution formula to this "tomato salad problem".

We here give a description of the problem and another derivation of a solution formula, using the formalism of fractional calculus, hoping this new derivation to be more transparent (at least to people familiar with fractional calculus) than Goldsmith's. Furthermore, we discuss aspects that, to a large extent, have been neglected by previous authors, namely the transition from the well-posed tomato salad problem to Wicksell's ill-posed problem (as the thickness of the slice tends to zero), and non-negative measures instead of classical functions as densities.

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1 Introduction

An important problem of stereology is the determination of the probability density of the sizes of randomly distributed particles in a solid opaque medium from that of the sizes of their intersections with a random plane cut.

Applications are, among others, the study of corpuscles embedded in the (solidified) tissue of a biological organ, in sedimentary petrography, in crystallography, in investigation of carbide particles in steel, in statistics of sizes of bubbles in Swiss cheese. See references given in [9] and literature on stochastic geometry, e.g. [3], [12], [16], [17]. In spherical stereology the case of spherical particles is considered. The pioneering paper in this field is that by Wicksell [18], published in 1925. As one of the most lucid papers we recommend [15].

The tomato salad problem has been treated by Bach in a series of pioneering papers of which we quote [1], [2]. A solution formula has been derived by Goldsmith [5]. A random slice with positive thickness t, bounded by two parallel planes, is cut out from the solid opaque medium. The thickness t should be so small that the slice (apart from parts of spheres contained in it) is still transparent. The cut-out parts of spheres are orthogonally projected on another plane (lying parallel to the slice), where they produce circles as shadows whose radii are observable and are called "apparent" radii in distinction from the "actual" radii of the spheres. A simplifying assumption is that the slice is so thin or the spatial density of spheres is so small that overlapping of shadows is negligible. The problem is to determine the probability distribution of the actual radii from the observed one of the apparent radii.

Naturally, $t > 0$, but we formally require $t \geq 0$ so that Wicksell's classical problem of spherical stereology (described in the Abstract) is a special case. Whereas this classical problem is modelled by an Abel integral equation of first kind (ill-posed in customary function spaces) the case $t > 0$ leads to a well-posed Abel integral equation of second kind.

We imagine our slice cut out from the solid opaque medium as horizontal. This means that we can (and henceforth shall) speak of the "upper bounding plane" and the "lower bounding plane" (in case $t > 0$). The third plane, also horizontal, is the "observation plane". We imagine it as lying below the lower bounding plane. The image of a cut-out part of a sphere, produced by the orthogonal projection on the plane of observation, is called the "shadow" of this sphere. The reader is advised to make sketches of the geometrical configurations described in words. See Fig. 1.

We concentrate in this paper our attention to analytical aspects. The reader who is interested in numerical methods may look into [5], [6], [7], [8], [10] and [13].

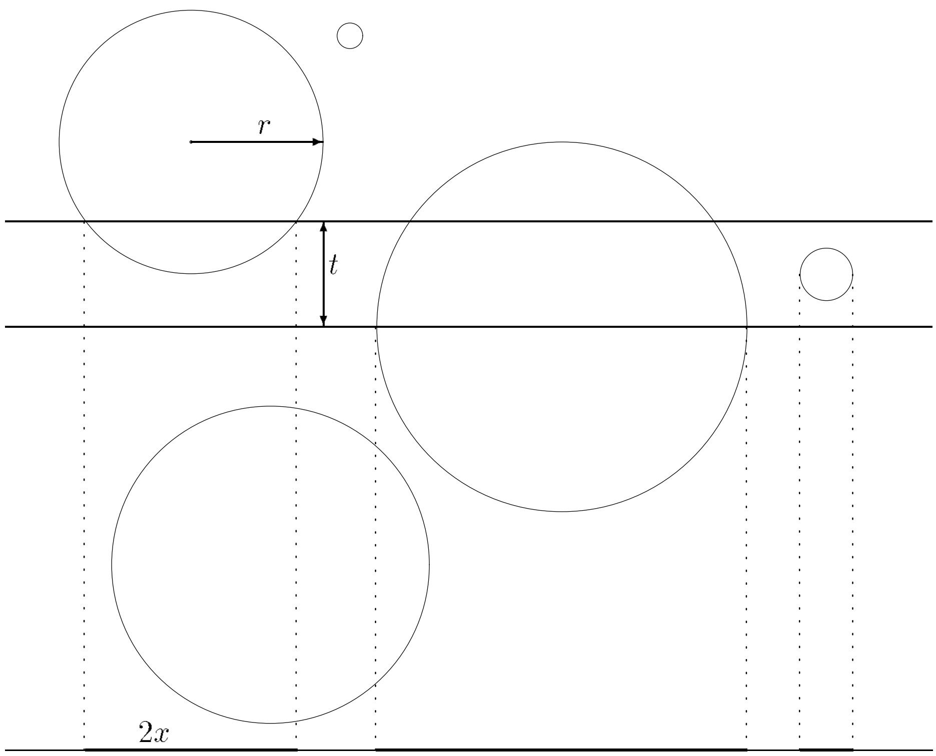

Fig. 1: General configuration

2 Notation and generalities

Let us first introduce some notations. By λ we denote the spatial density of spheres with centers in a unit volume of the medium, i.e. the expected number of spheres with centers in a unit volume. Now, let

$$r_2 > r_1 \geq 0,\ x_2 > x_1 \geq 0,\ \tau_2 > \tau_1 \geq 0,\ \sigma_2 > \sigma_1 \geq 0,$$

and understand $\int_\alpha^\beta$ for $\beta > \alpha$ to mean $\int_{\alpha-}^{\beta-}$. Then we define non-negative generalized functions $\tilde{f},\ f,\ \tilde{g},\ g,\ \tilde{p},\ p, \tilde{q},\ q$ by the following properties.

$\int_{r_1}^{r_2} \tilde{f}(r)dr$ = expected number of spheres, with centers in a unit volume, having radii in the semi-open interval $[r_1,\ r_2)$.

$\int_{x_1}^{x_2} \tilde{g}(x)dx$ = expected number of spheres meeting the slice whose centers are projected on a unit area of the observation plane and whose apparent radii are in $[x_1,\ x_2)$.

$\int_{r_1}^{r_2} f(r)dr$ = probability of actual radius in $[r_1,\ r_2)$.

$\int_{x_1}^{x_2} g(x)dx$ = probability of a sphere that meets the slice having apparent radius in $[x_1,\ x_2)$.

$$m = \int_0^\infty rf(r)dr = \text{mean actual radius of a sphere.} \tag{2.1}$$

REMARK: We shall always assume m to be finite.

$\int_{\tau_1}^{\tau_2} \tilde{p}(\tau)d\tau$ = expected number of spheres, with centers in a unit volume, having central cross-section area in $[\tau_1,\ \tau_2)$.

EXPLANATION: A sphere with radius r has central cross-section area πr^2.

$\int_{\sigma_1}^{\sigma_2} \tilde{q}(\sigma)d\sigma$ = expected number of spheres meeting the slice whose centers are projected on a unit area of the observation plane and whose shadows have an area in $[\sigma_1,\ \sigma_2)$.

$\int_{\tau_1}^{\tau_2} p(\tau)d\tau$ = probability of a sphere having central cross-section area in $[\tau_1,\ \tau_2)$.

$\int_{\sigma_1}^{\sigma_2} q(\sigma)d\sigma$ = probability of a sphere that meets the slice having shadow area (of its cut-out part) in $[\sigma_1,\ \sigma_2)$.

REMARK: By admitting generalized functions (actually measures) we can treat by the same formalism continuous as well as more general probability distributions. By our understanding of definite integrals the primitives

$$F(r) = \int_0^r f(s)ds,\ G(x) = \int_0^x g(s)ds,\ P(\tau) = \int_0^\tau p(s)ds,\ Q(\sigma) = \int_0^\sigma q(s)ds$$

for nonnegative arguments are nonnegative non-decreasing left-continuous functions tending to 1 for their respective argument tending to infinity. If, for example, $f(r) = \delta(r-a)$ with $a \geq 0$, then $F(r) = 0$ for $r \leq a$, $F(r) = 1$ for $r > a$. A reader who does not like integration of generalized functions may rewrite this paper replacing $f(\tau)d\tau$ by $dF(\tau)$ etc.

We note the basic relations between our functions and their arguments. With r as actual and x as apparent radius of a sphere, τ as central cross-section area and σ as shadow area we have

$$\tau = \pi r^2,\ \sigma = \pi x^2,\ r = \sqrt{\tau/\pi},\ x = \sqrt{\sigma/\pi}, \tag{2.2}$$

and the equalities of differentials

$$f(r)dr = p(\tau)d\tau,\ g(x)dx = q(\sigma)d\sigma,$$

$$\tilde{f}(r)dr = \tilde{p}(\tau)d\tau,\ \tilde{g}(x)dx = \tilde{q}(\sigma)d\sigma,$$

$$d\tau = 2\pi r dr,\ d\sigma = 2\pi x dx$$

imply

$$\left.\begin{array}{ll} p(\tau) = f(r)/(2\pi r), & q(\sigma) = g(x)/(2\pi x), \\ \tilde{p}(\tau) = \tilde{f}(r)/(2\pi r), & \tilde{q}(\sigma) = \tilde{g}(x)/(2\pi x). \end{array}\right\} \tag{2.3}$$

Further, we observe the relations

$$\tilde{f}(r) = \lambda f(r), \tag{2.4}$$

$$\tilde{g}(x) = \lambda(t + 2m)g(x). \tag{2.5}$$

The validity of (2.4) is obvious, and (2.5) follows from the fact that the total expected number $\int_0^\infty \tilde{g}(x)dx$ of spheres that meet the slice and whose centers are projected on a unit area of the observation plane is $\lambda(m+t+m)$, with m as mean radius of a sphere. Here t comes from the sphere centers inside the slice whereas $m+m$ comes from centers above the upper and below the lower bounding plane. The requirement $\int_0^\infty g(x)dx = 1$ and the natural scaling relation

$$g(x) = \left(\int_0^\infty \tilde{g}(s)ds\right)^{-1} \tilde{g}(x) \tag{2.6}$$

also yield (2.5). Note that scaling relations analogous to (2.6) hold for f and $\tilde{f}$, q and $\tilde{q}$, p and $\tilde{p}$.

Another derivation of (2.5) will be given in the next section in which we shall derive two integral equations: Bach's equation for determining $\tilde{f}$ from $\tilde{g}$, Goldsmith's equation for determining f from g.

We shall use the notion of characteristic function of a set. For $M \subset \mathbf{R}$ we define

$$\chi_M(s) = 1 \text{ if } s \in M, \ = 0 \text{ if } s \notin M. \tag{2.7}$$

In Sections 4 and 5 an essential role will be played by the Mittag-Leffler function with index 1/2. On general properties of Mittag-Leffler functions the reader can find orientation in Chapter 18 of [4]. In particular,

$$E_{1/2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(\frac{n}{2}+1\right)} \text{ for } z \in \mathbf{C}.$$

We need this function for non-positive real argument. It is known that

$$E_{1/2}(-s) = \frac{2}{\sqrt{\pi}} \exp(s^2) \int_s^\infty \exp(-x^2)dx \text{ for } s \in \mathbf{R}. \tag{2.8}$$

For completeness we present a short nice proof (adapted from [5]) of (2.8). Indeed, for all $s \in \mathbf{C}$, we have $E_{1/2}(-s) = u(s) - v(s)$ with

$$u(s) = \sum_{k=0}^{\infty} \frac{s^{2k}}{\Gamma(k+1)} = \exp(s^2), \ v(s) = \sum_{k=0}^{\infty} \frac{s^{2k+1}}{\Gamma\left(k+\frac{3}{2}\right)}.$$

With $y = \sqrt{2}s$ we get, using $\Gamma(1/2) = \sqrt{\pi}$ and the functional equation of the gamma function, $v(s) = \sqrt{2/\pi}\, w(y)$ with

$$w(y) = y + \frac{1}{3}y^3 + \frac{1}{3 \times 5}y^5 + \frac{1}{3 \times 5 \times 7}y^7 + \ldots,$$

and readily we see that

$$w'(y) = yw(y) + 1 \text{ for } y \in \mathbf{C},\ w(0) = 0.$$

This initial value problem is uniquely solved by

$$w(y) = \exp\left(\frac{y^2}{2}\right) \int_0^y \exp\left(-\frac{z^2}{2}\right) dz.$$

Back substitution $x = z/\sqrt{2}$ gives

$$v(s) = \frac{2}{\sqrt{\pi}} \exp(s^2) \int_0^s \exp(-x^2) dx,$$

$$E_{1/2}(-s) = \exp(s^2) \left\{1 - \frac{2}{\sqrt{\pi}} \int_0^s \exp(-x^2) dx\right\}. \tag{2.9}$$

Using the well-known identity $2\int_0^\infty \exp(-x^2)dx = \sqrt{\pi}$ we now deduce (2.8) from (2.9).

It is not difficult to show that $E_{1/2}(-s)$ is strictly decreasing from 1 towards 0 as s runs from 0 towards ∞ (through real s).

3 Derivation of integral equations

In this section we assume $t > 0$ if not explicitly stated otherwise. First we consider the special case of all spheres having the same actual radius $\rho > 0$. Then, with Dirac's δ-function, $f(r) = \delta(r - \rho) =: f(r, \rho)$. Let us use the notations $g(x, \rho)$ and $\tilde{g}(x, \rho)$ for the corresponding functions $g(x)$ and $\tilde{g}(x)$, respectively.

The expected number of spheres with centers in the slice and projected center lying in a unit observational area is $t\lambda$. For shortness of expression we shall henceforth say "unit observational area" for "unit area of the observation plane". The expected number of spheres with centers outside the slice but meeting it and having projected center in a unit observational area is $2\rho\lambda$, the factor 2ρ stemming from the fact that intersection happens if the center of the sphere is at most distant by ρ from the slice but can lie above the upper bounding plane or below the lower bounding plane. See Fig. 2.

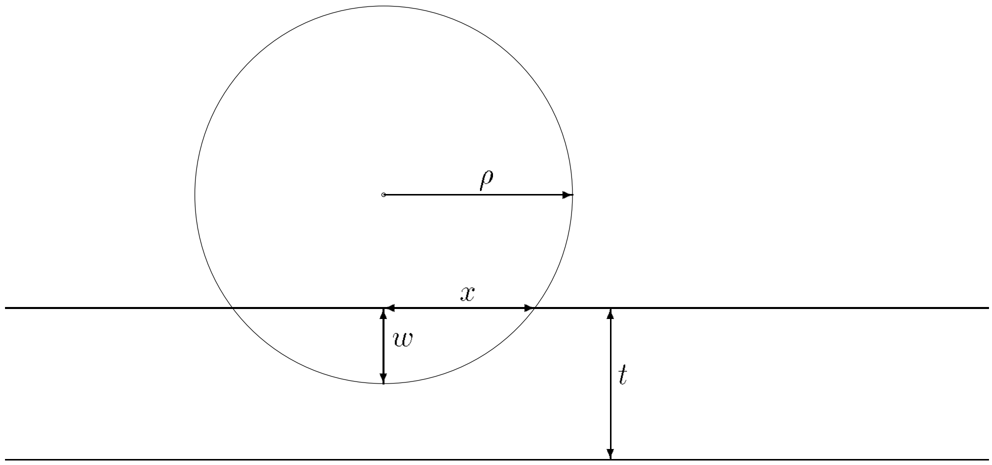

Fig. 2: A sphere immersed from above

Denoting by w the depth of immersion below the upper bounding plane (if the center is above this plane) or above the lower bounding plane (if the center is below this plane) we see, by a theorem of elementary geometry, that for the apparent radius x we have

$$x^2 = w(2\rho - w). \tag{3.1}$$

The random variable w being uniformly distributed in the interval $[0, \rho]$ the probability for immersion depth lying in the interval $[w,\ w + dw)$ is $\frac{dw}{\rho}$ for $0 \le w < \rho$ and positive differential dw. From (3.1) we get $2xdx = (2\rho - 2w)dw$, hence $dw = \frac{xdx}{\rho - w}$, and we conclude that the expected number of such spheres that produce apparent radii in $[x,\ x + dx)$ is

$$2\lambda dw = \frac{2\lambda x dx}{\rho - w} \text{ for } 0 \le w < \rho,$$

the factor 2 resulting from contributions from above and below the slice. From (3.1) or by invoking Pythagoras we obtain $\rho - w = (\rho^2 - x^2)^{1/2}$, hence

$$2\lambda dw = \frac{2\lambda x dx}{\sqrt{\rho^2 - x^2}} \text{ for } 0 \le x < \rho.$$

Remembering now in addition the expected number λt of sphere centers inside the slice and projected on a unit observational area (these spheres produce an apparent radius equal to the actual radius ρ) we find (using the notation (2.7)), for $x \geq 0$,

$$\tilde{g}(x,\rho) = \lambda t \delta(x-\rho) + \frac{2\lambda x}{\sqrt{\rho^2 - x^2}} \chi_{[0,\rho)}(x). \tag{3.2}$$

Using the scaling relation (2.6) and taking into account the expected number

$$\lambda(t+2\rho) = \int_0^\infty \tilde{g}(x,\rho)dx$$

of sphere centers inside the slice or at distance at most ρ from it that are projected on a unit observational area we arrive at

$$g(x,\rho) = \frac{1}{t+2\rho} \left\{ t\delta(x-\rho) + \frac{2x}{\sqrt{\rho^2 - x^2}} \chi_{[0,\rho)}(x) \right\}. \tag{3.3}$$

REMARK: It is known that very small spheres are difficult to detect in Wicksell's classical arrangement ($t = 0$). Let us look what happens with equation (3.2) in the limiting cases $t = 0,\ \rho = 0$. Observe that we have defined (3.2) and (3.3) under the assumption (i) $t > 0,\ \rho > 0$. Consider the three remaining cases (ii) $t > 0,\ \rho = 0$, (iii) $t = 0,\ \rho > 0$, (iv) $t = 0,\ \rho = 0$. A critical re-view shows that (3.2) becomes

$$\tilde{g}(x,0) = \lambda t \delta(x) \text{ in case (ii)},$$

$$\tilde{g}(x,\rho) = \frac{2\lambda x}{\sqrt{\rho^2 - x^2}} \chi_{[0,\rho)}(x) \text{ in case (iii)},$$

$$\tilde{g}(x,0) \equiv 0 \text{ in case (iv)}.$$

This last relation reveals that point-like particles are practically undetectable in Wicksell's arrangement (they are hit by the plane cut with probability zero). We shall return to this question in Section 6.

Now we have the situation that there is only one possible actual radius. Generalizing our treatment and working with positive differentials dx and dr we see (a) and (b).

(a) The expected number of spheres centered inside the slice above a unit observational area, producing apparent radii in $[x,\ x+dx)$ is $\lambda t f(x)dx$ (for such spheres actual and apparent radii coincide).

(b) The expected number of spheres meeting the slice but having their centers outside of it above a unit observational area and having actual radius in $[r,\ r+dr)$ that produce apparent radius in $[x,\ x+dx)$ is

$$\frac{2\lambda x}{\sqrt{r^2-x^2}}\chi_{[0,r)}(x)f(r)drdx.$$

Integrating this from $r=0$ to $r=\infty$ gives the expected number of spheres meeting the slice but centered outside of it above a unit observational area as

$$2\lambda x\int_{r=x}^{\infty}\frac{f(r)dr}{\sqrt{r^2-x^2}}dx.$$

Adding the contributions (a) and (b) and taking account of (2.4) $\tilde{f}=\lambda f$ gives us Bach's integral equation connecting the densities $\tilde{g}$ and $\tilde{f}$ of expected numbers

$$\tilde{g}(x)=t\tilde{f}(x)+2x\int_{r=x}^{\infty}\frac{\tilde{f}(r)dr}{\sqrt{r^2-x^2}}. \tag{3.4}$$

For given $\tilde{g}$ this is an Abel type integral equation of second kind for determining $\tilde{f}$ from $\tilde{g}$.

To find an integral relation combining the probability densities f and g we remember the scaling condition (2.6) $g(x)=\tilde{g}(x)/\mu$ with $\mu=\int_0^\infty\tilde{g}(x)dx$. Using $\int_0^\infty f(r)dr=1$ and again (2.4) we obtain by integrating (3.4) from $x=0$ to $x=\infty$ after interchanging the order of integrations $\mu=\lambda(t+2m)$ and thus have another proof of (2.5). Dividing (3.4) by λ we obtain Goldsmith's integral equation

$$(t+2m)g(x)=tf(x)+2x\int_{r=x}^{\infty}\frac{f(r)dr}{\sqrt{r^2-x^2}} \tag{3.5}$$

which for determination of f from given g is to be complemented by the scaling condition

$$\int_0^\infty f(r)dr=1. \tag{3.6}$$

We can solve the system $\{(3.5),\,(3.6)\}$ for the unknown function f and the unknown mean radius m by the following procedure. Solve (3.5) for $\phi(r):=f(r)/(t+2m)$. Then, with $\gamma=\int_0^\infty\phi(r)dr$, we have $f(r)=\phi(r)/\gamma,\ m=(\frac{1}{\gamma}-t)/2$.

4 Solution of Bach's integral equation

The essential work to be carried out is to solve Bach's integral equation (3.4), $\tilde{g}$ given, $\tilde{f}$ unknown, or the integral equation resulting from (3.5) by substitution $f(r) = (t+2m)\phi(r)$ and division by $(t+2m)$. Both integral equations have the same form, and so we concentrate on the solution of (3.4),

$$\tilde{g}(x) = t\tilde{f}(x) + 2x\int_{r=x}^{\infty} \frac{\tilde{f}(r)dr}{\sqrt{r^2 - x^2}},$$

treating the case $t > 0$ of positive thickness of the slice. At the end of this section we shall make a short remark on Wicksell's case $t = 0$.

In analogy to [15] (where only the case $t = 0$ is treated) and deviating from Goldsmith [5] we take as independent variables the central cross-section area $\tau = \pi r^2$ and the shadow area $\sigma = \pi x^2$ of a sphere instead of its actual and apparent radius r and x, respectively. See Section 2 for the notations. Bach's equation (3.4) is by this substitution equivalent to the integral equation

$$\tilde{q}(\sigma) = t\tilde{p}(\sigma) + \frac{1}{\sqrt{\pi}}\int_{\tau=\sigma}^{\infty} \frac{\tilde{p}(\tau)d\tau}{\sqrt{\tau - \sigma}}, \tag{4.1}$$

whereas Goldsmith's equation (3.5) with the scaling condition (3.6) is equivalent to

$$(t+2m)q(\sigma) = tp(\sigma) + \frac{1}{\sqrt{\pi}}\int_{\tau=\sigma}^{\infty} \frac{p(\tau)d\tau}{\sqrt{\tau - \sigma}} \tag{4.2}$$

with the scaling condition

$$\int_0^{\infty} p(\tau)d\tau = 1. \tag{4.3}$$

The system { (4.2), (4.3)} can be treated in analogy to the procedure described at the end of the preceding Section 3. After having calculated $\tilde{p}(\tau)$ from (4.1) or $p(\tau)$ from {(4.2), (4.3)} we get by back-substitution, see (2.3),

$$\tilde{f}(r) = 2\pi r\tilde{p}(\pi r^2),\ f(r) = 2\pi r p(\pi r^2). \tag{4.4}$$

To solve (4.1) formally we apply the Weyl fractional calculus described in [14] and then discuss conditions for validity of the formal solution obtained. We use Weyl operators V^α, $\alpha > 0$, according to

$$(V^\alpha u)(y) = \frac{1}{\Gamma(\alpha)}\int_{s=y}^{\infty} (s-y)^{\alpha-1}u(s)ds,\ y \geq 0. \tag{4.5}$$

This expression is meaningful if the generalized function $u(s)$ behaves sufficiently well, e.g. if it is bounded and tends to zero faster than every negative power of s as $s \to \infty$. Let us remark that Miller and Ross in [14] write $W^{-\alpha}$ instead of V^{α}. With $V^0 := I =$ identity operator the operators V^{α} form a semigroup:

$$V^{\alpha+\beta} = V^{\alpha}V^{\beta} \text{ for } \alpha \geq 0,\ \beta \geq 0.$$

Writing (4.1) in the form

$$\tilde{q} = (tI + V^{1/2})\tilde{p} := B_t\tilde{p} \tag{4.6}$$

we obtain formally

$$\tilde{p} = (tI + V^{1/2})^{-1}\tilde{q} = B_t^{-1}\tilde{q}$$

and by Neumann series expansion

$$B_t^{-1} = t^{-1}(I + t^{-1}V^{1/2})^{-1} = t^{-1}\sum_{n=0}^{\infty}(-t)^{-n}V^{n/2},$$

$$\tilde{p}(\tau) = t^{-1}\left\{\tilde{q}(\tau) + \sum_{n=1}^{\infty}\frac{(-t)^{-n}}{\Gamma(n/2)}\int_{\tau}^{\infty}(\sigma-\tau)^{\frac{n}{2}-1}\tilde{q}(\sigma)d\sigma\right\}.$$

If $\tilde{q}(\tau)$ is so well behaved that the series converges and $\sum$ and $\int$ can be interchanged we obtain

$$\tilde{p}(\tau) = \frac{1}{t}\left\{\tilde{q}(\tau) + \int_{\tau}^{\infty} w_t(\sigma-\tau)\tilde{q}(\sigma)d\sigma\right\} = (B_t^{-1}\tilde{q})(\tau) \tag{4.7}$$

with

$$w_t(y) = \sum_{n=1}^{\infty}\frac{(-t)^{-n}}{\Gamma(n/2)}y^{\frac{n}{2}-1} = \frac{d}{dy}E_{1/2}\left(-\frac{\sqrt{y}}{t}\right). \tag{4.8}$$

Here $E_{1/2}$ is the Mittag-Leffler function of index 1/2, see Section 2 for details. We see that asymptotically

$$w_t(y) \sim -\frac{t^{-1}y^{-1/2}}{\sqrt{\pi}} \text{ for } y \to 0,$$

and from formulas (1.17), (2.5), (3.4), (4.3) of [9] we deduce

$$w_t(y) \sim -\frac{ty^{-3/2}}{2\sqrt{\pi}} \text{ for } y \to \infty.$$

Hence (4.7) is meaningful (the convolution integral converges) if $\tilde{q}(\tau)$ is integrable and bounded. Appropriately interpreted, (4.7) is valid as solution formula under much wider conditions.

In situations arising in practice the density $\tilde{p}$ may have the form

$$\tilde{p}(\tau) = \gamma(\tau) + \sum_{j\in M} c_j \delta(\tau - a_j),\ \tau \geq 0.$$

Here $\gamma(\tau)$ is non-negative, piecewise continuous, exponentially decreasing to zero for $\tau \to \infty$ (or having bounded support), M is a finite or denumerably infinite index set, all $a_j > 0$, all $c_j > 0$, and $\sum_{j\in M} c_j < \infty$. See [6], [7], [8] and [13].

By careful direct calculation it can be shown that our formalism, in particular formula (4.7), does work correctly if applied to a generalized function $\tilde{q}$ corresponding by (4.1) to a generalized function $\tilde{p}$ of this special form.

Remark: In the special case $t = 0$ we have, with $Du = u'$, formally

$$B_0^{-1} = V_{1/2}^{-1} = -DV^{1/2} \tag{4.9}$$

(see [14]).

5 Noise amplification

Again, assume $t > 0$ if not stated explicitly otherwise.

In practice, the function $\tilde{q}(\sigma)$ is given by statistical measuring and counting procedures, hence unavoidably numerically not precisely. It is therefore important to investigate the amplification of noise effected in applying the inverse operator B_t^{-1} to noisy data. In Banach space setting the worst case amplification factor is given by the operator norm $\|B_t^{-1}\|$.

If we work in the Lebesgue space $L_\beta = L_\beta[0,\infty)$, $\beta \in [1,\ \infty]$, of functions defined on $[0,\ \infty)$ then we observe that the operator B_t of (4.6) may be applicable only in part of L_β. For example,

$$\int_\sigma^\infty (\tau - \sigma)^{-1/2} \min(1, \tau^{-\alpha}) d\tau \equiv \infty,\ \text{if } 0 \leq \alpha \leq 1/2 \text{ and } \sigma \geq 0,$$

but the function $\min(1, \tau^{-\alpha})$ is in L_β if $\beta > 1/\alpha$. However, B_t is applicable to all functions in L_β with bounded support.

We shall now show that the inverse operator B_t^{-1} of the solution formula (4.7) is a bounded linear operator from L_β into L_β, for all $\beta \in [1,\ \infty]$. Our argumentation will be analogous to one used in [11].

In fact, by appropriate changes of variables and extension by zero of the involved functions, the integral in (4.7) can be written as a convolution of functions defined on $\mathbf{R}$, and using Young's inequality for convolutions,

$$\|u * v\|_{L_\beta(\mathbf{R})} \leq \|u\|_{L_1(\mathbf{R})} \|v\|_{L_\beta(\mathbf{R})}$$

we find, with $\|.\|_\beta = \|.\|_{L_\beta[0,\ \infty)}$, from (4.7)

$$\|\tilde{p}\|_\beta \leq \frac{1}{t}\|q\|_\beta (1 + \|w_t\|_1).$$

Because $w_t(x) < 0$ for $x > 0$ we have

$$\|w_t\|_1 = \int_0^\infty |w_t(x)| dx = -\int_0^\infty \frac{d}{dx} E_{1/2}\left(-\frac{\sqrt{x}}{t}\right) dx = E_{1/2}(0) = 1.$$

Hence

$$\|\tilde{p}\|_\beta \leq \frac{2}{t}\|q\|_\beta, \ \ \|B_t^{-1}\| \leq \frac{2}{t}. \tag{5.1}$$

This bound tends to ∞ as $t \to 0$, in accordance with the well-known fact that $B_0 = V^{1/2}$ is a smoothing operator whose inverse is unbounded.

When we consider B_t^{-1} as an operator acting only on functions with support in an interval $[0,\ S]$, with $S \in (0,\ \infty)$, an analogous argument teaches us that

$$\|B_t^{-1}\|_{L_\beta[0,\ S]} \leq \frac{1}{t}\left(2 - E_{1/2}\left(-\frac{\sqrt{S}}{t}\right)\right).$$

Let us illustrate by a simple example that noise amplification by a factor like $1/t$ is realistic. For $0 \leq a < b < \infty$ the function $\tilde{q}(\sigma) = \chi_{(a,b]}(\sigma)$ is in L_β for every $\beta \in [1,\ \infty]$.

If $t > 0$ we find from (4.7)

$$\tilde{p}(\tau) = \frac{1}{t}\left\{E_{1/2}\left(-\frac{\sqrt{b-\tau}}{t}\right)\chi_{(0,b]}(\tau) - E_{1/2}\left(-\frac{\sqrt{a-\tau}}{t}\right)\chi_{(0,a]}(\tau)\right\},$$

in particular

$$\tilde{p}(a-) = \frac{1}{t}\left\{E_{1/2}\left(-\frac{\sqrt{b-a}}{t}\right) - 1\right\}, \ \ E_{1/2}(b-) = 1/t.$$

In the special case $t = 0$ by (4.9)

$$\tilde{p}(\tau) = \frac{1}{\sqrt{\pi}} \left\{ (b-\tau)^{-1/2} \chi_{(0,b]}(\tau) - (a-\tau)^{-1/2} \chi_{(0,a]}(\tau) \right\},$$

and we see that $\tilde{p}$ is unbounded although $\tilde{q}$ is bounded. This illustrates ill-posedness. We also recognize the transition from well-posedness to ill-posedness in the factor $1/t$ tending to ∞ as t tends to 0.

6 Superposition of δ-functions as density of actual radius

In the preceding section we have seen the advantage of the positive thickness slice method ($t > 0$) versus Wicksell's method of a plane cut ($t = 0$) with respect to noise amplification (well-posedness versus ill-posedness). In what follows we shall exhibit another advantage (already hinted at in Section 3) of taking $t > 0$, namely: pointlike particles (spheres with radius 0, in practice spheres with extremely small radius) are undetectable if $t = 0$. They are hit by a plane cut with probability 0, however can be found with positive probability (if they exist) in a slice of positive thickness. This seems clear to naive intuition, but we shall make it precise.

Consider first the case of finitely many possible values of the actual radius r, equivalently the case of finitely many possible values $a_1,\ a_2,\ a_3, \ldots, a_N$ of the central cross-section area of a sphere. For definiteness assume $N \in \mathbf{N}$,

$$0 \leq a_1 < a_2 < \ldots < a_N < \infty,$$

and, with all $c_j > 0$,

$$\tilde{p}(\tau) = \sum_{j=1}^{N} c_j \delta(\tau - a_j). \tag{6.1}$$

Then by (4.1)

$$\tilde{q}(\sigma) = t \sum_{j=1}^{N} c_j \delta(\sigma - a_j) + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{N} \frac{c_j}{\sqrt{a_j - \sigma}} \chi_{(0,a_j]}(\sigma). \tag{6.2}$$

Now consider the special case $N = 1$, $a_1 = 0$, $c_1 = c > 0$. Then $\tilde{q}(\sigma) = ct\delta(\sigma)$, and for the expected number $Q(t) = \int_0^\infty \tilde{q}(\sigma)d\sigma$ of particles producing a shadow in a unit observational area we get

$$Q(t) = tc \tag{6.3}$$

which is positive if $t > 0$, but 0 if $t = 0$. Hence pointlike particles can be expected to be seen if and only if $t > 0$.

If you do not trust manipulations with δ-functions consider the case of extremely small central cross-section areas (equivalent to extremely small actual radii). For $h > 0$ let $\tilde{p}_h(\tau) = \frac{c}{h}\chi_{(0,h]}(\tau)$. Then the spatial density of sphere-centers is $\lambda = \int_0^\infty \tilde{p}_h(\tau)d\tau = c > 0$, and by (4.1) we obtain

$$\tilde{q}_h(\sigma) = \frac{c}{h}\left\{t + \frac{2}{\sqrt{\pi}}(h-\sigma)^{1/2}\right\}\chi_{(0,h]}(\sigma).$$

For the expected number $Q_h(t) = \int_0^\infty \tilde{q}_h(\sigma)d\sigma$ of spheres producing a shadow center in a unit observational area we find

$$Q_h(t) = c\left(t + \frac{4}{3\sqrt{\pi}}h^{1/2}\right),$$

and $Q_h(t) \to tc$ as $h \to 0$ in agreement with (6.3).

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Supplement (not contained in the paper to be published)

For the interested reader's convenience we add a few more references in which problems of spherical stereology are treated. This list illustrates the wide range of applicability.

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