COMPLETION OF SEMIUNIFORM CONVERGENCE SPACES

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For the sixtieth birthday of my friend and colleague Horst Herrlich

Abstract. Semiuniform convergence spaces form a common generalization of filter spaces (including symmetric convergence spaces [and thus symmetric topological spaces] as well as Cauchy spaces) and uniform limit spaces (including uniform spaces) with many convenient properties such as cartesian closedness, hereditariness and the fact that products of quotients are quotients. Here, for each semiuniform convergence space a completion is constructed, called the simple completion. This one generalizes Császár's λ–completion of filter spaces. Thus, filter spaces are characterized as subspaces of convergence spaces. Furthermore, Wyler's completion of separated uniform limit spaces can be easily derived from the simple completion.

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1. Introduction

In 1967 Cook and Fischer [5] introduced uniform convergence spaces as a generalization of Weil's uniform spaces [18]. With respect to the formation of function spaces, these spaces are not better behaved than uniform spaces, in other words: They do not form a cartesian closed topological category. But a slight modification of the definition of uniform convergence spaces, proposed by Wyler [20], leads to a cartesian closed topological category (cf. [12]). Therefore we will use Wyler's definition of uniform convergence spaces and call them uniform limit spaces. Omitting two of their defining axioms, one obtains semiuniform convergence spaces. They form a topological category, which is not only cartesian closed, but also hereditary and in which products of quotients are quotients. Hereditariness has been studied extensively by H. Herrlich (cf. e.g.[7]), and it is very useful in the theory of connection and disconnection (cf. [17] and [16]). On the other hand the category **ULim** of uniform limit spaces is not hereditary (cf. [2]). Since the category **SUConv** of semiuniform convergence spaces contains important topological categories as nicely embedded subcategories, e.g. the categories **Fil** of filter spaces (introduced by Katětov [8]) and **Conv** of convergence spaces (=symmetric Kent convergence spaces [10]) respectively and their subcategories **Chy** of Cauchy spaces and $\mathbf{Top_s}$ of symmetric topological spaces respectively, it is suitable for studying continuity, Cauchy continuity and uniform continuity as well as convergence structures in function spaces such as simple convergence, continuous convergence and uniform convergence (cf. [16]). Since topological and uniform

concepts are avaible in **SUConv**, completeness can be defined, and the question arises, whether every semiuniform convergence space has a completion, i.e. whether it can be densely embedded into a complete semiuniform convergence space. The corresponding question for uniform limit spaces has been solved by Wyler [19], whereas completions in the realm of filter spaces have been constructed by Kent and Rath [11] as well as by Császár [6]. Here a completion is constructed for every semiuniform convergence space, called the simple completion. It generalizes one of Császár's completions of filter spaces, namely his λ–completion. By the way, it turns out that every filter space is a subspace (in **SUConv**) of some convergence space. Wyler's completion for separated uniform limit spaces can be easily derived from our simple completion, and its universal property follows directly from an extension lemma for the simple completion.
The terminology of this article corresponds to [1] and [13].

Conventions 1) A filter on a set X is not allowed to contain the empty set.
2) Subcategories are always assumed to be full and isomorphism–closed.

2. Some basic definitions and results

2.1 Definitions. 1) A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that the following are satisfied:

UC_1) $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in A\}$,

UC_2) $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \supset \mathcal{F}$,

UC_3) $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$, where $F^{-1} = \{(y,x) : (x,y) \in F\}$.

2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *semiuniform limit space* provided that the following is satisfied:

UC_4) $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$.

3) A semiuniform limit space is called a *uniform limit space* provided that the following is satisfied:

UC_5) $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$ (whenever $\mathcal{F} \circ \mathcal{G}$ exists, i.e. $F \circ G = \{(x,y) : \exists z \in X \text{ with } (x,z) \in G \text{ and } (z,y) \in F\} \neq \phi$ for every $F \in \mathcal{F}, G \in \mathcal{G}$), where $\mathcal{F} \circ \mathcal{G}$ is the filter generated by the filter base $\{F \circ G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$.

4) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{J}_X) \subset \mathcal{J}_Y$.

5) The category of semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**.

2.2 Remark. **SUConv** is a cartesian closed and hereditary topological category in which quotients are productive, i.e. a strong topological universe (cf. [15]). Initial and final structures can be easily described, e.g. if X in a set $((X_i, \mathcal{J}_{X_i}))_{i\in I}$ a family of semiuniform convergence spaces and $(f_i : X \to X_i)_{i\in I}$ a family of maps, then $\mathcal{J}_X = \{\mathcal{F} : \mathcal{F}$ is a filter on $X \times X$ and $(f_i \times f_i)(\mathcal{F}) \in \mathcal{J}_{X_i}$ for each $i \in I\}$ is the initial **SUConv**–structure on X with respect to the given data.

2.3 Definitions. 1) A *Kent convergence space* is a pair (X, q), where X is a set and $q \subset F(X) \times X$ (provided $F(X)$ denotes the set of all filters on X), such that the following are satisfied:

C_1) $(\dot{x}, x) \in q$ for each $x \in X$,

C_2) $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{G} \supset \mathcal{F}$,

C_3) $(\mathcal{F} \cap \dot{x}, x) \in q$ whenever $(\mathcal{F}, x) \in q$.

If (X, q) is a Kent convergence space, then we say that $\mathcal{F} \in F(X)$ *converges* to $x \in X$ provided that $(\mathcal{F}, x) \in q$.

2) A Kent convergence space (X, q) is called *symmetric* provided that the following is satisfied:

$$(\mathcal{F}, x) \in q \quad \text{and} \quad y \in \bigcap_{F\in\mathcal{F}} F \quad \text{imply} \quad (\mathcal{F}, y) \in q\ .$$

3) A map $f : (X, q) \to (X, q')$ between Kent convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.

2.4 Remarks. 1)a) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying (symmetric) Kent convergence space* $(X, q_{\gamma_{\mathcal{J}_X}})$ defined as follows:

$$(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}} \Leftrightarrow \mathcal{F} \cap \dot{x} \in \gamma_{\mathcal{J}_X}$$

where $\gamma_{\mathcal{J}_X} = \{\mathcal{H} \in F(X) : \mathcal{H} \times \mathcal{H} \in \mathcal{J}_X\}$.

b) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *convergence space* provided that

$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there are some $x \in X$ and $\mathcal{G} \in F(X)$ with $(\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}}$ such that $\mathcal{F} \supset \mathcal{G} \times \mathcal{G}\}$,

in other words: A convergence space is a semiuniform convergence space, which is 'generated' by all convergent filters. The category **Conv** of convergence spaces (and uniformly continuous maps) is isomorphic to the category $\mathbf{KConv_s}$ of symmetric Kent convergence spaces (and continuous maps); it is a bicoreflective subcategory of **SUConv** (cf. [16;3.2,3.3 and 3.4]) and thus closed under formation of quotients and coproducts (formed in **SUConv**).

2) If (X, q) is a Kent convergence space, then for each subset A of X we define the *closure* $\overline{A}$ of A by

$$\overline{A} = \{x \in X\text{: there is some } \mathcal{G} \in F(X) \text{ with } (\mathcal{G}, x) \in q \text{ and } A \in \mathcal{G}\}.$$

2.5 Definitions. 1) A *filtermerotopic space* (shortly: a *filter space*) is a pair (X, γ), where X is a set and $\gamma \subset F(X)$ such that the following are satisfied:

(1) $\dot{x} \in \gamma$ for each $x \in X$,

(2) $\mathcal{G} \in \gamma$ whenever $\mathcal{F} \in \gamma$ and $\mathcal{F} \subset \mathcal{G}$.

2) A filter space (X, γ) is called a *Cauchy space*, provided that the following is satisfied:

b) If $\mathcal{F}$ and $\mathcal{G}$ belong to γ such that every member of $\mathcal{F}$ meets every member of $\mathcal{G}$ (i.e. $\sup\{\mathcal{F}, \mathcal{G}\}$ exists), then $\mathcal{F} \cap \mathcal{G}$ belongs to γ.

3) A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that the filter $f(\mathcal{F})$ generated by $\{f[F] : F \in \mathcal{F}\}$ belongs to γ' for each $\mathcal{F} \in \gamma$.

4) The category of filter spaces (and Cauchy continuous maps) is denoted by **Fil**, whereas its full subcategory of all Cauchy spaces is denoted by **Chy**.

2.6 Remarks. 1) **Chy** is a bireflective subcategory of the strong topological universe **Fil**, and it is a cartesian closed topological category in which quotients are neither hereditary nor productive (cf. [4], [3] and [14]).

2)a) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an *underlying filter space* $(X, \gamma_{\mathcal{J}_X})$ defined as follows:

$$\gamma_{\mathcal{J}_X} = \{\mathcal{F} \in F(X) : \ \mathcal{F} \times \mathcal{F} \in \mathcal{J}_X\} .$$

The elements of $\gamma_{\mathcal{J}_X}$ are called $\mathcal{J}_X$*–Cauchy filters.*

b) If $(X, \mathcal{J}_X)$ is a uniform limit space, then the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is a Cauchy space (cf. [9]).

3) If a semiuniform convergence space $(X, \mathcal{J}_X)$ is called **Fil**–*determined* provided that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there is some $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$ (i.e. a semiuniform convergence space $(X, \mathcal{J}_X)$ is **Fil**–determined iff it is 'generated' by all $\mathcal{J}_X$–Cauchy filters), then the category **Fil–D–SUConv** of **Fil**–determined semiuniform convergence spaces (and uniformly continuous maps) is isomorphic to **Fil** and a bireflective and bicoreflective subcategory of **SUConv** (cf. [16;2.4]). **Conv** can be embedded into **Fil** as a bicoreflective subcategory (cf. [16;2.8,2.11.3) and 3.2]).

2.7 Definitions. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called
1) *complete* provided that each $\mathcal{J}_X$–Cauchy filter converges in $(X, q_{\gamma_{\mathcal{J}_X}})$,
2) a T_1*–space* provided that for each pair $(x, y) \in X \times X, (\dot{x}, y) \in q_{\gamma_{\mathcal{J}_X}}$ implies $x = y$,
3) a T_2*–space* (or *separated*) provided that each filter on X converges to at most one element of X in $(X, q_{\gamma_{\mathcal{J}_X}})$.

2.8 Proposition. (cf. [16;3.8]). *A semiuniform convergence space is a convergence space iff it is complete and* **Fil**–*determined.*

3. The simple completion

3.1 Definition. Let (X,γ) be a filter space. Then $\mathcal{F}\in\gamma$ is called *γ–free* provided that there is no fixed filter $\mathcal{G}\in\gamma$ with $\mathcal{G}\subset\mathcal{F}$.

3.2 Proposition. *If (X,γ) is a filter space, then there is an equivalence relation $\sim$ on γ defined by*

$$\begin{aligned}\mathcal{F}\sim\mathcal{G}\Leftrightarrow &\text{ There exist finitely many } \mathcal{F}_0,\dots,\mathcal{F}_n\in\gamma \text{ with } \mathcal{F}_0=\mathcal{F} \text{ and } \mathcal{F}_n=\mathcal{G}\\ &\text{ such that } \sup\{\mathcal{F}_{i-1},\mathcal{F}_i\} \text{ exists for each } i\in\{1,\dots,n\}.\end{aligned}$$

3.3. If (X,γ) is a filter space and $\mathcal{F}\in\gamma$, then the equivalence class of $\mathcal{F}$ with respect to $\sim$ is denoted by $[\mathcal{F}]$. If (X,γ) is a Cauchy space, then

$$\mathcal{F}\sim\mathcal{G}\Leftrightarrow\mathcal{F}\cap\mathcal{G}\in\gamma\ .$$

3.4 Theorem. *Let $(X,\mathcal{J}_X)$ be a semiuniform convergence space. Put $Y=X\cup\{[\mathcal{F}]:\mathcal{F}\ \gamma_{\mathcal{J}_X}\text{-free}\}$. If $i:X\to Y$ denotes the inclusion map, then a* **SUConv***–structure $\mathcal{J}_Y$ on Y is defined by*

$\mathcal{F}\in\mathcal{J}_Y$ *iff there is some* $\mathcal{G}\in\mathcal{B}_Y$ *with* $\mathcal{F}\supset\mathcal{G}$,

where $\mathcal{B}_Y=\{i\times i(\mathcal{H}):\mathcal{H}\in\mathcal{J}_X\}\cup\{(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}]):\mathcal{F}\ \gamma_{\mathcal{J}_X}\text{–free}\}$, and $(Y,\mathcal{J}_Y)$ is a complete semiuniform convergence space containing $(X,\mathcal{J}_X)$ as a dense subspace (i.e. $\overline{X}=Y$ where the closure is formed with respect to $(Y,q_{\gamma_{\mathcal{J}_Y}})$). The set $\gamma_{\mathcal{J}_Y}$ of all $\mathcal{J}_Y$–Cauchy filters is generated by

$$\{i(\mathcal{F}):\mathcal{F}\in\gamma_{\mathcal{J}_X}\ \ not\ \ \gamma_{\mathcal{J}_X}-free\}\cup\{i(\mathcal{F})\cap[\dot{\mathcal{F}}]:\ \mathcal{F}\ \gamma_{\mathcal{J}_X}-free\}\ .$$

Proof. It is easy to check that $(Y,\mathcal{J}_Y)\in|\mathbf{SUConv}|$. Furthermore, $(X,\mathcal{J}_X)$ is a subspace of $(Y,\mathcal{J}_Y)$: Since $i\times i(\mathcal{H})\in\mathcal{J}_Y$ for each $\mathcal{H}\in\mathcal{J}_X,\mathcal{J}_X\subset(\mathcal{J}_Y)_X=\{\mathcal{K}\in F(X\times X):i\times i(\mathcal{K})\in\mathcal{J}_Y\}$. Conversely, if $\mathcal{K}\in(\mathcal{J}_Y)_X$, then 1) $i\times i(\mathcal{K})\supset i\times i(\mathcal{H})$ with $\mathcal{H}\in\mathcal{J}_X$ or 2) $i\times i(\mathcal{K})\supset(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}])$ with $\mathcal{F}\ \gamma_{\mathcal{J}_X}$–free.
In the first case we obtain $\mathcal{K}=(i\times i)^{-1}(i\times i(\mathcal{K}))\supset\mathcal{H}=(i\times i)^{-1}(i\times i(\mathcal{K}))$ and thus $\mathcal{H}\in\mathcal{J}_X$. In the second case we get

$$\mathcal{K}\supset(i\times i)^{-1}((i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}]))=i^{-1}(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times i^{-1}(i(\mathcal{F})\cap[\dot{\mathcal{F}}])=\mathcal{F}\times\mathcal{F}\ .$$

Since $\mathcal{F}\times\mathcal{F}\in\mathcal{J}_X,\mathcal{K}\in\mathcal{J}_X$. Next we prove

$$(*)\left\{\begin{array}{l}\gamma_{\mathcal{J}_X}=\gamma' \text{ with } \gamma'=\{\mathcal{K}\in F(Y):\text{ there is some } \mathcal{G}\in\mathcal{B} \text{ mit } \mathcal{K}\supset\mathcal{G}\}\\ \text{where } \mathcal{B}=\{i(\mathcal{F}):\mathcal{F}\in\gamma_{\mathcal{J}_X} \text{ not } \gamma_{\mathcal{J}_X}\text{–free }\}\cup\{i(\mathcal{F})\cap[\dot{\mathcal{F}}]:\mathcal{F}\ \gamma_{\mathcal{J}_X}\text{–free}\}\end{array}\right.$$

a) If $\mathcal{K}\in\gamma_{\mathcal{J}_Y}$, then $\mathcal{K}\times\mathcal{K}\in\mathcal{J}_Y$. Thus, 1) $\mathcal{K}\times\mathcal{K}\supset i\times i(\mathcal{H})$ or 2) $\mathcal{K}\times\mathcal{K}\supset(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}])$ with $\mathcal{F}\ \gamma_{\mathcal{J}_X}$–free. In the first case X belongs to $\mathcal{K}$ (namely: $X\times X\in\mathcal{H}\subset i\times i(\mathcal{H})\subset\mathcal{K}\times\mathcal{K}$ means $X\times X\supset K\times K$ with $K\in\mathcal{K}$ and consequently $X\supset K$, i.e. $X\in\mathcal{K}$). Thus, $i^{-1}(\mathcal{K})=\mathcal{F}$ exists and, since $\mathcal{K}$ is a $\mathcal{J}_Y$–Cauchy filter, it is a $\mathcal{J}_X$–Cauchy filter. Furthermore, $i(\mathcal{F})=\mathcal{K}$ (since $X\in\mathcal{K}$), i.e. $\mathcal{K}\in\gamma'$. In the second case

$\mathcal{K} \supset i(\mathcal{F}) \cap [\dot{\mathcal{F}}]$ with $\mathcal{F}$ $\gamma_{\mathcal{J}_X}$–free, i.e. $\mathcal{K} \in \gamma'$.
b) If $\mathcal{K} \in \gamma'$, then 1) $\mathcal{K} \supset i(\mathcal{F})$ with $\mathcal{F} \in \gamma_{\mathcal{J}_X}$ not $\gamma_{\mathcal{J}_X}$–free or 2) $\mathcal{K} \supset i(\mathcal{F}) \cap [\dot{\mathcal{F}}]$ with $\mathcal{F}$ $\gamma_{\mathcal{J}_X}$–free. In the first case $\mathcal{K} \times \mathcal{K} \supset i(\mathcal{F}) \times i(\mathcal{F}) = i \times i(\mathcal{F} \times \mathcal{F})$, i.e. $\mathcal{K} \times \mathcal{K} \in \mathcal{J}_Y$ since $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$. Thus $\mathcal{K} \in \gamma_{\mathcal{J}_X}$. In the second case $\mathcal{K} \times \mathcal{K} \supset (i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) \times (i(\mathcal{F}) \cap [\dot{\mathcal{F}}])$, i.e. $\mathcal{K} \times \mathcal{K} \in \mathcal{J}_Y$. Hence $\mathcal{K} \in \gamma_{\mathcal{J}_Y}$.
It follows from $(*)$ that the underlying filter space $(Y, \gamma_{\mathcal{J}_Y})$ of $(Y, \mathcal{J}_Y)$ is Császár's λ–completion of the filter space $(X, \gamma_{\mathcal{J}_X})$ (cf. [6;5.1]). Thus $(Y, \gamma_{\mathcal{J}_Y})$ is complete, i.e. $(Y, \mathcal{J}_Y)$ is complete, and X is dense in Y (note, that the definition of closure for filter spaces used by A. Császár [6] is equivalent to our definition of closure, i.e. for a filter space (X, γ) and a subset A of X we have $\overline{A} = c\,(A)$, where – denotes the closure with respect to the Kent convergence space (X, q_γ) with q_γ defined by $(\mathcal{F}, x) \in q_\gamma \iff \mathcal{F} \cap \dot{x} \in \gamma$ and c denotes Császár's closure $[x \in c(A) \iff \exists \mathcal{F} \in \gamma$ with $x \in \bigcap_{F \in \mathcal{F}} F$ and $A \cap F \neq \phi \; \forall \, F \in \gamma])$.

3.5 Definition. Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space and $(Y, \mathcal{J}_Y)$ the complete semiuniform convergence space constructed in 3.4. Then the inclusion map $i : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ is called the *simple completion* of $(X, \mathcal{J}_X)$. Occasionally, $(Y, \mathcal{J}_Y)$ is already called the simple completion of $(X, \mathcal{J}_X)$.

3.6 Theorem. *Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space. Then the following are equivalent:*

(1) *$(X, \mathcal{J}_X)$ is* **Fil**–*determined.*

(2) *The simple completion $(Y, \mathcal{J}_Y)$ of $(X, \mathcal{J}_X)$ is* **Fil**–*determined.*

(3) *$(X, \mathcal{J}_X)$ is a dense subspace (with respect to* **SUConv***) of a convergence space.*

(4) *$(X, \mathcal{J}_X)$ is a subspace (with respect to* **SUConv***) of a convergence space.*

Proof. (1) $\Rightarrow$ (2). Since $(X, \mathcal{J}_X)$ is **Fil**–determined, $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there is some $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ with $\mathcal{F} \supset \mathcal{G} \times \mathcal{G}\}$. We have to prove that $\mathcal{J}_Y \subset \mathcal{J}_{\gamma_{\mathcal{J}_Y}} = \{\mathcal{H} \in F(Y \times Y)$: there is some $\mathcal{K} \in \gamma_{\mathcal{J}_Y}$ with $\mathcal{H} \supset \mathcal{K} \times \mathcal{K}\}$ since $\mathcal{J}_{\gamma_{\mathcal{J}_Y}} \subset J_Y$ is always valid. Let $\mathcal{H} \in \mathcal{J}_Y$. Then 1) $\mathcal{H} \supset i \times i(\mathcal{K})$ with $\mathcal{K} \in \mathcal{J}_X$ or 2) $\mathcal{H} \supset (i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) \times (i(\mathcal{F}) \cap [\dot{\mathcal{F}}])$ with $\mathcal{F}$ $\gamma_{\mathcal{J}_X}$–free. In the first case there is some $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{K}$ and thus $i \times i(\mathcal{G} \times \mathcal{G}) = i(\mathcal{G}) \times i(\mathcal{G}) \subset i \times i(\mathcal{K}) \subset \mathcal{H}$. Since $i(\mathcal{G}) \in \gamma_{\mathcal{J}_Y}$ (cf. 3.4), $i(\mathcal{G}) \times i(\mathcal{G}) \in \mathcal{J}_Y$ and consequently $\mathcal{H} \in \mathcal{J}_Y$. In the second case $\mathcal{K} = i(\mathcal{F}) \cap [\dot{\mathcal{F}}]$ with $\mathcal{F}$ $\gamma_{\mathcal{J}_X}$–free belongs to $\gamma_{\mathcal{J}_Y}$ (cf. 3.4) and $\mathcal{K} \times \mathcal{K} \subset \mathcal{H}$, i.e. $\mathcal{H} \in \mathcal{J}_Y$.

(2) $\Rightarrow$ (3). By assumption $(X, \mathcal{J}_X)$ is a dense subspace of the **Fil**–determined space $(Y, \mathcal{J}_Y)$. Since $(Y, \mathcal{J}_Y)$ is also complete, it follows from 2.8 that $(Y, \mathcal{J}_Y)$ is a convergence space.

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). By 2.8 each convergence space is **Fil**–determined, and **Fil–D–SUConv** is closed under formation of subspaces. (cf. [15;3.8]).

3.7 Corollary. *If* $R_{\mathbf{SUConv}}$ **Conv** *denotes the epireflective hull of* **Conv** *with respect to* **SUConv**, *then*

$$R_{\mathbf{SUConv}}\mathbf{Conv} = \mathbf{Fil\text{–}D\text{–}SUConv}\ .$$

Proof. Use 3.6 and note that **Conv** is closed under formation of products [in **SUConv**] (cf. [16;3.13])

3.8 Corollary. a) *If* $(X, \mathcal{J}_X)$ *is a complete semiuniform convergence space, then* $(X, \mathcal{J}_X) = (Y, \mathcal{J}_Y)$.

b) *If* $(X, \mathcal{J}_X)$ *is a convergence space, then* $(X, \mathcal{J}_X) = (Y, \mathcal{J}_Y)$.

Proof. a) If $(X, \mathcal{J}_X)$ is complete, then for each $\mathcal{J}_X$–Cauchy filter $\mathcal{F}$ there is some $x \in X$ with $\mathcal{F} \cap \dot{x} \in \gamma_{\mathcal{J}_X}$, i.e. $\mathcal{F}$ is not $\gamma_{\mathcal{J}_X}$–free. Thus, $X = Y$ and $\mathcal{J}_X = \mathcal{J}_Y$.

b) By 2.8 each convergence space is complete.

3.9 Proposition. *A semiuniform convergence space* $(X, \mathcal{J}_X)$ *is a* T_1*–space if and only if its simple completion* $(Y, \mathcal{J}_Y)$ *is a* T_1*–space.*

Proof. "$\Rightarrow$": Since by 3.5 $(Y, \gamma_{\mathcal{J}_Y})$ is Császár's λ–completion of $(X, \gamma_{\mathcal{J}_X})$, this implication follows from [6;5.3].

"$\Leftarrow$": The class of all T_1–spaces is closed under formation of subspaces (in **SU-Conv**).

3.10 Remarks. 1) Another property being preserved under simple completion is connectedness (or more generally $\mathcal{E}$–connectedness provided that $\mathcal{E}$ is a class of T_1–spaces) (cf. [16;6.21]).

2) Since **Fil** is isomorphic to **Fil–D–SUConv** (cf. [15;3.6]), filter spaces and **Fil**–determined semiuniform convergence spaces may be identified. Thus, it follows from 3.4 and 3.6 that *the simple completion of filter spaces is Császár's* λ*–completion of filter spaces* (cf. [6]).

3.11 Extension lemma. *Let* $(X, \mathcal{J}_X)$ *be a uniform limit space,* $i : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ *its simple completion,* $(X', \mathcal{J}_{X'})$ *a separated complete semiuniform convergence space and* $f : (X, \mathcal{J}_X) \to (X', \mathcal{J}_{X'})$ *a uniformly continuous map. Then there is a unique uniformly continuous map* $\overline{f} : (Y, \mathcal{J}_Y) \to (X', \mathcal{J}_{X'})$ *such that* $\overline{f} \circ i = f$.

Proof. By 2.6. 2)b) $(X, \gamma_{\mathcal{J}_X})$ is a Cauchy space. Thus, $\mathcal{G} \in [\mathcal{F}]$ iff $\mathcal{F} \cap \mathcal{G} \in \gamma_{\mathcal{J}_X}$. Then a map $\overline{f} : Y \to X'$ is defined by

$$\overline{f}(y) = \begin{cases} f(x) & \text{if} \quad y = x \in X \\ x' & \text{if} \quad y \in Y \backslash X\ , \end{cases}$$

where $(f(\mathcal{F}), x') \in q_{\gamma_{\mathcal{J}_{X'}}}$ with $y = [\mathcal{F}]$ (x' exists, since $(X', \mathcal{J}_{X'})$ is complete, and it is uniquely determined, since $(X', \mathcal{J}_{X'})$ is a T_2–space; furthermore, it is independent of the choice of the representative $\mathcal{F}$). Obviously, $\overline{f} \circ i = f$. In order to prove that $\overline{f}$ is uniformly continuous, let $\mathcal{K} \in \mathcal{J}_Y$. Then 1) $\mathcal{K} \supset i \times i(\mathcal{H})$ with $\mathcal{H} \in \mathcal{J}_X$ or 2) $\mathcal{K} \supset (i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) \times (i(\mathcal{F}) \cap [\dot{\mathcal{F}}])$ with $\mathcal{F}$ $\gamma_{\mathcal{J}_X}$–free. In the first case $\overline{f} \times \overline{f}(\mathcal{K}) \supset$

$\overline{f}\times\overline{f}(i\times i(\mathcal{H}))=(\overline{f}\circ i\times\overline{f}\circ i)(\mathcal{H})=f\times f(\mathcal{H})\in\mathcal{J}_{X'}$, i.e. $\overline{f}\times\overline{f}(\mathcal{K})\in\mathcal{J}_{X'}$. In the second case $\overline{f}\times\overline{f}(\mathcal{K})\supset\overline{f}\times\overline{f}((i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}])=\overline{f}(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times\overline{f}(i(\mathcal{F})\cap[\dot{\mathcal{F}}])=(f(\mathcal{F})\cap\overline{f}[\dot{\mathcal{F}}]))\times(f(\mathcal{F})\cap\overline{f}[\dot{\mathcal{F}}])=(f(\mathcal{F})\cap\dot{x}')\times(f(\mathcal{F})\cap\dot{x}')\in\mathcal{J}_{X'}$ since $f(\mathcal{F})\cap\dot{x}'\in\gamma_{\mathcal{J}_{X'}}$ (i.e. $(f(\mathcal{F}),x')\in q_{\gamma_{\mathcal{J}_{X'}}}$). Thus, $\overline{f}\times\overline{f}(\mathcal{K})\in\mathcal{J}_{X'}$. Since now $\overline{f}:Y\to X'$ is uniformly continuous, it is also continuous with respect to the underlying Kent convergence spaces. Hence, $\overline{f}$ is uniquely determined by $\overline{f}\circ i=f$, since X is dense in Y and X' is a T_2–space.

4. Relationship between the simple completion and the Wyler completion in the realm of separated uniform limit spaces

4.1 Proposition. *Let (X,γ) be a filter space and $\mathcal{F}\in\gamma$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is γ–free.*

(2) *$\mathcal{F}$ does not converge in the underlying (symmetric) Kent convergence space (X,q_γ) defined by $(\mathcal{G},x)\in q_\gamma$ iff $\mathcal{G}\cap\dot{x}\in\gamma$.*

If (X,γ) is a Cauchy space, then each of the above conditions is equivalent to

(3) *$[\mathcal{F}]\neq[\dot{x}]$ for each $x\in X$.*

Proof. "(1) $\Rightarrow$ (2)" (indirectly). If $\mathcal{F}\in\gamma$ converges to $x\in X$ in (X,q_γ), then $\mathcal{F}\cap\dot{x}\in\gamma$. Since $\mathcal{F}\cap\dot{x}\subset\mathcal{F}$ is fixed, $\mathcal{F}$ is not γ–free.

"(2) $\Rightarrow$ (1)" (indirectly). If $\mathcal{F}\in\gamma$ is not γ–free, then there is a fixed $\mathcal{F}_0\in\gamma$ such that $\mathcal{F}_0\subset\mathcal{F}$. Choose $x\in\cap\mathcal{F}_0$. Then $\mathcal{F}_0\cap\dot{x}=\mathcal{F}_0\subset\mathcal{F}\cap\dot{x}$ and consequently $\mathcal{F}\cap\dot{x}\in\gamma$, i.e. $(\mathcal{F},x)\in q_\gamma$.

Let (X,γ) be Cauchy space:

"(1) $\Rightarrow$ (3)" (indirectly). If there is some $x\in X$ such that $[\mathcal{F}]=[\dot{x}]$, then $\mathcal{F}\cap\dot{x}\in\gamma$ is a fixed filter. Thus $\mathcal{F}$ is not γ–free.

"(3) $\Rightarrow$ (1)" (indirectly). If $\mathcal{F}$ is not γ–free, then $\mathcal{F}\cap\dot{x}\in\gamma$ (cf. "(2) $\Rightarrow$ (1)"), i.e. $(\mathcal{F},x)\in q_\gamma$.

4.2 Remark. If $(X,\mathcal{J}_X)$ is a separated uniform limit space and $Y'=\{[\mathcal{F}]:\mathcal{F}\in\gamma_{\mathcal{J}_X}\}$ (i.e. Y' is the underlying set of the Wyler completion of $(X,\mathcal{J}_X)$), then $j:X\to Y'$ defined by $j(x)=[\dot{x}]$ is injective ($[\dot{x}]=[\dot{y}]$ means $\dot{x}\cap\dot{y}\in\gamma_{\mathcal{J}_X}$, i.e. $\dot{x}$ converges to y in the T_1–space $(X,q_{\gamma_{\mathcal{J}_X}})$; consequently $x=y$). Thus, X and $\{[\dot{x}]:x\in X\}$ may be identified. Then, using 4.1., Y' may be identified with $Y=X\cup\{[\mathcal{F}]:\ \mathcal{F}\ \gamma_{\mathcal{J}_X}\text{–free}\}$.

4.3 Proposition (cf.[2;1.3.4]). *The category* **SULim** *of semiuniform limit spaces (and uniformly continuous maps) is a bireflective subcategory of* **SUConv**. *If $(X,\mathcal{J}_X)$ is a semiuniform convergence space and $\mathcal{J}_X^*=\{\mathcal{F}\in F(X\times X)$: there exist finitely many $\mathcal{F}_1,\dots,\mathcal{F}_n\in\mathcal{J}_X$ such that $\mathcal{F}\supset\bigcap_{i=1}^{n}\mathcal{F}_i\}$, then $1_X:(X,\mathcal{J}_X)\to(X,\mathcal{J}_X^*)$ is the bireflection of $(X,\mathcal{J}_X)$ with respect to* **SULim**.

4.4 Theorem. *Let $(X,\mathcal{J}_X)$ be a separated uniform limit space and $(Y,\mathcal{J}_Y)$ its simple completion. Then the bireflective* **SULim***–modification of $(Y,\mathcal{J}_Y)$, namely $(Y,\mathcal{J}_Y^*)$, is the Wyler completion of $(X,\mathcal{J}_X)$, i.e. a separated complete uniform limit space containing $(X,\mathcal{J}_X)$ as a dense subspace and being characterized (up to isomorphism) by the following universal property: If $f:(X,\mathcal{J}_X)\to(X',\mathcal{J}_{X'})$ is a uniformly continuous map from $(X,\mathcal{J}_X)$ into a separated complete uniform limit space, then there is a unique uniformly continuous map $\overline{f}:(Y,\mathcal{J}_Y^*)\to(X',\mathcal{J}_{X'})$ such that $\overline{f}\circ i=f$, where $i:X\to Y$ denotes the inclusion map.*

Proof. $\mathcal{J}_Y^*$ is generated by $\{i\times i(\mathcal{H})\cap[(i(\mathcal{F}_1)\cap[\dot{\mathcal{F}}_1])\times(i(\mathcal{F}_1)\cap[\dot{\mathcal{F}}_1])]\cap\ldots\cap[(i(\mathcal{F}_n)\cap[\dot{\mathcal{F}}_n])\times(i(\mathcal{F}_n)\cap[\dot{\mathcal{F}}_n])]:\mathcal{H}\in\mathcal{J}_X$ and $\mathcal{F}_1,\ldots,\mathcal{F}_n\in\gamma_{\mathcal{J}_X}$ non–convergent$\}$.
If $(Y,\mathcal{J}_Y^w)$ denotes the Wyler completion of $(X,\mathcal{J}_X)$ (cf. [19]), $\mathcal{J}_Y^w$ is generated by $\{i\times i(\mathcal{H})\cap[(i(\mathcal{F}_1)\times[\dot{\mathcal{F}}_1])\cap([\dot{\mathcal{F}}_1]\times i(\mathcal{F}_1))]\cap\ldots\cap[(i(\mathcal{F}_n)\times[\dot{\mathcal{F}}_n])\cap([\dot{\mathcal{F}}_n]\times i(\mathcal{F}_n))]:\mathcal{H}\in\mathcal{J}_X$ and $\mathcal{F}_1,\ldots,\mathcal{F}_n\in\gamma_{\mathcal{J}_X}$ non–convergent$\}$ (it might be assumed additionally that the filters $\mathcal{F}_i$ are pairwise non–equivalent, i.e. $[\mathcal{F}_i]\neq[\mathcal{F}_k]$ for $i\neq k$). For each $\mathcal{F}\in\gamma_{\mathcal{J}_X}$ the following is valid:

$$(*)\quad (i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}])=(i(\mathcal{F})\times i(\mathcal{F}))\cap(i(\mathcal{F})\times i[\dot{\mathcal{F}}])\cap$$
$$\cap([\dot{\mathcal{F}}]\times i(\mathcal{F}))\cap([\dot{\mathcal{F}}]\times[\dot{\mathcal{F}}])\subset(i(\mathcal{F}\times[\dot{\mathcal{F}}])\cap([\dot{\mathcal{F}}]\times i(\mathcal{F}))$$

If follows immediately from $(*)$ that $\mathcal{J}_Y^w\subset\mathcal{J}_Y^*$. According to Wyler [19; Satz 3.2] $i(\mathcal{F})\cap[\dot{\mathcal{F}}]$ is a $\mathcal{J}_Y^w$–Cauchy filter for each $\mathcal{F}\in\gamma_{\mathcal{J}_X}$, i.e. $(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\times(i(\mathcal{F})\cap[\dot{\mathcal{F}}])\in\mathcal{J}_Y^w$. Furthermore, for each $\mathcal{H}\in\mathcal{J}_X, i\times i(\mathcal{H})\in J_Y^w$. Since $\mathcal{J}_Y^w$ is a **SULim**–structure (even a **ULim**–structure), $\mathcal{J}_Y^*\subset\mathcal{J}_Y^w$.

4.5 Remark. The universal property of the Wyler completion follows immediately from 3.11 and the universal property of the **SULim**–bireflection: If $(X,\mathcal{J}_X)$ is a separated uniform limit space , $(X',\mathcal{J}_{X'})$ a separated complete uniform limit space and $f:(X,\mathcal{J}_X)\to(X',\mathcal{J}_{X'})$ a uniformly continuous map, then there is a unique uniformly continuous map $\overline{f}:(Y,\mathcal{J}_Y)\to(X',\mathcal{J}_{X'})$ such that $\overline{f}\circ i=f$ (cf. 3.11). If $1_Y:(Y,\mathcal{J}_Y)\to(Y,\mathcal{J}_Y^*)$ is the bireflection of $(Y,\mathcal{J}_Y)$ with respect to **SULim**, then there is a unique uniformly continuous $g:(Y,\mathcal{J}_Y^*)\to(X',\mathcal{J}_{X'})$ such that $\overline{f}\circ 1_Y=g$ (i.e. $g=\overline{f}$). Thus $\overline{f}$ is the unique uniformly continuous extension of f to the Wyler completion of $(X,\mathcal{J}_X)$:

$$\begin{tikzcd}
(X,\mathcal{J}_X) \arrow[rr, "f"] \arrow[dr, "i"'] & & (X',\mathcal{J}_{X'}) \\
& (Y,\mathcal{J}_Y) \arrow[ur, "\overline{f}"] \arrow[d, "1_Y"'] & \\
& (Y,\mathcal{J}_Y^*) \arrow[uur, "\overline{f}"'] &
\end{tikzcd}$$

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