# Mittag-Leffler type functions: notes on growth properties and distribution of zeros

**R.Gorenflo**[1)], **Yu.Luchko**[1)] **and S.Rogosin**[1),2)]

[1)] Fachbereich Mathematik und Informatik,
Freie Universität Berlin, D-14195 Berlin, Germany

[2)] Department of Mathematics and Mechanics,
Belarusian State University, BY-220050 Minsk, Belarus

**Abstract**

Integral representations, asymptotic behaviour and distribution of zeros for classical Mittag-Leffler functions as well as for the functions of Mittag-Leffler type are discussed. They are obtained by different approaches. Special attention is paid to the questions of growth regularity of these functions.

1991 **Mathematics Subject Classification:** 30D10, 30D15, 30E20, 33B20
**Key Words:** Asymptotics, Completely Regular Growth, Integral Representation, Mittag-Leffler Function, Distribution of Zeros.

# 1 Introduction

Recently it turned out that the role of the well-known *Mittag-Leffler function* [1]

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \ \alpha > 0, \tag{1}$$

in different branches of analysis is more essential than it seemed before (see e.g. [2-6]). A number of problems in applications were found for which the knowledge of its properties are of great importance. Some of these problems are studied in [7-16] and in the papers referenced there. Mainly these applications are due to connections of Mittag-Leffler functions with fractional calculus (see e.g. [5]). At the beginning of the 50's a certain generalization of the function (1) was introduced by Agarval, Humbert and others [25], [26]

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ \alpha > 0, \ \beta \in \mathbf{C}, \tag{2}$$

called by M.M. Djrbashian the *Mittag-Leffler type function*. The main properties of the function (1) as well as the function (2) are reviewed in the article [17] from the point of view of these functions' key role in the solution of certain problems of fractional calculus (see also [27]). Some of these results were obtained in spirit of Djrbashian-Sedletskii's approach ([2], [3], [18]) based on deep properties of analytic functions.

Here we try to give some explanations and extensions of the results from [17], [18].

# 2 Integral representations

As it is well-known the functions (1) and (2) are entire functions of the order $\rho = 1/\alpha$ and of the type $\sigma = 1$ (this can be ensured by direct calculations of the latter parameters using the corresponding formulas in terms of Taylor coefficients and the Stirling asymptotic formula for the $\Gamma$-function (see e.g. [2], [19])).

Let us consider first the known integral representation of the Mittag-Leffler type function (see e.g. [2], [19])

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta,\ \ z \in G^{(-)}(\varepsilon;\delta), \tag{3}$$

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}} + \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta,\ \ z \in G^{(+)}(\varepsilon;\delta), \tag{4}$$

where

$$0 < \alpha < 2,\ \ \Re(\beta) < 1 + \alpha, \tag{5}$$

and a number $\delta$ is chosen to satisfy the inequality

$$\frac{\pi\alpha}{2} < \delta \le \min\{\pi\alpha, \pi\}; \tag{6}$$

the contour $\gamma(\varepsilon;\delta)$ is shown on the Fig. 1 below and consists of parts of two rays and a part $C_\delta(0;\varepsilon)$ of circumference with the center at 0 and with radius $r_0 = \varepsilon$; the domains $G^{(-)}(\varepsilon;\delta)$ and $G^{(+)}(\varepsilon;\delta)$ are situated at the left and at the right of $\gamma(\varepsilon;\delta)$, respectively.

Let us denote for shortness the right-hand sides of (3) and (4) by $E^{-}_{\alpha,\beta}(z)$ and $E^{+}_{\alpha,\beta}(z)$, respectively. Our aim here is to give sense to the right-hand sides of formulas (3)-(4) as there are multivalued functions in them. On the

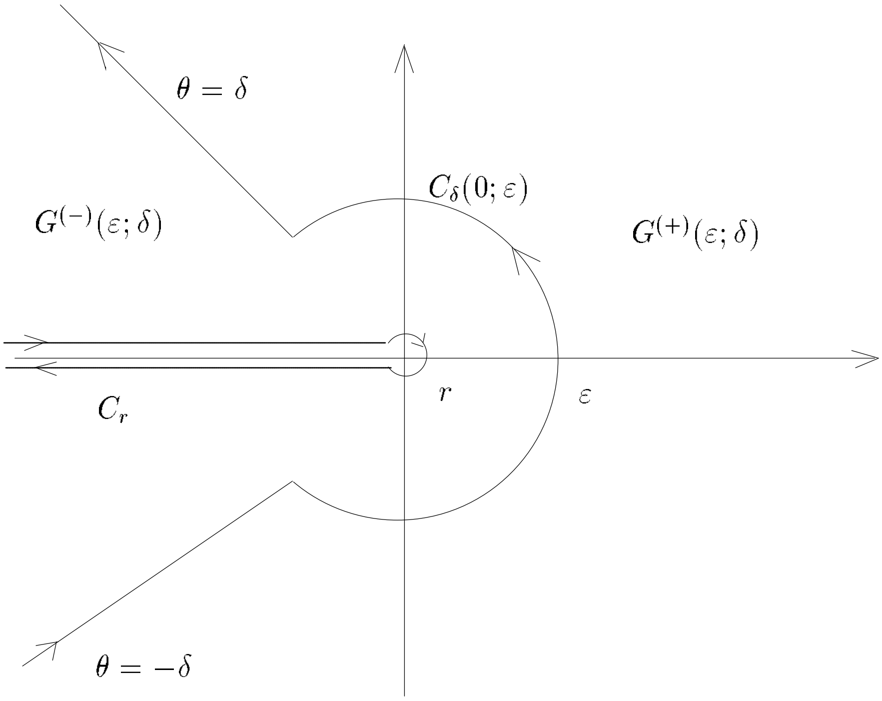


Figure 1: The contour $\gamma(\varepsilon;\delta)$

base of our discussion we obtain some new representation formulas for the Mittag-Leffler type function.

First we have to make sense for multivalued functions $\zeta^{1/\alpha}$ and $\zeta^{(1-\beta)/\alpha}$ in formulas (3), (4), if at least one of the numbers $1/\alpha$ and $(1-\beta)/\alpha$ is not a real integer. Let us take so called *principle branches* of these functions supposing further ($\zeta = re^{i\phi},\ -\pi < \phi < \pi$)

$$\zeta^{1/\alpha} = r^{1/\alpha}e^{i\phi/\alpha};\ \zeta^{(1-\beta)/\alpha} = r^{(1-\Re(\beta))/\alpha}e^{\Im(\beta)\phi/\alpha}e^{i[(1-\Re(\beta))\phi/\alpha-\Im(\beta)\log r/\alpha]}. \quad (7)$$

After this the functions $\zeta^{1/\alpha}$, $\zeta^{(1-\beta)/\alpha}$ are to be considered as single-valued analytic functions in the complex plane $\mathbf{C}$ cut along the negative real semi-axis. On the positive real axis $\zeta^{1/\alpha}$ is positive-real, and $\zeta^{(1-\beta)/\alpha}$ is positive-real if $\beta$ is real.

The question now is: whether there exists the analytic continuation of the functions $E^{\mp}_{\alpha,\beta}(z)$ defined by (3), (4) under conditions (7) through the lines $\gamma(\varepsilon;\delta)$ and $(-\infty,0)$?

Let us now take a point $\omega \in \gamma(\varepsilon;\delta)$. Due to the *Sokhotsky-Plemelj formulas for Cauchy-type integrals* (see e.g. [20]) we can obtain (taking into account that the chosen notation for domains are opposite to those usual for Sokhotsky-Plemelj formulas):

$$E^{-}_{\alpha,\beta}(\omega^{(-)}) = \lim_{\substack{z\to\omega\\ z\in G^{(-)}(\varepsilon;\delta)}} \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\alpha}\omega^{(1-\beta)/\alpha}e^{\omega^{1/\alpha}} + \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - \omega}\,d\zeta;$$

$$E^{+}_{\alpha,\beta}(\omega^{(+)}) = \lim_{\substack{z\to\omega\\ z\in G^{(+)}(\varepsilon;\delta)}} \left[\frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta + \frac{1}{\alpha}z^{(1-\beta)/\alpha}e^{z^{1/\alpha}}\right]$$

$$= -\frac{1}{2\alpha}\omega^{(1-\beta)/\alpha}e^{\omega^{1/\alpha}} + \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - \omega}\,d\zeta + \frac{1}{\alpha}\omega^{(1-\beta)/\alpha}e^{\omega^{1/\alpha}}.$$

All the integrals in the right-hand sides of the last formulas here and in the further discussions are to be understood as Cauchy principal values. So, the values of $E^{-}_{\alpha,\beta}(\omega^{(-)})$ and $E^{+}_{\alpha,\beta}(\omega^{(+)})$ are equal.

Let us consider now the neighbourhood of the negative real semi-axis. Two situations are to be distinguished here:

1) $1/\alpha \in (\frac{1}{2}+2k; \frac{3}{2}+2k)$, $k \in \mathbf{N}^{+}$: in this case the function $e^{\zeta^{1/\alpha}}$ decreases exponentially along both sides of the cut $(-\infty, 0)$, when $r = |\zeta| \to +\infty$;

2) $1/\alpha \in [\frac{3}{2}+2k; \frac{5}{2}+2k]$, $k \in \mathbf{N}^{+}$: in this case the function $e^{\zeta^{1/\alpha}}$ is bounded or increasing along both sides of the cut $(-\infty, 0)$, when $r = |\zeta| \to +\infty$.

Introduce now an integer $m$:

$$m = \begin{cases} 0, & 1/\alpha \in (\frac{1}{2}+2k; \frac{3}{2}+2k),\ k \in \mathbf{N}^{+}, \\ [4/\alpha]\,, & 1/\alpha \in [\frac{3}{2}+2k; \frac{7}{4}+2k),\ k \in \mathbf{N}^{+}, \\ & 1/\alpha \in [2+2k; \frac{9}{4}+2k),\ k \in \mathbf{N}^{+}, \\ [4/\alpha]-1, & 1/\alpha \in [\frac{7}{4}+2k; 2+2k),\ k \in \mathbf{N}^{+}, \\ & 1/\alpha \in [\frac{9}{4}+2k; \frac{5}{2}+2k],\ k \in \mathbf{N}^{+}. \end{cases} \tag{8}$$

and then add to and subtract from the right side of the formula (3) the integral

$$I(z) = \frac{e^{z^m}}{2\pi\alpha i}\int_{C_r} \frac{e^{\zeta^{1/\alpha}-\zeta^m}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta,\ \ z\in G^{(-)}(\varepsilon;\delta)\setminus(-\infty;0], \tag{9}$$

where $C_r$ can be seen in Fig. 1 ($C_r = (-\infty;-r)\cup C_\pi(0,r)\cup(-r;-\infty)$) with corresponding orientation ($0<r<|z|<\varepsilon$):

$$E^-_{\alpha,\beta}(z) = \frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta + \frac{e^{z^m}}{2\pi\alpha i}\int_{C_r} \frac{e^{\zeta^{1/\alpha}-\zeta^m}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta \tag{10}$$

$$-\frac{e^{z^m}}{2\pi\alpha i}\int_{C_r} \frac{e^{\zeta^{1/\alpha}-\zeta^m}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta = \frac{e^{z^m}}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)\cup C_r} \frac{e^{\zeta^{1/\alpha}-\zeta^m}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta$$

$$+\frac{1}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta - z}(1-e^{z^m-\zeta^m})\,d\zeta - I(z) := \Phi(z)+\Psi(z)-I(z).$$

Immediate from the Cauchy theorem we deduce

$$\Psi(z) = 0,\ \ z\in G^{(-)}(\varepsilon;\delta)\setminus(-\infty,0). \tag{11}$$

Further, we can apply the Cauchy integral formula to $\Phi(z)$ and obtain

$$\Phi(z) = \frac{e^{z^m}}{2\pi\alpha i}\int_{\gamma(\varepsilon;\delta)\cup C_r} \frac{e^{\zeta^{1/\alpha}-\zeta^m}\zeta^{(1-\beta)/\alpha}}{\zeta - z}\,d\zeta = \frac{1}{\alpha}e^{z^{1/\alpha}}z^{(1-\beta)/\alpha}, \tag{12}$$

$$z\in G^{(-)}(\varepsilon;\delta)\setminus(-\infty,0).$$

Using the properties of the integrand near the point $\zeta=0$ we can conclude that the formula (12) (as well as the formula for $I(z)$ in the further discussion) holds for all $r>0$. So we can let in it tend $r\to+0$.

Let us use in the following calculations the notation:

$$\phi_m(z) = e^{z^{1/\alpha}-z^m}z^{(1-\beta)/\alpha},\ \text{for } m\neq 0,\ \ \phi_0(z) = e^{z^{1/\alpha}}z^{(1-\beta)/\alpha}. \tag{13}$$

If $z$ tends to $x^{(+)} = |x|e^{i\pi}$ or $x^{(-)} = |x|e^{-i\pi}$ in the formula (10), then we have for different parts of it (again using the Sokhotsky-Plemelj formula)

$$\lim_{z\to x^{(\pm)}}\Phi(z) = \frac{1}{\alpha}\phi_0(x^{(\pm)}), \tag{14}$$

$$\lim_{z\to x^{(+)}} I(z) = \frac{1}{2\alpha}\phi_0(x^{(+)}) - \frac{e^{(x^{(+)})^m}}{2\pi\alpha i}\int_0^\infty \frac{\phi_m(\tau^{(+)})}{\tau-|x|}\,d\tau \tag{15}$$

$$-\frac{1}{2\alpha}\phi_0(x^{(-)}) + \frac{e^{(x^{(-)})^m}}{2\pi\alpha i}\int_0^\infty \frac{\phi_m(\tau^{(-)})}{\tau-|x|}\,d\tau,$$

$$\lim_{z\to x^{(-)}} I(z) = -\frac{1}{2\alpha}\phi_0(x^{(+)}) - \frac{e^{(x^{(+)})^m}}{2\pi\alpha i}\int_0^\infty \frac{\phi_m(\tau^{(+)})}{\tau-|x|}\,d\tau \tag{16}$$

$$+\frac{1}{2\alpha}\phi_0(x^{(-)}) + \frac{e^{(x^{(-)})^m}}{2\pi\alpha i}\int_0^\infty \frac{\phi_m(\tau^{(-)})}{\tau-|x|}\,d\tau.$$

Combining the results of (11), (14-16) we conclude

$$E^-_{\alpha,\beta}(x^{(+)}) = E^-_{\alpha,\beta}(x^{(-)}) = \frac{1}{2\alpha}\phi_0(x^{(+)}) + \frac{1}{2\alpha}\phi_0(x^{(-)}) \tag{17}$$

$$+\frac{1}{2\pi\alpha i}\int_0^\infty \frac{e^{(x^{(+)})^m}\phi_m(\tau^{(+)}) - e^{(x^{(-)})^m}\phi_m(\tau^{(-)})}{\tau-|x|}\,d\tau.$$

Inserting here the formulas (13) and (7) we can obtain the following representation of the function $E_{\alpha,\beta}(z)$ of Mittag-Leffler type on the negative real semi-axis:

$$E_{\alpha,\beta}(-x) = \frac{1}{\alpha}e^{x^{1/\alpha}\cos(\pi/\alpha)+(1-\Re(\beta))\log(x)/\alpha}x^{-i\Im(\beta)/\alpha}$$

$$\times[\cos(A(x))\cosh(B) - i\sin(A(x))\sinh(B)] \tag{18}$$

$$+\frac{1}{\pi\alpha i}\int_0^\infty e^{(-1)^m(x^m-\tau^m)+\tau^{1/\alpha}\cos(\pi/\alpha)+(1-\Re(\beta))\log(\tau)/\alpha}\tau^{-i\Im(\beta)/\alpha}$$

$$\times[-\cos(A(\tau))\sinh(B) - i\sin(A(\tau))\cosh(B)]\frac{d\tau}{\tau-x},\ x>0,$$

where

$$A(x) = \pi(1-\Re(\beta))/\alpha + x^{1/\alpha}\sin(\pi/\alpha),\ B = \pi\Im(\beta)/\alpha.$$

REMARK 1. The second condition (5) (i.e. $\Re(\beta) < 1+\alpha$) guarantees the existence of the limit

$$\lim_{\varepsilon\to+0}\int_{C(\varepsilon;\delta)} \frac{e^{\zeta^{1/\alpha}}\zeta^{(1-\beta)/\alpha}}{\zeta-z}\,d\zeta = 0. \tag{19}$$

In fact this was proved when we discussed the behaviour of the integrand near $\zeta=0$. Consequently, we can put $\varepsilon=0$ in the initial formulas (3), (4).

Then the integral representations (3), (4) coincide with that proposed in [3] (only if $\Re(\beta) < 1+\alpha$).
REMARK 2. The integral representation formulas (3), (4) are valid also for $\Re(\beta) \geq 1+\alpha$. Anyway the above proposed method cannot be applied in this situation because we cannot go to the limit in (19) (as well as in (12)) by $\varepsilon \to 0$.

In the case $1/\alpha = n$, $n \in \mathbf{N}$ , the following simpler formulas for the function of Mittag-Leffler type can be used (see e.g. [2], [3]):

$$E_{1/n,\beta}(z) = z^{(1-\beta)n} e^{z^n} \left\{ z_0^{(\beta-1)n} e^{-z_0^n} E_{1/n,\beta}(z_0) \right. \tag{20}$$

$$\left. + n \int_{z_0}^{z} e^{-\tau^n} \left( \sum_{k=1}^{n} \frac{\tau^{-k}}{\Gamma(\beta - k/n)} \tau^{\beta n - 1} \right) d\tau \right\},\ n \in \mathbf{N},$$

where $z_0$ is an arbitrary complex number, $z_0 \neq 0$. The most interesting cases of the latter formula are:

$$E_{1,\beta}(z) = z^{1-\beta} e^{z} \left\{ z_0^{\beta-1} e^{-z_0} E_{1,\beta}(z_0) + \int_{z_0}^{z} e^{-\tau} \frac{\tau^{\beta-2}}{\Gamma(\beta-1)} d\tau \right\}, \tag{21}$$

$$E_{1/2,\beta}(z) = z^{2(1-\beta)} e^{z^2} \left\{ z_0^{2(\beta-1)} e^{-z_0^2} E_{1/2,\beta}(z_0) \right. \tag{22}$$

$$\left. + 2\int_{z_0}^{z} e^{-\tau^2} \left[ \frac{\tau^{2\beta-2}}{\Gamma(\beta-1/2)} + \frac{\tau^{2\beta-3}}{\Gamma(\beta-1)} \right] d\tau \right\}.$$

Formula (20) is true with $z_0 = 0$, if $\beta = 1$. In this case we have

$$E_{1/n,1}(z) = e^{z^n} \left\{ 1 + n \int_{0}^{z} e^{-\tau^n} \left( \sum_{k+1}^{n-1} \frac{\tau^{k-1}}{\Gamma(\frac{k}{n})} \right) d\tau \right\},\ n \geq 2.$$

In particular,

$$E_{1/2,1}(z) = e^{z^2} \left\{ 1 + \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-\tau^2} d\tau \right\} = \omega(-iz) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t+iz} dt. \tag{23}$$

Many analytical properties of the latter function are well-known due to its connection with the *probability integral.* There exist some tables for the function $\omega(z)$ appearing in (23) (see e.g. [21]).

# 3 A Couple of Asymptotic Results

The asymptotic behaviour of an entire function is usually studied via its restriction to rays. Let us point out here some asymptotic results for the function $E_{1/2,1}(z)$ choosing the most important rays for it. These can be found by comparing this function with the function $e^{z^2}$. In order to do it we have to introduce the so called *indicator function* of an entire function of order $\rho$:

$$h(\theta) = \limsup_{r \to \infty} \frac{\log \mid f(re^{i\theta}) \mid}{r^\rho}. \tag{24}$$

As already seen both functions compared are of the order $\rho = 2$. Simple calculations show us that the function $e^{z^2}$ has the indicator function (see also Fig. 2):

$$h(\theta) = \cos 2\theta, \ -\pi \le \theta \le \pi. \tag{25}$$

If we take as a working hypothesis the closeness of $E_{1/2,1}(z)$ to $e^{z^2}$, then the characteristic rays for our function should be the following: $\arg z = \theta = 0;\ \pm\pi/4;\ \pm\pi/2;\ \pm 3\pi/4;\ \pm\pi$. Because of the symmetry

$$E_{1/2,1}(\overline{z}) = \overline{E_{1/2,1}(z)}, \tag{26}$$

it is sufficient to consider only rays in the upper half-plane:

$$\theta = 0;\ \pi/4;\ \pi/2;\ 3\pi/4;\ \pi.$$

1) From the formula (23) we can get the results

$$E_{1/2,1}(x) = e^{x^2}\left(1 + \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right) \sim 2e^{x^2}, x \to +\infty; \tag{27}$$

2) For any fixed $\eta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ $\left(= (-\frac{\pi\alpha}{2}, \frac{\pi\alpha}{2})\right)$, due to Cauchy's theorem, we have for all $z = xe^{i\eta},\ x > 0$

$$\int_0^z e^{-\zeta^2}\,d\zeta = \int_0^x e^{-t^2}\,dt + \int_0^\eta e^{-x^2\cos 2\phi} ixe^{i\phi}e^{-ix^2\sin 2\phi}\,d\phi. \tag{28}$$

The right-most integral in (28) is tending to zero when $x$ is tending to $+\infty$. Therefore

$$E_{1/2,1}(xe^{i\eta}) \sim 2e^{x^2\cos 2\eta}, x \to \infty\ (\eta \in (-\frac{\pi}{4}, \frac{\pi}{4})); \tag{29}$$

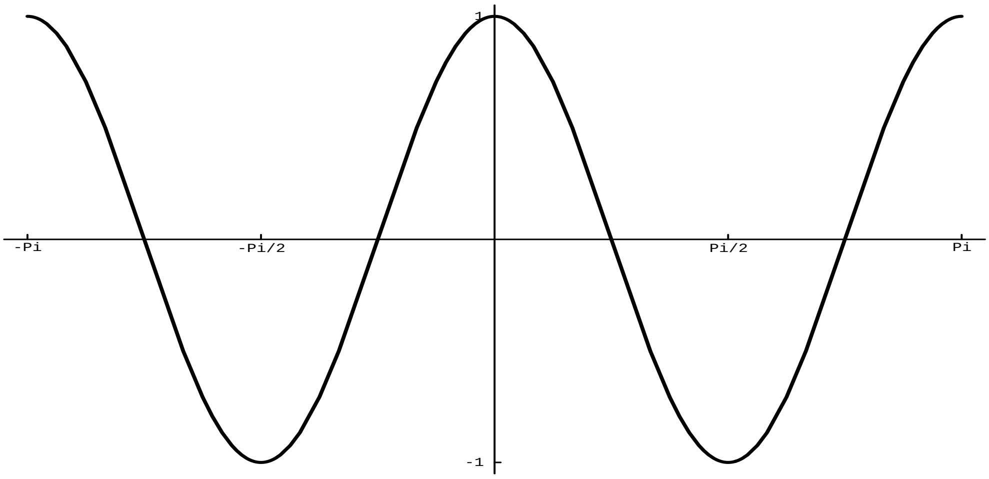


Figure 2: The indicator function for $e^z$

3) If $\theta = \frac{\pi}{4}$, then the asymptotics for integral in (23) can be obtained via the *method of stationary phase* (see e.g. [22]):

$$E_{1/2,1}(xe^{i\pi/4}) = e^{ix^2}\left\{1 + \frac{\sqrt{2}(i+1)}{\sqrt{\pi}}\int_0^x e^{-it^2}\,dt\right\} \sim 2e^{ix^2},\ x\to\infty. \tag{30}$$

As a matter of fact the same approach can be used also for $\theta = \frac{3\pi}{4}$ :

$$E_{1/2,1}(xe^{\frac{3\pi i}{4}}) = e^{-ix^2}\left\{1 + \frac{\sqrt{2}(i-1)}{\sqrt{\pi}}\int_0^x e^{it^2}\,dt\right\} \sim \frac{e^{\frac{\pi i}{4}}}{\sqrt{\pi}}\frac{1}{x}; \tag{31}$$

4) If $\theta = \frac{\pi}{2}$, then

$$E_{1/2,1}(ix) = E_{1/2,1}(xe^{\frac{\pi i}{2}}) = e^{-x^2}\left\{1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right\}.$$

The direct calculations show us that

$$\begin{cases} \Re E_{1/2,1}(ix) = e^{-x^2} \to 0,\ x \to +\infty, \\ \Im E_{1/2,1}(ix) = \frac{2e^{-x^2}}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt \sim \frac{1}{\sqrt{\pi}}\frac{1}{x},\ x \to +\infty; \end{cases} \tag{32}$$

5) If at last $\theta = \pi$, then

$$E_{1/2,1}(-x) = E_{1/2,1}(xe^{\pi i}) = e^{x^2}\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right\} \tag{33}$$

$$= \frac{2e^{x^2}}{\sqrt{\pi}}\int_x^{+\infty} e^{-t^2}\,dt \sim \frac{1}{\sqrt{\pi}}\frac{1}{x},\ x \to +\infty.$$

In this case the main terms of the asymptotics within the brackets are canceled.

The presented formulas (27-33) above are in principle not new. They are only used here for illustration of the difference between the functions $E_{1/2,1}(z)$ and $e^{z^2}$, as well as the starting point for further discussion of the asymptotic behaviour of the Mittag-Leffler type function.

The applied approaches are useful for any integer $\rho = \frac{1}{\alpha}$ (except of $\rho$ equal to 1). For instance the function $E_{1/3,1}(z)$ also has two characteristics rays, for one of which the main terms of the asymptotics are added, but for another one - canceled:

$$E_{1/3,1}(x) = e^{x^3}\left\{1 + \frac{3}{\Gamma(1/3)}\int_0^x e^{-t^3}\,dt\right. \tag{34}$$

$$\left. + \frac{3}{\Gamma(2/3)}\int_0^x te^{-t^3}\,dt\right\} \sim 3e^{x^3},\ x \to +\infty,$$

$$E_{1/3,1}(-x) = e^{-x^3}\left\{1 - \frac{3}{\Gamma(1/3)}\int_0^x e^{t3}\,dt + \frac{3}{\Gamma(2/3)}\int_0^x te^{t^3}\,dt\right\} \tag{35}$$

$$= e^{-x^3} - \frac{1}{\Gamma(1/3)}\frac{1}{x^2} + \frac{1}{\Gamma(2/3)}\frac{1}{x} + O\left(\frac{1}{x^3}\right),\ x \to +\infty.$$

# 4 Distribution of Zeros

A couple of the most recent results about asymptotics of zeros of the function of Mittag-Leffler type were obtained by A.M.Sedletskii [18]. Let us formulate one of them and make some comments to it (see [18, p.118]):

Let $0 < \alpha < 2$, $\beta \in \mathbf{C}$ and $\beta - \alpha \neq 1, 0, -1, \ldots$. Then all zeros $z_n$ of the function $E_{\alpha,\beta}(z)$ with large enough absolute value are simple and

$$z_n^{1/\alpha} = 2\pi n i - (\tau\alpha)\{\log(2\pi|n|) + \frac{\pi i}{2}\mathrm{sign}(n)\} + \log(c) + \delta_n, \qquad (36)$$

where

$$\tau = 1 + (1-\beta)/\alpha, \ c = \alpha/\Gamma(\beta - \alpha), \qquad (37)$$

$$\delta_n = O(|n|^{-\alpha}) + O\left(\frac{\log(|n|)}{|n|}\right), \ n \to \pm\infty.$$

REMARK 3. ([18], see also [2,3]). As it follows from (36)-(37) the zeros of $E_{\alpha,\beta}(z)$, $z = re^{i\theta}$, are situated in the curvilinear strip $D_\gamma(\alpha)$ for sufficiently big $|n|$ :

$$D_\gamma(\alpha) = \{re^{i\theta} : -\gamma \leq r^{1/\alpha}\cos(\theta/\alpha) + \tau\log(r) - \log(c) \leq \gamma, \ \gamma > 0\}, \qquad (38)$$

i.e. in "a neighbourhood" of the rays $\arg(z) = \theta = \pm\pi\alpha/2$ (see Fig. 3).

The above described domain can be divided into the following parts (left to the rays, right to the rays and the rays themselves (see again Fig. 3)):

$$D_\gamma(\alpha) = D_\gamma^{(-)}(\alpha) \cup D_\gamma^{(+)}(\alpha) \cup \{z : \arg(z) = \theta = \pm\pi\alpha/2\}.$$

REMARK 4. Let $0 < \alpha < 2$, $\beta \in \mathbf{C}$ and $\beta - \alpha \neq 1, 0, -1, \ldots$. Then
1) If $\Re(\beta) < 1 + \alpha$, then all the zeros of $E_{\alpha,\beta}(z)$ with large enough $|n|$ are situated in the domain $D_\gamma^{(-)}(\alpha)$.
2) If $\Re(\beta) > 1 + \alpha$, then all the zeros of $E_{\alpha,\beta}(z)$ with large enough $|n|$ are situated in the domain $D_\gamma^{(+)}(\alpha)$.
3) If $\Re(\beta) = 1 + \alpha$, then the distribution of zeros depends on the relation between the parameters $\alpha$, $\beta$, $\tau$, $c$ and even the second term of the asymptotic formulas (36)-(37).

The mentioned results follow from (36)-(37), more precisely, from the asymptotic equality

$$\arg(z_n) = \alpha \arctan\left(\frac{2\Re(\tau)\log(2\pi|n|) - \pi\Im(\tau)\mathrm{sign}(n) - 2\log(|c|)/\alpha}{\frac{4\pi n}{\alpha} - 2\Im(\tau)\log(2\pi|n|) - \pi\Re(\tau)\mathrm{sign}(n) + \frac{2\arg c}{\alpha}}\right) \quad (39)$$

$$+\frac{\pi\alpha}{2}\mathrm{sign}(n) + \Delta_n,\ \Delta_n + O(1/|n|) + O(\log(|n|)/n^{2-\alpha}),\ n \to \pm\infty.$$

The three different cases appear due to the values of the constant $\tau$:

$$\Re(\tau) = 1 + (1 - \Re(\beta))/\alpha. \quad (40)$$

REMARK 5. If $\beta \in \mathbf{R}$ then the non-real seros zeros of the function $E_{\alpha,\beta}(z)$ appear as pairs of complex conjugate numbers. This can be seen from the very definition of the function of Mittag-Leffler type (2).

REMARK 6. Let $0 < \alpha < 2$, $\beta \in \mathbf{C}$, $\beta - \alpha \neq 1, 0, -1, \ldots$ and $\Re(\beta) \neq 1 + \alpha$. Then all zeros $z_n$ of the function $E_{\alpha,\beta}(z)$ with large enough $|n|$ do not lie on the rays $\arg(z) = \pm\pi\alpha/2$.

This is a straightforward consequence of the formula (39). Anyway in some simple cases this and an even more stronger statement follows from

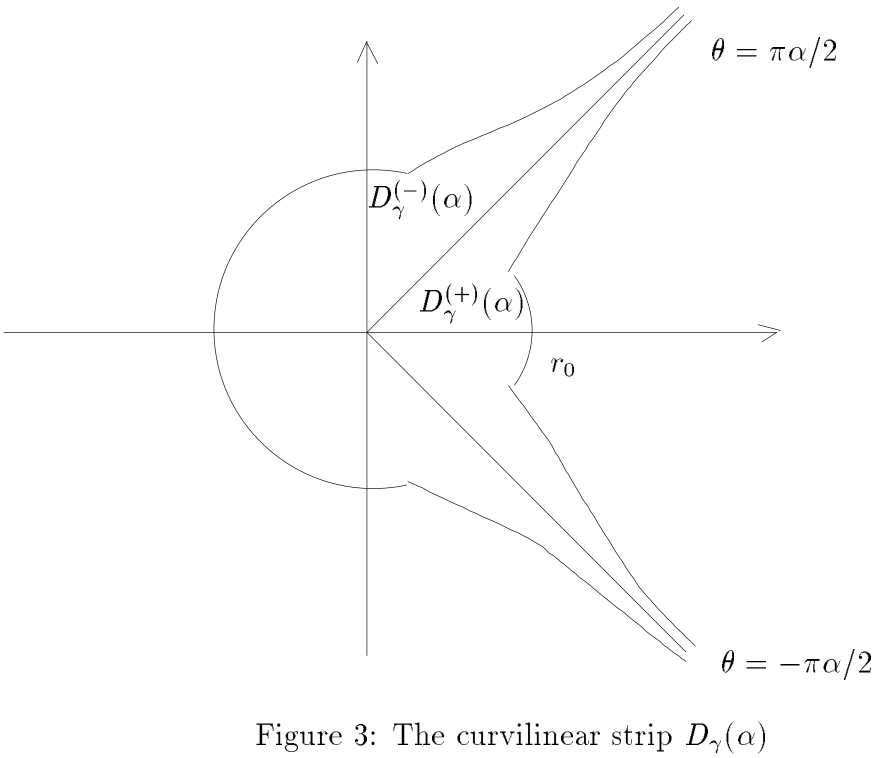


Figure 3: The curvilinear strip $D_\gamma(\alpha)$

another interesting consideration. Let us pay attention again to the function $E_{1/2,1}(z)$ and prove that there are no seros of this function on the rays $\arg(z) = \pm\pi/4$. We suppose at first the contrary statement that there exists a zero $z = pe^{i\pi/4}$. Then $\overline{z} = pe^{-i\pi/4}$ should also be a zero of $E_{1/2,1}(z)$. The formula (23) gives us then the following system of equations (the first pair for $z$, the second one for $\overline{z}$):

$$\cos(p^2) + \frac{2}{\sqrt{\pi}} \int_0^p \cos(p^2 - u^2 + \pi/4)\, du = 0, \tag{41}$$

$$sin(p^2) + \frac{2}{\sqrt{\pi}} \int_0^p \sin(p^2 - u^2 + \pi/4)\, du = 0, \tag{42}$$

$$cos(p^2) + \frac{2}{\sqrt{\pi}} \int_0^p \sin(p^2 - u^2 + \pi/4)\, du = 0, \tag{43}$$

$$sin(p^2) + \frac{2}{\sqrt{\pi}} \int_0^p \cos(p^2 - u^2 + \pi/4)\, du = 0. \tag{44}$$

From the second and third of these we get that the necessary condition for $z = pe^{i\pi/4}$ to be a zero of $E_{1/2,1}(z)$ is

$$p = \sqrt{\pi/4 + \pi n},\ \ n = 0, 1, 2, \ldots. \tag{45}$$

Another necessary condition is then (from the first and the second equation):

$$\int_0^p \sin(p^2 - u^2) du = 0, \tag{46}$$

or in equivalent form

$$S(q) = \int_0^1 \sin(q(1 - u^2)) du = 0,\ \ q = p^2. \tag{47}$$

Let us represent the function $S(q)$ in the form:

$$S(q) = \int_0^1 \sin(q(1-u^2))du = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k (q(1-u^2))^{2k+1}}{(2k+1)!} du$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k+1}}{(2k+1)!} \int_0^1 (1-u^2)^{2k+1} du = \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k+1}}{(2k+1)!} \frac{1}{2} B(2k+2, 1/2)$$

$$= q\frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-q^2)^k}{\Gamma(2k+2+1/2)} = \frac{\sqrt{\pi}}{2} q E_{2,5/2}(-q^2).$$

Using the asymptotic formula for $E_{2,\beta}(-x)$ from [2] we get

$$S(q) = \frac{\sqrt{\pi}}{2} q E_{2,5/2}(-q^2) = \frac{\sqrt{\pi}}{2} q[q^{-3/2}\cos(q - 3\pi/4) + q^{-2}/\sqrt{\pi} + O(q^{-4})],$$

and for $q = p^2 = \pi/4 + \pi n,\ n = 0, 1, 2, \ldots$

$$S(q) = S(\pi/4 + \pi n) = \frac{1}{2(\pi/4 + \pi n)} + O(1/n^3). \tag{48}$$

It is clear from the last representation that $S(\pi/4 + \pi n) \neq 0$ for sufficiently large $n$ and, consequently, according to the (47) all zeros $z_n$ of the function $E_{1/2,1}(z)$ with sufficiently large $|n|$ do not lie on the rays $\arg(z) = \pm\pi/4$. For small values of $n$ ($n = 1, 2, \ldots, 50$) we display here the results of numerical evaluation of $S(\pi/4 + \pi n)$ via the integral representation (47) by using the program "Mathematica":

| | | | | |
|---|---|---|---|---|
| 0.122840, | 0.069807, | 0.048642, | 0.037297, | 0.030234, |
| 0.025416, | 0.021921, | 0.019270, | 0.017190, | 0.015516, |
| 0.014138, | 0.012985, | 0.012006, | 0.011164, | 0.010432, |
| 0.009791, | 0.009224, | 0.008718, | 0.008266, | 0.007858, |
| 0.007488, | 0.007151, | 0.006844, | 0.006562, | 0.006302, |
| 0.006062, | 0.005839, | 0.005633, | 0.005440, | 0.005260, |
| 0.005092, | 0.004934, | 0.004786, | 0.004646, | 0.004514, |
| 0.004390, | 0.004272, | 0.004160, | 0.004054, | 0.003953, |
| 0.003858, | 0.003766, | 0.003679, | 0.003596, | 0.003517, |
| 0.003441, | 0.003368, | 0.003298, | 0.003231, | 0.003167 |

As we can see, $S(\pi/4 + \pi n) \neq 0$ for $n = 1, 2, \ldots, 50$. Hence the first fifty zeros of the function $E_{1/2,1}(z)$ do not lie on the rays $\arg(z) = \pm\pi/4$ as well. For all other zeros this statement seems to be also true due to the asymptotic formula (48) and the result of numerical calculation of the coefficient of $1/n^3$ in the error term of (48), i.e., $n^3|S(\pi/4+\pi n) - 1/(\pi/2+2\pi n)|,\ n = 1, 2, \ldots, 50$

| | | | | |
|---|---|---|---|---|
| 0.004483, | 0.007425, | 0.008849, | 0.009643, | 0.010137, |
| 0.010471, | 0.010710, | 0.010889, | 0.011027, | 0.011138, |
| 0.011228, | 0.011302, | 0.011365, | 0.011419, | 0.011465, |
| 0.011506, | 0.011541, | 0.011573, | 0.011601, | 0.011626, |
| 0.011649, | 0.011670, | 0.011689, | 0.011706, | 0.011722, |
| 0.011736, | 0.011750, | 0.011763, | 0.011774, | 0.011785, |
| 0.011795, | 0.011805, | 0.011814, | 0.011822, | 0.011830, |
| 0.011837, | 0.011844, | 0.011851, | 0.011857, | 0.011863, |
| 0.011869, | 0.011875, | 0.011880, | 0.011885, | 0.011889, |
| 0.011894, | 0.011898, | 0.011902, | 0.011906, | 0.011910 |

# 5 The growth of the Mittag-Leffler type function

One more immediate consequence of Sedletskii's results ([18]) is the fact that $E_{\alpha,\beta}(z)$ is a function of *completely regular growth* (in Levin-Pfluger's sense).

Recall (see e.g. [19],[23]) that for an entire function $f$ of order $\rho$ a ray $\arg z = \theta$ is a ray of completely regular growth (CRG) if there exists the weak limit

$$\lim{}^*_{r\to\infty} r^{-\rho} \log \mid f(re^{i\theta}) \mid, \tag{49}$$

where the term "weak limit" ($\lim^*$ in notation) means that $r$ tends to infinity omissing the values from a set $E_\theta \subset \mathbf{R}_+$ such that $\lim_{r\to\infty} r^{-1}\mathrm{mes}E_\theta \bigcap (0,r) = 0$. If all the rays are CRG-rays of the function $f$ then it is called a *function of completely regular growth.* The main characterization of CRG-functions is the following:

An entire function $f$ of order $\rho$ is a CRG-function iff its set of zeros has

a) in the case of noninteger $\rho$ - *an angular density*:

$$\lim_{r\to\infty} r^{-\rho} n(r,\phi,\theta) =: \Delta(\phi,\theta), \tag{50}$$

b) in the case of an integer $\rho$ - an angular density and besides *an angular symmetry*:

$$\lim_{r\to\infty} \{q_\rho + \frac{1}{\rho} \sum_{0<|z_n|<r} z_n^{-\rho}\} =: a_\rho. \tag{51}$$

Here the $z_n$ are zeros of the function $f$; $n(r,\phi,\theta)$ is their number in the domain $\{z = se^{i\alpha} : 0 < s < r,\ \phi \le \alpha < \theta\}$; $q_\rho$ means the coefficient of the $\rho$-th power in the exponential factor of the *Weierstrass product representation* for $f$. The function $\Delta(\theta) = \Delta(\theta_0, \theta)$ with $\theta_0 \in [0, 2\pi]$ arbitrary, but fixed, is then called the *angular density* and the number $a_\rho$ is called *coefficient of angular symmetry*.

Let us consider the Weierstrass product representation for the Mittag-Leffler type function. It has the form

$$E_{\alpha,\beta}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \frac{z^2}{2z_n^2} + \ldots + \frac{z^{[1/\alpha]}}{[1/\alpha] z_n^{[1/\alpha]}}} \tag{52}$$

$$\times \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_{-n}}\right) e^{\frac{z}{z_{-n}} + \frac{z^2}{2z_{-n}^2} + \ldots + \frac{z^{[1/\alpha]}}{[1/\alpha] z_{-n}^{[1/\alpha]}}} e^{q_0 + q_1 z + \ldots + q_{[1/\alpha]} z^{[1/\alpha]}},$$

where $z_n, \ z_{-n}$ are the zeros of $E_{\alpha,\beta}(z)$ in the neighbourhoods of the rays $\arg z = \frac{\pi\alpha}{2}$, $\arg z = \frac{-\pi\alpha}{2}$ respectively.

This formula can be rewritten in terms of *simple Weierstrass factors*:

$$G(\zeta, p) = (1 - \zeta) e^{\zeta + \frac{\zeta^2}{2} + \ldots + \frac{\zeta^p}{p}}, \ p \in \mathbf{N}. \tag{53}$$

Then (52) becomes

$$E_{\alpha,\beta}(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}, [1/\alpha]\right) \prod_{n=1}^{\infty} G\left(\frac{z}{z_{-n}}, [1/\alpha]\right) e^{q_0 + q_1 z + \ldots + q_{[1/\alpha]} z^{[1/\alpha]}}. \tag{54}$$

The formula (54) (or (52)) is not too informative for obtaining for instance asymptotic results. But using the techniques developed for CRG-functions (see e.g. [19],[23],[24]) we can compare the asymptotic behaviour of $E_{\alpha,\beta}(z)$ with that of more simple functions. In order to formulate it in stronger form let us introduce two pairs of sequences:

$$w_n = \mid z_n \mid e^{i\frac{\pi\alpha}{2}}, \ n = 1, 2, \ldots, \ w_{-n} = \mid z_{-n} \mid e^{-i\frac{\pi\alpha}{2}}, \ n = 1, 2, \ldots; \tag{55}$$

$$\omega_n = (2\pi n)^\alpha e^{i\frac{\pi\alpha}{2}}, \ n = 1, 2, \ldots, \ \omega_{-n} = (2\pi n)^\alpha e^{-i\frac{\pi\alpha}{2}}, \ n = 1, 2, \ldots; \tag{56}$$

and construct the corresponding Weierstrass products for them (putting for simplicity $\rho = 1/\alpha$ )

$$W(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{w_n}, [\rho]\right) \prod_{n=1}^{\infty} G\left(\frac{z}{w_{-n}}, [\rho]\right) e^{q_0 + q_1 z + \ldots + q_{[\rho]} z^{[\rho]}}, \tag{57}$$

$$\Omega(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{\omega_n}, [\rho]\right) \prod_{n=1}^{\infty} G\left(\frac{z}{\omega_{-n}}, [\rho]\right) e^{q_0 + q_1 z + \ldots + q_{[\rho]} z^{[\rho]}}. \tag{58}$$

**Lemma 1** *The functions $E_{\alpha,\beta}(z), W(z), \Omega(z)$ are functions of completely regular growth with the same characteristics, i.e. with the same angular densities and in the case of integer $\rho$ also with the same coefficients of angular symmetry.*

For proving this result it suffices to examine the corresponding properties of the sequences $(z_{\pm n}), (w_{\pm n}), (\omega_{\pm n})$.

**Lemma 2** *The functions $E_{\alpha,\beta}(z), W(z), \Omega(z)$ have the same asymptotic behaviour, i.e. there exist the two weak limits (see (49)):*

$$\lim\nolimits^*_{r\to\infty} \frac{\mid E_{\alpha,\beta}(z) \mid}{\mid W(z) \mid} = \lim\nolimits^*_{r\to\infty} \frac{\mid W(z) \mid}{\mid \Omega(z) \mid} = 1. \tag{59}$$

Comparing the formulas (52) and (57) we arrive at the conclusion that the quotient under the first limit is in fact a *Blaschke type product* for two rays. Such products were studied by N.V. Govorov [24] under the same conditions as ours. So we only have to repeat the calculations from [24] to get the existence of the first limit and its value.

The second result follows also from a quite general consideration. Comparing the sequences $(w_{\pm n})$ and $(\omega_{\pm n})$ we can see that there exists (see e.g. [19], [23]) a *proximate order* $\rho(r)$ (i.e. a nonnegative, nondecreasing function such that $\lim_{r\to\infty} \rho(r) = \rho,\ \lim_{r\to\infty} \rho'(r) r \log r = 0$ ) for which

$$\mid w_{\pm n} \mid^{\rho(r)} = \mid \omega_{\pm n} \mid^{\rho} . \tag{60}$$

One possible way to construct it in our situation is to use an interpolating $\mathbf{C}^1$-function $\rho(r)$ obeying the interpolation conditions

$$\rho(n) = \rho \frac{\log \mid \omega_n \mid}{\log \mid w_n \mid},\ n = 1, 2, \ldots . \tag{61}$$

As a matter of fact we need only the asymptotic behaviour but not the equality in certain points. Hence the interpolation problem can be simplified and even solved in closed form.

After constructing the proximate order it remains to apply the Valiron-Levin theory of entire functions of a proximate order (see again [19]).

Now we are in the position to discuss the global behaviour of the Mittag-Leffler type function $E_{\alpha,\beta}(z)$. As it is a CRG-function, we would expect quite a simple formula for its indicator function. Our attempt to compare this function for precise values of $\alpha$ and $\beta$ with the corresponding exponential function fails. The known asymptotic results (see e.g. [2],[3],[17],[18]) show that this failure is general. Moreover, the indicator function for $E_{\alpha,\beta}(z)$ ($0 < \alpha < 2; \beta \neq 1, 0, -1, \ldots$, when $\alpha = 1$) has the following form (see Fig. 4):

$$h_{E_{\alpha,\beta}}(\theta) = \begin{cases} \cos\theta/\alpha, & 0 \leq \mid \theta \mid < \frac{\pi\alpha}{2}, \\ 0 & \frac{\pi\alpha}{2} \leq \mid \theta \mid \leq \pi. \end{cases} \tag{62}$$

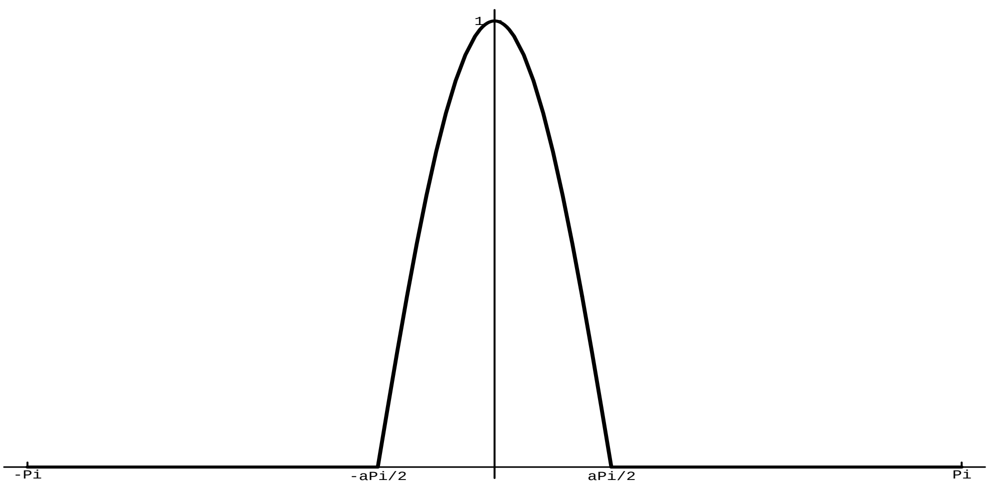


Figure 4: The indicator function for $E_{\alpha,\beta}(z)$ ($aPi/2$ means $\alpha\pi/2$)

Let us try to explain this "strange" variant of the asymptotic behaviour of a CRG-function. First of all we can consider only the case of the function $E_{1/2,\beta}(z)$. The fact is that in the general situation (i.e. for $E_{\alpha,\beta}(z)$) we can always use conformal mapping arguments. Really, as it will be seen further, the results for the function $E_{1/2,\beta}(z)$ are obtained (separately!) for two domains: $D^{(-)}_{1/2} = \{z : \pi/4 \leq\mid \arg z \mid\leq \pi\}$ and $D^{(+)}_{1/2} = \{z : 0 \leq\mid \arg z \mid\leq \pi/4\}$. We can come to these domains from those for $E_{\alpha,\beta}(z)$ using (again separately!) two conformal mappings $w = z^{1/(2\alpha)}$ with different choices of the single-valued branches. Therefore, without loss of generality it is possible to consider the only function $E_{1/2,\beta}(z)$ (the function $E_{1,\beta}$ is less characteristic; for instance $E_{1,1}(z) = e^z$ has no zeros at all).

Let us introduce, following [19], a function

$$V(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{\mid\omega_n\mid}, 2\right) \prod_{n=1}^{\infty} G\left(\frac{ze^{-i\pi/2}}{\mid\omega_{-n}\mid}, 2\right) \tag{63}$$

and consider it in the domain $D^{(-)}_{1/2}$. Here $G(u,p)$ are simple Weierstrass factors and the sequences $(\omega_{\pm n})$ are those constructed above. The sequence of positive numbers $(\mid\omega_n\mid)$ has the density $\Delta = 1/(2\pi)$ and the order $\rho = 2$. Then using the known example by B.Ya.Levin (see [19]) we get:

$$\lim_{r\to\infty} \frac{\log V(re^{i\theta})}{r^2} = \frac{i\pi\Delta}{2} e^{2i\theta},\ \ \pi/2 < \theta < 2\pi. \tag{64}$$

Hence

$$\lim_{r\to\infty} = \frac{\log \Omega(re^{i\theta})}{r^2} = (q_2 - \pi\Delta/2)e^{2i\theta},\ \ \pi/4 <\mid \theta \mid\leq \pi, \tag{65}$$

because of the evident identity

$$\Omega(z) = V(ze^{-i\pi/4})e^{q_0+q_1 z+q_2 z^2}. \tag{66}$$

So, as already seen, $q_2 = 1/4$.

In the opposite domain $D^{(+)}_{1/2}$ we can use Govorov's theory of the CRG-functions in an angle (see [24]). Three different cases appear in our situation ($\Re\beta < 3/2,\ \Re\beta > 3/2, \Re\beta = 3/2$), according to the different location of zeros of the Mittag-Leffler type function.

Let us consider only the first case, which (as following from (39)) means that all zeros $z_n$ of $E_{1/2,\beta}(z)$ are situated in the domain $D^{(-)}_{1/2}$ for large enough $n$. The second case gives in principle the same results but with much more details in formulation. One more possible way here is to reverse the approaches, applying the first one to the domain $D^{(+)}_{1/2}$ and the second to the domain $D^{(-)}_{1/2}$ (of course after using conformal mappings). As for the third case it appeared that technical complexity there is too big. The thing is that it is necessary there to separate the zeros of $E_{1/2,\beta}(z)$ between two domains for calculation of its characteristics. This plan is not yet completed.

Let us return to the case of $E_{1/2,\beta}(z)$, $\Re\beta < 3/2$, $0 < \alpha < 2$, $0 \leq| \arg z |\leq \pi/4$. In [24] was given the following formula for the indicator function of a CRG-function in an angle (here it is presented with certain simplifications corresponding to the considered case):

$$h_{E_{1/2}}(\theta) = -2\{\sigma - 1/2q_2\}\cos 2\theta, \ 0 \leq| \theta |\leq \pi/4, \tag{67}$$

where

$$\sigma = \lim_{r\to\infty}\left\{-\frac{1}{2\pi}\int_1^r \frac{\log | E_{1/2,\beta}(xe^{i\pi/4}) |}{x^{3/2}}\,dx \right. \tag{68}$$
$$\left. -\frac{1}{2\pi}\int_1^r \frac{\log | E_{1/2,\beta}(xe^{-i\pi/4}) |}{x^{3/2}}\,dx\right\}.$$

From (62) it follows then that the number $\sigma$ should be equal to $-\frac{3}{8} = -0,375$. Calculations with the main term of asymptotics (cf. (30)) give us the approximation $\sigma \approx -0,441$. It is not too close to the exact value. The thing is that not only the first term of asymptotics makes an input into the exact value of $\sigma$. Anyway we can say that the qualitative result is more or less satisfactory.

In the conclusion we have to point out the following. If $\Re\beta < 1+\alpha$, then for the left domain $D^{(-)}_{1/2}$ the product's term of the Weierstrass representation (52) of $E_{1/2,\beta}(z)$ has a cos-like indicator function. This indicator function is annihilated with that corresponding to the exponential term of (52). In the right domain $D^{(+)}_{1/2}$ there are no zeros of $E_{1/2,\beta}(z)$ in the case $\Re\beta < 1+\alpha$. Nevertheless they are accumulating near the boundary of this domain from outside. Their influence and the symmetry of their location appears in the additional term $-2\sigma\cos\rho\theta$ which is not too big but positive. Its combination with that corresponding to the exponential term in (52) gives us the final picture of the indicator function (see Fig. 4).

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