ON EXPLICIT VISCOSITY SOLUTIONS FOR $\partial u/\partial t + H(u, \nabla_x u) = 0$

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Abstract. We prove that the explicit formula [2] for viscosity solutions of Hamilton-Jacobi equation $\partial u/\partial t + H(u, \nabla_x u) = 0$ in $(0, +\infty) \times \mathbb{R}^n$ with $u(0, x) = \sigma(x)$ is still valid while the initial data $\sigma(x)$ is continuous in $\mathbb{R}^n$ (not necessarily Lipschitz continuous and bounded in $\mathbb{R}^n$). The solution is given by

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left(h\left(\frac{x - y}{t}\right) \vee \sigma(y) \right),$$

where the quasiconvex function h is the quasiconvex dual of H, i.e., $h(q) = \inf\{\gamma \in \mathbb{R} : H(\gamma, p) \geq \langle p, q\rangle, \forall p \in \mathbb{R}^n\}$ and $a \vee b = \max\{a, b\}$.

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1. INTRODUCTION

It is well-known that the Hopf first formula [7] for globally Lipschitz solution of the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} + H\big(\nabla_x u(t, x)\big) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^n, \quad u(0, x) = \sigma(x),$$

is given by

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left(\sigma(y) + tH^*\left(\frac{x - y}{t}\right)\right), \tag{1.1}$$

where the function $H : \mathbb{R}^n \to \mathbb{R}$ is convex and satisfies the growth condition

$$\lim_{\|p\| \to \infty} \frac{H(p)}{\|p\|} = +\infty,$$

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$$|\sigma(x)-\sigma(y)| \leq K\|x-y\|, \quad \text{for some} \quad K>0.$$

The function H^* is the Fenchel conjugate of H:

$$H^*(q) \quad = \quad \sup_{p\in \mathbb{R}^n} \{\langle p,q\rangle - H(p)\}.$$

In a recent paper [2], Barron, Jensen and Liu found the explicit viscosity solution of the problem

$$\frac{\partial u(t,x)}{\partial t} \; + \; H\big(u(t,x), \nabla_x u(t,x)\big) \; = \; 0, \; (t,x)\in(0,+\infty)\times\mathbb{R}^n,$$

$$u(0,x)=\sigma(x), \; x\in\mathbb{R}^n$$

to be given by

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left(h\left(\frac{x-y}{t}\right) \vee \sigma(y)\right), \tag{1.2}$$

where

$$h(q) = \inf\{\gamma\in\mathbb{R} : H(\gamma,p) \geq \langle p,q\rangle, \forall p\in\mathbb{R}^n\}$$

is quasiconvex dual of H. Their assumptions are: $H(\gamma,p)$ is continuous in $\mathbb{R}^{n+1}$, non-decreasing in $\gamma\in\mathbb{R}$, convex and positively homogeneous of degree 1 in $p\in\mathbb{R}^n$, the initial data $\sigma(x)$ is Lipschitz continuous and bounded in $\mathbb{R}^n$.

Our main aim is to show that the function $u(t,x)$ given by the formula (1.2) still is a viscosity solution of the problem, if the initial data $\sigma(x)$ is just assumed to be continuous in $\mathbb{R}^n$.

We refer the readers to the papers [1, 3 - 6] for the study of viscosity solutions related to general first-order nonlinear PDE's and to the papers [1, 2, 7, 11 - 16] for the Hopf formulas of generalized solutions (Lipschitz solutions, viscosity solutions, quasiclassical solutions, minimax solutions,...).

Throughout this paper we will use the following notations

$$a\vee b = \max\{a,b\}, \quad a\wedge b = \min\{a,b\}.$$

2. PRELIMINARIES

Consider the following Cauchy problem for Hamilton-Jacobi equation

$$\frac{\partial u(t,x)}{\partial t} \; + \; H\big(u(t,x), \nabla_x u(t,x)\big) \; = \; 0 \qquad (t,x)\in\Omega=(0,+\infty)\times\mathbb{R}^n, \tag{2.1}$$

with initial data

$$u(0,x) \qquad = \; \sigma(x) \quad x\in\mathbb{R}^n. \tag{2.2}$$

The Hamiltonian $H : \mathbb{R}^{n+1} \to \mathbb{R}$ is assumed to satisfy the following conditions

(A1) $H(., p)$ is nondecreasing in $\mathbb{R}$ for any $p \in \mathbb{R}^n$;
(A2) $H(\gamma, .)$ is convex in $\mathbb{R}^n$ for any fixed $\gamma \in \mathbb{R}$;
(A3) $H(\gamma, \lambda p) = \lambda H(\gamma, p)$ for all $\lambda \geq 0$, i.e., H is positively homogeneous of degree 1 in $p \in \mathbb{R}^n$;
(A4) $H(., p)$ is upper semicontinuous in $\mathbb{R}$.

Remark 1. From (A2), as an immediate consequence of Corollary 10.1.1, [10], the function $H(\gamma, .)$ is continuous in $\mathbb{R}^n$ for all $\gamma \in \mathbb{R}$.

For our purpose, we modify the definition of the conjugate functions in [2] as follows

Definition 1.1. Define $h : \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, +\infty]$ by

$$h(q) = \inf\{\gamma \in \mathbb{R} : H(\gamma, p) \geq \langle p, q\rangle, \forall p \in \mathbb{R}^n\}, \tag{2.3}$$

and $H^{**} : \mathbb{R} \times \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$H^{**}(\gamma, p) = \sup\{\langle p, q\rangle : q \in \mathbb{R}^n, h(q) \leq \gamma\} \tag{2.4}$$

We now recall the definition of quasiconvexity which was introduced by many authors (see, for example, Mangasarian [9]).

Definition 1.2. Let $C \subset \mathbb{R}^n$ be a convex set, and $f : C \to \overline{\mathbb{R}}$. The function f is said to be quasiconvex if the level sets of f are all convex sets in C. In other words, f is quasiconvex if

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}, \quad \forall 0 \leq \lambda \leq 1, x, y \in C.$$

Examples.

1) If $H(\gamma, p) = e^{\gamma}\|p\|$, then

$$h(q) = \log\|q\|, \quad H^{**}(\gamma, p) = H(\gamma, p).$$

2) Let $(\gamma, p) \in \mathbb{R}^2$,

$$H(\gamma, p) = \begin{cases} \gamma|p| & \text{if} \quad \gamma \geq 0, \\ 0 \quad \text{if} & \gamma < 0. \end{cases}$$

A direct calculation gives us

$$h(q) = \begin{cases} q & \text{if} \quad q > 0, \\ -\infty & \text{if} \quad q = 0, \\ -q & \text{if} \quad q < 0. \end{cases}$$

$$H^{**}(\gamma, p) = H(\gamma, p) = \gamma|p|, \quad \forall(\gamma, p) \in \mathbb{R}^2.$$

The following theorem summarizes some basic properties of the above functions.

Theorem 2.1. [Th. 2.1 and Lem. 2.2, 2] *Assume (A1)-(A4).*

i) The function h is quasiconvex, lower semicontinuous on $\mathbb{R}^n$ and

$$\lim_{\|q\|\to\infty} h(q) = +\infty, \tag{2.5}$$

$$\inf\{h(q) : \quad q \in \mathbb{R}^n\} \quad = \quad -\infty; \tag{2.6}$$

ii)

$$H(\gamma, p) = H^{**}(\gamma, p) \qquad \text{for all} \quad (\gamma, p) \in \mathbb{R}^{n+1}.$$

For our purpose we need some results concerning lower semicontinuity. The next lemmas are stated for real-valued functions.

Lemma 2.2. [Th. Sec. 1, App. C, 9] *Let $(\theta_i)_{i\in I}$ be a (finite or infinite) family of lower semicontinuous functions on $\Gamma \subset \mathbb{R}^n$. Its least upper bound*

$$\theta(x) = \sup_{i\in I} \theta_i(x)$$

is lower semicontinuous on Γ. If I is finite, then its greatest lower bound

$$\phi(x) = \inf_{i\in I} \theta_i(x)$$

is also lower semicontinuous on Γ.

Remark 2. Lemma 2.2 remains valid if $\theta, \theta_i, i \in I$ are extended-real-valued functions.

Lemma 2.3. [Th. Sec. 4, App. C, 9] *A lower (upper) semicontinuous real function θ defined on a compact set Γ in $\mathbb{R}^n$ is bounded from below (above) and attains in Γ the value*

$$\overline{\alpha} = \inf_{x\in\Gamma} \theta(x) \qquad \left(\overline{\beta} = \sup_{x\in\Gamma} \theta(x)\right).$$

In other words, there exists a point $\overline{x} \in \Gamma$ such that

$$x \in \Gamma \quad \Rightarrow \quad \theta(x) \quad \geq \quad \theta(\overline{x}) \qquad (x \in \Gamma \quad \Rightarrow \quad \theta(x) \leq \theta(\overline{x}))\,.$$

The following lemma plays an important role in our works, though the proof is simple.

Lemma 2.4. *Under the assumptions (A1)-(A4), there must exist some $q^* \in \mathbb{R}^n$ such that $h(q^*) = -\infty$.*

Proof. Assume the contrary, that $h(q) > -\infty, \forall q \in \mathbb{R}^n$. We define the function

$$h^-(q) := \min\left(h(q), 0\right), \qquad q \in \mathbb{R}^n.$$

By virtue of Lemma 2.2, one says that $h^-(q)$ is lower semicontinuous on $\mathbb{R}^n$. Further, $h^-(q)$ always takes the finite values. That is, $h^-(q)$ is lower semicontinuous on $\mathbb{R}^n$, and $-\infty < h^-(q) < +\infty, \quad \forall q \in \mathbb{R}^n$.

It follows from Lemma 2.3 applied to $h^-(q)$ that, for every number $K > 0$, there exists a number $M \in \mathbb{R}$ such that

$$\inf_{q \in \overline{B}(0,K)} h^-(q) = M > -\infty.$$

Hence

$$\inf_{q \in \overline{B}(0,K)} h(q) \quad \geq \quad \inf_{q \in \overline{B}(0,K)} h^-(q) = M > -\infty. \tag{2.7}$$

On the other hand, from (2.5), there exists a number $N > 0$ so that

$$h(q) > 0, \qquad \text{for all} \quad \|q\| > N. \tag{2.8}$$

But from (2.6) and (2.8) we get

$$-\infty = \inf\{h(q); \quad q \in \mathbb{R}^n\} \quad = \quad \inf\{h(q); \quad q \in \overline{B}(0, N)\}.$$

That means

$$\inf_{q \in \overline{B}(0,N)} h(q) = -\infty, \tag{2.9}$$

which contradicts (2.7) where $K = N$. This contradiction proves Lemma 2.4. □

3. HOPF'S TYPE FORMULA

In this section, we will assume instead of (A4) the following condition

(A'4) $H(., p)$ is continuous in $\mathbb{R}$ for all $p \in \mathbb{R}^n$.

The initial data function $\sigma(x)$ is supposed to be only continuous in $\mathbb{R}^n$. We proceed now to establish the main result of this paper.

Theorem 3.1. *Assume (A1)-(A3), (A'4) and $\sigma(x)$ is continuous in $\mathbb{R}^n$. Then the function*

$$u(t, x) = \min_{y \in \mathbb{R}^n} \left(h\left(\frac{x - y}{t}\right) \vee \sigma(y) \right), \qquad (t, x) \in \Omega, \tag{3.1}$$

is a viscosity solution of Problem (2.1)-(2.2).

Proof. The proof is divided into four steps.

Step 1. We will begin with showing that the infimum in (3.1) is actually attained. First, we rewrite (3.1) in the form

$$u(t, x) = \inf_{z \in \mathbb{R}^n} \left(h(z) \vee \sigma(x - tz)\right). \tag{3.2}$$

By virtue of Lemma 2.4, there exists some $q^* \in \mathbb{R}^n$ so that $h(q^*) = -\infty$. Let us select a such q^* and keep it fixed. It is easily seen that for every fixed $(T, x_0) \in \Omega$, and for all $(t, x) \in (0, T) \times B(x_0, 1)$ we have

$$\begin{aligned}\|x - tq^*\| = \|x - x_0 + x_0 - tq^*\| &\leq \|x - x_0\| + \|x_0\| + t\|q^*\| \\ &\leq 1 + \|x_0\| + T\|q^*\|.\end{aligned}$$

Then, for all $(t, x) \in (0, T) \times B(x_0, 1)$, it follows that

$$\begin{aligned}u(t, x) &\leq h(q^*) \vee \sigma(x - tq^*) = \sigma(x - tq^*) \\ &\leq \max_{\|y\| \leq 1 + \|x_0\| + T\|q^*\|} \sigma(y) = K < +\infty.\end{aligned}$$

Hence, from (2.5) there exists $M > 0$ that depends only on T and x_0, such that, for all $(t, x) \in (0, T) \times B(x_0, 1)$

$$h(z) \geq K + 1 > u(t, x), \quad \forall \|z\| > M.$$

So, for all $(t, x) \in (0, T) \times B(x_0, 1)$, we have

$$h(z) \vee \sigma(x - tz) \geq h(z) > u(t, x), \quad \forall \|z\| > M.$$

Thus,

$$u(t, x) = \inf_{z \in \overline{B}(0, M)} (h(z) \vee \sigma(x - tz)) \quad \forall (t, x) \in (0, T) \times B(x_0, 1), \tag{3.3}$$

where M depends only on T and x_0.

¿From the above arguments, one can recognize that for all $(t, x) \in (0, T) \times B(x_0, 1)$, we get

$$\begin{aligned}u(t, x) &= \inf_{z \in \overline{B}(0, M)} [(h(z) \vee \sigma(x - tz))] \wedge K \\ &= \inf_{z \in \overline{B}(0, M)} [(h(z) \vee \sigma(x - tz)) \wedge K].\end{aligned}$$

Now in view of Lemma 2.2, we see that the function $z \mapsto (h(z) \vee \sigma(x - tz)) \wedge K, z \in \overline{B}(0, M)$ is lower semicontinuous. Apart from that, this function is bounded on the closed ball $\overline{B}(0, M)$, which is a compact set. Hence, Lemma 2.3 implies that the infimum in (3.1) is actually attained.

Step 2. We have to show that the function $u(t, x)$ defined above is a continuous function in $\mathbb{R}^n$.

In fact, for every $(t_0, x_0) \in \Omega$, from the representation (3.2) we may choose $\xi_0 \in \mathbb{R}^n$ such that

$$u(t_0, x_0) = h(\xi_0) \vee \sigma(x_0 - t_0\xi_0).$$

Let us keep such a point ξ_0 fixed. Then,

$$\begin{aligned}u(t, x) - u(t_0, x_0) &= \inf_{z \in \mathbb{R}^n} (h(z) \vee \sigma(x - tz)) - h(\xi_0) \vee \sigma(x_0 - t_0\xi_0) \\ &\leq h(\xi_0) \vee \sigma(x - t\xi_0) - h(\xi_0) \vee \sigma(x_0 - t_0\xi_0).\end{aligned}$$

$$u(t,x) - u(t_0,x_0) \le |\sigma(x - t\xi_0) - \sigma(x_0 - t_0\xi_0)|. \tag{3.4}$$

Interchanging (t_0, x_0) and (t, x), we obtain

$$u(t_0,x_0) - u(t,x) \le |\sigma(x - t\xi) - \sigma(x_0 - t_0\xi)|. \tag{3.5}$$

Puting $T = t_0 + 1$, from (3.3), we can assume that ξ in (3.5) satisfies $\|\xi\| \le M$. Then, let us combines (3.4) and (3.5) to get

$$-|\sigma(x - t\xi) - \sigma(x_0 - t_0\xi)| \le u(t,x) - u(t_0,x_0) \le |\sigma(x - t\xi_0) - \sigma(x_0 - t_0\xi_0)|$$
$$\forall (t,x) \in (0, t_0+1) \times B(x_0, 1). \tag{3.6}$$

On the other hand, for all $(t,x) \in (0, t_0+1) \times B(x_0,1)$ we have

$$\begin{aligned}\|(x - t\xi) - (x_0 - t_0\xi)\| &\le \|x - x_0\| + \|\xi\||t - t_0| \\ &\le \|x - x_0\| + M|t - t_0| \to 0 \quad \text{as} \quad (t,x) \to (t_0,x_0).\end{aligned} \tag{3.7}$$

At the same time

$$\begin{aligned}\|x - t\xi - (x_0 - t_0\xi_0)\| &\le \|x - x_0\| + \|t\xi - t_0\xi + t_0\xi - t_0\xi_0\| \\ &\le 1 + \|\xi\||t - t_0| + t_0\|\xi - \xi_0\| \\ &< 1 + M(t_0+1) + t_0(\|\xi\| + \|\xi_0\|) \\ &\le 1 + M(t_0+1) + 2Mt_0 = 1 + M(1 + 3t_0),\end{aligned}$$

and

$$\|x_0 - t_0\xi - (x_0 - t_0\xi_0)\| = t_0\|\xi - \xi_0\| \le 2Mt_0 < 1 + M(1+3t_0).$$

Therefore, we can see that all of $x - t\xi$ and $x_0 - t_0\xi$ belong to $\overline{B}(x_0 - t_0\xi_0, 1 + M(1+3t_0))$. From the fact that the function $\sigma(.)$ is continuous in $\mathbb{R}^n$, we conclude that $\sigma(.)$ is uniformly continuous on the closed ball $\overline{B}(x_0 - t_0\xi_0, 1 + M(1+3t_0))$. Hence, from (3.7) and the uniform continuity of $\sigma(.)$ on $\overline{B}(x_0 - t_0\xi_0, 1 + M(1+3t_0))$ we get

$$|\sigma(x - t\xi) - \sigma(x_0 - t_0\xi)| \to 0 \quad \text{as} \quad (t,x) \to (t_0,x_0).$$

Consequently, in view of (3.6), by letting $(t,x) \to (t_0,x_0)$, we obtain

$$\lim_{(t,x)\to(t_0,x_0)} u(t,x) = u(t_0,x_0).$$

Step 3. We prove that $u(t,x)$ is actually a viscosity solution of (2.1). For convenience, let us show that $u(t,x)$ given by (3.1) can be expressed for any $x \in \mathbb{R}^n$ and $0 \le s < t$ as follows

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left(h\left(\frac{x-y}{t-s}\right) \vee u(s,y) \right). \tag{3.8}$$

Indeed, as in [2, p. 56-57], we have

$$u(t,x) \le \min_{y\in\mathbb{R}^n}\left(h\left(\frac{x-y}{t-s}\right) \vee u(s,y)\right). \tag{3.8a}$$

To get (3.8) we need to show

$$u(t,x) \ge \min_{y\in\mathbb{R}^n}\left(h\left(\frac{x-y}{t-s}\right) \vee u(s,y)\right). \tag{3.8b}$$

Let us choose $q \in \mathbb{R}^n$ such that $u(t,x) = h((x-q)/t) \vee \sigma(q)$. Then

$$\begin{aligned}
\min_{y\in\mathbb{R}^n}\left(h\left(\frac{x-y}{t-s}\right) \vee u(s,y)\right) &= \min_{y\in\mathbb{R}^n}\left[h\left(\frac{x-y}{t-s}\right) \vee \min_{z\in\mathbb{R}^n}\left(h\left(\frac{y-z}{s}\right) \vee \sigma(z)\right)\right]\\
&\le \min_{y\in\mathbb{R}^n}\left(h\left(\frac{x-y}{t-s}\right) \vee h\left(\frac{y-q}{s}\right) \vee \sigma(q)\right)\\
&\le \min_{\{y=x-\frac{t-s}{t}(x-q)\}}\left(h\left(\frac{x-y}{t-s}\right) \vee h\left(\frac{y-q}{s}\right) \vee \sigma(q)\right)\\
&= h\left(\frac{x-q}{t}\right) \vee h\left(\frac{x-q}{t}\right) \vee \sigma(q) = u(t,x).
\end{aligned}$$

Combining (3.8a) and (3.8b) we obtain (3.8).

Next, we will show that $u(t,x)$ is actually a viscosity solution of (2.1).

i) u is a subsolution, i.e., if $\varphi \in C^1(\Omega)$ and $u-\varphi$ attains its local maximum at $(t_0,x_0) \in \Omega$, then

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0), \nabla_x\varphi(t_0,x_0)) \le 0.$$

Assume the contrary, that u is not a subsolution. Then, there exist $\varepsilon_0 > 0, (t_0,x_0) \in \Omega$, a neighbourhood $V(t_0,x_0)$ of (t_0,x_0) and $\varphi \in C^1(\Omega)$ so that $u-\varphi$ attains its maximum M at (t_0,x_0) on $V(t_0,x_0)$:

$$u-\varphi \le M, \quad (t,x) \in V(t_0,x_0),$$

$$u(t_0,x_0) - \varphi(t_0,x_0) = M \tag{3.9}$$

and

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0), \nabla_x\varphi(t_0,x_0)) > \varepsilon_0 > 0. \tag{3.10}$$

¿From (3.10), and by the continuity of $H(.,p)$, using $H = H^{**}$ we conclude that there exists $\delta > 0$ so that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0)-\delta, \nabla_x\varphi(t_0,x_0))$$

$$= \frac{\partial \varphi(t_0, x_0)}{\partial t} + \sup_{\{q: h(q) \le u(t_0, x_0) - \delta\}} \langle \nabla_x \varphi(t_0, x_0), q \rangle > \varepsilon_0 .$$

Hence, there exists $q_0 \in \mathbb{R}^n$, $h(q_0) \le u(t_0, x_0) - \delta$ such that

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q_0 \rangle > \varepsilon_0 > 0. \tag{3.11}$$

Set $\mu = t_0 - t$, $x = x_0 - \mu q_0$ in the inequality obtained from (3.8) to get

$$\begin{aligned} u(t_0, x_0) &\le h\left(\frac{x_0 - x}{t_0 - t}\right) \vee u(t, x) \\ &= h(q_0) \vee u(t_0 - \mu, x_0 - \mu q_0). \end{aligned}$$

Since $h(q_0) \le u(t_0, x_0) - \delta < u(t_0, x_0)$, by the continuity of $u(t, x)$ at (t_0, x_0), there exists a number $\mu' > 0$ so that, for all $0 < \mu < \mu'$

$$u(t_0 - \mu, x_0 - \mu q_0) - \delta/2 > u(t_0, x_0) - \delta \ge h(q_0).$$

Let us choose a $\mu", 0 < \mu" < \mu'$ so that $(t_0 - \mu", x_0 - \mu" q_0)$ still belongs to $V(t_0, x_0)$ in (3.9). Then,

$$\begin{aligned} \varphi(t_0, x_0) + M = u(t_0, x_0) &\le h(q_0) \vee u(t_0 - \mu, x_0 - \mu q_0) \\ &= u(t_0 - \mu, x_0 - \mu q_0) \\ &\le \varphi(t_0 - \mu, x_0 - \mu q_0) + M, \quad \forall \mu, 0 < \mu < \mu". \end{aligned}$$

This implies

$$\varphi(t_0 - \mu, x_0 - \mu q_0) - \varphi(t_0, x_0) \ge 0, \quad \forall \mu, 0 < \mu < \mu".$$

Equivalently,

$$\frac{\varphi(t_0 - \mu, x_0 - \mu q_0) - \varphi(t_0, x_0)}{-\mu} \le 0, \quad \forall \mu, 0 < \mu < \mu". \tag{3.12}$$

But in the inequality (3.12), sending $\mu \to 0+$ gives us

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q_0 \rangle \le 0, \tag{3.13}$$

which contradicts (3.11). This proves that u is a subsolution.

ii) u is a supersolution, i.e., if $\varphi \in C^1(\Omega)$ and $u - \varphi$ attains its local minimum at $(t_0, x_0) \in \Omega$, then

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + H(u(t_0, x_0), \nabla_x \varphi(t_0, x_0)) \ge 0.$$

Assume the contrary, then there exist $\varepsilon_0 > 0, (t_0, x_0) \in \Omega$, a neighbourhood $V(t_0, x_0)$ of (t_0, x_0) and $\varphi \in C^1(\Omega)$ so that $u - \varphi$ attains its minimum m at (t_0, x_0) on $V(t_0, x_0)$:

$$u - \varphi \ge m, \quad (t, x) \in V(t_0, x_0),$$

$$u(t_0, x_0) - \varphi(t_0, x_0) = m \tag{3.14}$$

and

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + H(u(t_0, x_0), \nabla_x \varphi(t_0, x_0)) < -\varepsilon_0 < 0. \tag{3.15}$$

¿From (3.15) and the continuity of $H(., p)$ implies that there is a $\delta > 0$ so that

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + H(u(t_0, x_0) + \delta, \nabla_x \varphi(t_0, x_0)) < -\varepsilon_0.$$

Then, using $H = H^{**}$, we obtain

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + \sup_{\{q : h(q) \leq u(t_0, x_0) + \delta\}} \langle \nabla_x \varphi(t_0, x_0), q \rangle < -\varepsilon_0.$$

This means that for all $q \in \mathbb{R}^n, h(q) \leq u(t_0, x_0) + \delta$, it follows that

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q \rangle < -\varepsilon_0 < 0. \tag{3.16}$$

On the other hand, from (3.8) we can rewrite $u(t, x)$ in the form

$$u(t, x) = \min_{z \in \mathbb{R}^n} \left(h(z) \vee u(s, x - (t - s)z) \right).$$

Hence, for every $0 < t < t_0$, we may select $q \in \mathbb{R}^n$ such that

$$u(t_0, x_0) = h(q) \vee u(t, x_0 - (t_0 - t)q) \geq h(q).$$

Take an arbitrary sequence $(t_i)_i \subset (0, t_0), t_i \to t_0$ as $i \to \infty$ and put $\nu_i = t_0 - t_i > 0, i \in \mathbf{N}$. For every $i \in \mathbf{N}$, we choose a $q_i \in \mathbb{R}^n$ such that

$$\begin{aligned} u(t_0, x_0) &= h(q_i) \vee u(t_i, x_0 - (t_0 - t_i)q_i) \\ &= h(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i) \geq h(q_i). \end{aligned}$$

It follows from (2.5) that the sequence $(q_i)_i$ must be bounded. Without loss of generality, we can suppose that $q_i \to q_0$ as $i \to \infty$. Therefore, $(t_0 - \nu_i, x_0 - \nu_i q_i) \to (t_0, x_0)$ as $i \to \infty$. By the lower semicontinuity of h and the continuity of u, we can see that

$$\begin{aligned} u(t_0, x_0) \leq h(q_0) \vee u(t_0, x_0) &\leq \lim_{i \to \infty} \inf \left(h(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i) \right) \\ &\leq \lim_{i \to \infty} \inf u(t_0, x_0) = u(t_0, x_0). \end{aligned}$$

This means

$$h(q_0) \leq u(t_0, x_0) < u(t_0, x_0) + \delta. \tag{3.17}$$

Since $(t_0 - \nu_i, x_0 - \nu_i q_i) \to (t_0, x_0)$ as $i \to \infty$, there exists a number $N_1 \in \mathbf{N}$ such that $(t_0 - \nu_i, x_0 - \nu_i q_i) \in V(t_0, x_0)$, for all $i > N_1$. So, from (3.14) we have

$$\begin{aligned}\varphi(t_0, x_0) + m = u(t_0, x_0) &= h(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i) \\ &\geq u(t_0 - \nu_i, x_0 - \nu_i q_i) \\ &\geq \varphi(t_0 - \nu_i, x_0 - \nu_i q_i) + m, \forall i > N_1.\end{aligned}$$

Consequently,

$$\varphi(t_0 - \nu_i, x_0 - \nu_i q_i) - \varphi(t_0, x_0) \leq 0, \forall i > N_1.$$

Now, it follows from the Mean Value Theorem that

$$\begin{aligned}\varphi(t_0 - \nu_i, x_0 - \nu_i q_i) - \varphi(t_0, x_0) &= \langle \nabla\varphi(t_0 - \theta_i, x_0 - \theta_i q_i), (-\nu_i, -\nu_i q_i)\rangle \\ &= -\nu_i \left(\frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \langle \nabla_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i \rangle \right),\end{aligned}$$

where $0 \leq \theta_i \leq \nu_i, \forall i \in \mathbf{N}$. This implies

$$\frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \langle \nabla_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i \rangle \geq 0, \forall i > N_1.$$

On the other hand, we have

$$\begin{aligned}|(t_0 - \theta_i) - t_0| + \|(x_0 - \theta_i q_i) - x_0\| + \|q_i - q_0\| &= \theta_i + \theta_i \|q_i\| + \|q_i - q_0\| \\ &\leq \nu_i + \nu_i \|q_i\| + \|q_i - q_0\| \to 0 \\ &\quad \text{as} \quad i \to \infty,\end{aligned}$$

i.e., $(t_0 - \theta_i, x_0 - \theta_i q_i, q_i) \to (t_0, x_0, q_0)$ as $i \to \infty$. Since $\varphi \in C^1$, it is easy to see that the function $(t, x, q) \mapsto (\partial\varphi/\partial t)(t, x) + \langle \nabla_x \varphi(t, x), q\rangle$, $(t, x, q) \in \Omega \times \mathbb{R}^n$ must be continuous at the point (t_0, x_0, q_0). Hence, there exists a number $N_2 \in \mathbf{N}$ such that

$$\begin{aligned}&\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q\rangle + \varepsilon_0/2 \\ &\qquad > \frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \langle \nabla_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i\rangle, \forall i > N_2.\end{aligned}$$

Putting $N = \max\{N_1, N_2\}$, for all $i > N$, we obtain

$$\begin{aligned}&\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q\rangle + \varepsilon_0/2 \\ &\qquad > \frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \langle \nabla_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i\rangle \geq 0.\end{aligned}$$

Consequently,

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + \langle \nabla_x \varphi(t_0, x_0), q\rangle \geq -\varepsilon_0/2. \tag{3.18}$$

¿From (3.17) and (3.18), we have the fact that conflicts with (3.16). This proves that u is a supersolution.

Step 4. In the rest of the proof, we will claim that $u(t,x)$ satisfies the initial data (2.2)

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = \sigma(x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{3.19}$$

Indeed, for every $x_0 \in \mathbb{R}^n$, using the representation (3.2), by virtue of Lemma 2.4, we can select a $q^* \in \mathbb{R}^n, h(q^*) = -\infty$ and keep it fixed. Then

$$\begin{aligned} u(t,x) &\le h(q^*) \vee \sigma(x - tq^*) \\ &= \sigma(x - tq^*), \quad \forall (t,x) \in \Omega. \end{aligned}$$

Consequently,

$$\begin{aligned} \lim_{(t,x)\to(0,x_0)} u(t,x) &\le \lim_{(t,x)\to(0,x_0)} \sigma(x - tq^*). \\ &= \sigma(x_0). \end{aligned} \tag{3.20}$$

Choosing $T = 1$, from (3.3), for every $(t,x) \in (0,1)\times B(x_0,1)$, we may select $\xi \in \overline{B}(0,M)$ so that

$$\begin{aligned} u(t,x) &= h(\xi) \vee \sigma(x - t\xi) \\ &\ge \sigma(x - t\xi). \end{aligned}$$

On the other hand, for all $(t,x) \in (0,1) \times B(x_0,1)$ we have

$$\begin{aligned} \|x - t\xi - x_0\| &\le \|x - x_0\| + \|\xi\| t \quad \forall t > 0, \forall x \in V(x_0), \\ &\le \|x - x_0\| + Mt \to 0 \quad \text{as} \quad (t,x) \to (0,x_0). \end{aligned}$$

Hence,

$$\lim_{(t,x)\to(0,x_0)} u(t,x) \ge \lim_{(t,x)\to(0,x_0)} \sigma(x - t\xi) = \sigma(x_0). \tag{3.21}$$

A combination of (3.20) and (3.21) gives us (3.19). The proof is complete. □

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