EXPLICIT VISCOSITY SOLUTIONS FOR $\partial u/\partial t + H(t, u, \nabla_x u) = 0$

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Abstract. An explicit viscosity solution for a Cauchy problem is established. The Hamiltonian $H(t, u, \nabla_x u)$ is supposed to depend on t, u and $\nabla_x u$, and not necessarily Lipschitz continuous in $\nabla_x u$. The solution is given by

$$u(t,x) = \inf\Big\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p)d\tau\Big) \leq 0\Big\},$$

where $\sigma^{\#}(\gamma, p)$ is the first quasiconvex conjugate of $\sigma(x)$ which is being assumed to be quasiconvex.

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1. INTRODUCTION

In a recent paper [5], E. N. Barron, R. Jensen, and W. Liu found the explicit viscosity solution of the problem

$$u_t + H(u, \nabla_x u) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \quad u(T,x) = g(x),$$

to be given by

$$u(t,x) = \inf\Big\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n}\Big(p.x - g^{\#}(\gamma, p) + (T-t)H(\gamma, p)\Big) \leq 0\Big\}, \tag{1.1}$$

where

$$g^{\#}(\gamma, p) = \sup\{p.x : x \in \mathbb{R}^n, g(x) \leq \gamma\}, \quad \gamma \in \mathbb{R}, p \in \mathbb{R}^n,$$

is the first quasiconvex conjugate of the terminal function $g(x)$. This is an important result generalizing the Hopf formula to the case where the Hamiltonian H depends on u and the terminal function may not be convex.

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In this paper, we consider the Cauchy problem for some first order nonlinear PDEs, where the Hamiltonians depend on t, u and $\nabla_x u$, namely:

$$\frac{\partial u(t,x)}{\partial t} + H\big(t, u(t,x), \nabla_x u(t,x)\big) = 0, \quad (t,x) \in \Omega = (0,T) \times \mathbb{R}^n, \tag{1.2}$$

with initial data

$$u(0,x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{1.3}$$

The viscosity solution is given by

$$u(t,x) = \inf\Big\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n}\Big(\langle p,x\rangle - \sigma^{\#}(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\Big\}, \tag{1.4}$$

where $\sigma^{\#}(\gamma,p)$ is the first quasiconvex conjugate of $\sigma(x)$ in the sense of [3].

The formula (1.1) in [5] was proved to be a viscosity solution under the assumption that the Hamiltonian $H(\gamma,p)$ is Lipschitz continuous in the variable p. Here, apart from the t dependence in the Hamiltonian $H(t,\gamma,p)$, we remove this assumption and require that the initial function $\sigma(x)$ is quasiconvex and has *L-lsc property.* As we will prove in the section 2, the class of functions having L-lsc property is quite broad. It contains the continuous functions f which are convex or strictly quasiconvex, and satisfy the coercivity condition: $f(x) \to +\infty$ as $\|x\| \to +\infty$.

To construct the solution, apart from using the theory on quasiconvex conjugates according to the view of the paper [3] and many techniques of [5], we use the theory of multifunctions to define the class of initial functions. We want to refer the readers to [4]-[8], [14], [24] for the study of viscosity solutions, [1], [9], [17], [18] for multifuntions, [3] for quasiconvex conjugate, [2], [12], [17]-[23] for minimax solutions, globally Lipschitz solutions, quasiclassical solutions. and [10], [11], [13], [19] for the facts concerning convex functions, difference of convex functions, semicontinuous functions and normal cones.

2. PRELIMINARIES

In this section, we recall the definition of quasiconvex conjugates according to the view of the paper [3], and refer the readers to the book [9] for the definition of multifunctions.

Given a quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$, the first quasiconvex conjugate is defined by

$$f^{\#}(\gamma,p) = \sup\{\langle p,x\rangle : x \in \mathbb{R}^n, f(x) \le \gamma\}, \quad \gamma \in \mathbb{R}, p \in \mathbb{R}^n.$$

If $\{x : f(x) \le \gamma\} = \emptyset$ for some γ, then $f^{\#}(\gamma,p) = -\infty$.

The second conjugate is defined by

$$f^{\#\#}(x) = \inf\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n}(\langle p,x\rangle - f^{\#}(\gamma,p)) \le 0\}.$$

Throughout this paper, we will use the following notations for any $\gamma \in \mathbb{R}$, $g : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}$, and $x_0 \in \mathbb{R}^n, r > 0$.

$$E_{g,\gamma} = \{x \in \Omega : g(x) \leq \gamma\},$$

is the γ - *level set* of g.

$$B(0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\},$$

is the ball with center x_0 and radius r.

Next, we summary some basic properties of the above conjugates. The following proposition was proved in [3] without assuming the level sets being bounded.

Proposition 2.1. [Prop. 2.1, 3] *If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex, then*
(1) $\gamma_1 \leq \gamma_2$ implies $f^{\#}(\gamma_1, p) \leq f^{\#}(\gamma_2, p), \forall p \in \mathbb{R}^n$, i. e., $f^{\#}$ is nondecreasing in γ for p fixed.
(2) $f^{\#}(\gamma, p)$ is convex in $p \in \mathbb{R}^n$ for each fixed $\gamma \in \mathbb{R}$.
(3) $f^{\#}(\gamma, \lambda p) = \lambda f^{\#}(\gamma, p), \lambda \geq 0$, i. e., $f^{\#}(\gamma, p)$ is positively homogeneous of degree 1 in p.
(4) $f^{\#}(\gamma, p)$ is upper semicontinuous in γ.

Proposition 2.2. [Lem. 2.1, 5] *If $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and lower semicontinuous, then, $f^{\#\#}(x) = f(x)$.*

Let us describe some properties of semicontinuity which will be used in the proof of the main result.

Lemma 2.3.
(a) [Cor. 7 App. 1, 15] *Let $(\theta_i)_{i \in I}$ be a (finite or infinite) family of upper semicontinuous functions (usc) on $\Gamma \subset \mathbb{R}^n$. Its greatest lower bound*

$$\phi(x) = \inf_{i \in I} \theta_i(x)$$

is upper semicontinuous on Γ.
(b) Let $f(x, y), (x, y) \in A \times B$ be a real valued upper semicontinuous function and B be a compact metric space. Its least upper bound

$$g(x) = \sup_{y \in B} f(x, y), \quad x \in A$$

is upper semicontinuous.

Proof. Assume, for the sake of contradiction, that $g(x)$ is not upper semicontinuous. Passing to a subsequence, we may assume that there exists $(x_i)_i \subset A, x \in A, x_i \to x, \varepsilon_0 > 0$ such that

$$g(x_i) > g(x) + \varepsilon_0. \tag{2.1}$$

Since $f(x,y)$ is upper semicontinuous in $y \in B$ and B compact, $f(x,.)$ attains its maximum for each fixed x. Let

$$g(x_i) = f(x_i, y_i), g(x) = f(x,y), \quad y, y_i \in B, i \in \mathbf{N}. \tag{2.2}$$

Since $(y_i)_i$ is bounded, by passing to a subsequence, we may assume $y_i \to y' \in B$. From the upper semicontinuity of f, we have for all large i:

$$f(x_i, y_i) < f(x, y') + \varepsilon_0/2. \tag{2.3}$$

¿From (2.1), (2.2) and (2.3) we get

$$f(x,y') > f(x,y) + \varepsilon_0/2,$$

which conflicts with the fact that $f(x,y)$ is the maximum value. The proof is complete. □

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, set $\gamma^* = \inf_{x \in \mathbb{R}^n} f(x)$. Consider the multifunction L defined by

$$\begin{aligned} L : (\gamma^*, +\infty) \quad &\to \quad 2^{\mathbb{R}^n} \backslash \emptyset \\ \gamma \quad &\mapsto \quad E_{f,\gamma}, \end{aligned}$$

which will be accordingly called the *generated multi* (by f).

Definition 2.4. The function f is said to *have L-lsc property* if the generated multi L is $\varepsilon - \delta$-lsc.

Our purpose in the rest of this section is that some precious properties of the initial function should be attained. In the next section, we will have to assume the initial function is continuous, and satisfies the *coercivity condition*: $f(x) \to +\infty$ as $\|x\| \to \infty$. Clearly, the continuity of a function guarantees that the level sets are closed, and the coercivity condition guarantees their boundedness. Hence, the above conditions on the initial data provide us with the compactness of the level sets. So, for our function f, we assume it is continuous, satisfies the coercivity condition. Then, $E_{f,\gamma}$ is a compact set for each $\gamma \in \mathbb{R}$. This means that the multi L has compact values. Therefore, by virtue of Proposition 2.1, [9], we obtain the following.

Proposition 2.5. *If f is continuous, and satisfies the coercivity condition, then the multi L is $\varepsilon - \delta$-lsc if and only if L is lsc.*

The class of functions having L-lsc property is quite broad as can be seen by the following.

Proposition 2.6. *Let f be continuous, not attaine a local minimum in any open subset of $\mathbb{R}^n \backslash I$, and satisfy the coercivity condition, where I denotes the set on which f attains its (global) infimum. Then, f has L-lsc property.*

Proof. In view of Proposition 2.5, it is sufficient to prove that L is lsc. To do this, assume $(\gamma_i)_i \subset (\gamma^*, +\infty)$, V is open in $\mathbb{R}^n$ and $\gamma_i \to \gamma \in (\gamma^*, +\infty)$, and $L(\gamma) \cap V \neq \emptyset$. We have to show that $L(\gamma_i) \cap V \neq \emptyset$ for all large i.

Actually, since $L(\gamma) \cap V \neq \emptyset$, there exists $x \in L(\gamma) \cap V$. By the assumption that f does not attaine a local minimum in V, there is $x' \in V$ such that

$$f(x') < f(x) \leq \gamma_0.$$

Since $\gamma_i \to \gamma$, for $\gamma - f(x') > 0$, there exists $N \in \mathbf{N}$ such that for all $i > N$ we have

$$\gamma - \gamma_i < \gamma - f(x').$$

Equivalently,

$$\gamma_i > f(x') \quad \text{for all} \quad i > N,$$

which means $x' \in E_{f,\gamma_i} = L(\gamma_i)$ for all $i > N$. Hence, $L(\gamma_i) \cap V \neq \emptyset$ for all $i > N$. □

In the next section, we will also assume that the initial function is quasiconvex. Concerning this concept, firstly, we recall a result on minimum of a quasiconvex function.

Proposition 2.7. [Th. 4, Sec. 2, Chap. 9, 15] *Let θ be a real-valued function defined on the convex set Γ in $\mathbb{R}^n$, and let $\overline{x} \in \Gamma$ be a local minimum point. If θ is strictly quasiconvex at $\overline{x}$, then $\theta(\overline{x})$ is a global minimum of θ on Γ.*

Of course, this result is correct if the function θ is convex. Hence, from Proposition 2.6, we obtain the following result.

Corollary 2.8. *Let the continuous function f be strictly quasiconvex (or convex), and satisfy the coercivity condition. Then, f has L-lsc property.*

Lemma 2.9. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and has L-lsc property. The family $(f^{\#}(\gamma, p))_{\|p\|=1}$ is equi-lower semicontinuous on $(\gamma^*, +\infty)$, and $f^{\#}(\gamma, p) = -\infty$ if $\gamma < \gamma^*$.*

Proof. Let $\gamma_0 \in (\gamma^*, +\infty)$ and $\varepsilon > 0$ be arbitrary. By the lower semicontinuity of L, there exists $0 < \delta < \gamma_0 - \gamma^*$ such that

$$L(\gamma_0) \subset L(\gamma) + B_\varepsilon(0) \quad \text{on} \quad (\gamma_0 - \delta, \gamma_0 + \delta). \tag{2.4}$$

¿From the definition of f and (2.3), it holds for $\|p\| = 1$

$$\begin{aligned}
f^{\#}(\gamma_0, p) &= \sup(\langle p, x\rangle, f(x) \leq \gamma_0) \\
&= \sup(\langle p, x\rangle, x \in L(\gamma_0)) \\
&\leq \sup(\langle p, x\rangle, x \in L(\gamma) + B_\varepsilon(0)) \\
&= \sup(\langle p, x\rangle, x = y + z, y \in L(\gamma), z \in B_\varepsilon(0)) \\
&\leq \sup(\langle p, y\rangle, y \in L(\gamma)) + \varepsilon|p| \\
&= f^{\#}(\gamma, p) + \varepsilon, \quad \forall \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta).
\end{aligned} \tag{2.5}$$

Now, (2.5) means that $(f^{\#}(\gamma,p))_{\|p\|=1}$ is equilower semicontinuous at γ_0.

The second statement is obvious. The proof is complete. □

3. AN EXPLICIT VISCOSITY SOLUTION OF THE PROBLEM

Let us consider a Cauchy problem for Hamilton-Jacobi equations of the form

$$\frac{\partial u(t,x)}{\partial t} + H\big(t, u(t,x), \nabla_x u(t,x)\big) = 0, \quad (t,x) \in \Omega = (0,T) \times \mathbb{R}^n, \tag{3.1}$$

with initial data

$$u(0,x) = \sigma(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

In this paper, we will assume the following.

(A) The initial function $\sigma \in C(\mathbb{R}^n)$ is quasiconvex, has the L-lsc property, and satisfies the coercivity condition $\sigma(x) \to +\infty$ as $\|x\| \to \infty$.

Remark 1. Observe that the continuity of σ and the coercivity condition yield that $\gamma^* = \inf\{\sigma(x) : x \in \mathbb{R}^n\}$ is finite.

(B) The Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and
(i) $H(t,\gamma,\lambda p) = \lambda H(t,\gamma,p)$ for all $(t,\gamma,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n, \lambda \geq 0$;
(ii) $H(t,\gamma,p)$ is nondecreasing in $\gamma \in \mathbb{R}$ for all $(t,p) \in [0,T] \times \mathbb{R}^n$.

(C) The Hamitonian H satisfies one of two following conditions:
(i) To every fixed $t_0 \in (0,T)$, there exists a function $h : [0,T] \times \mathbb{R} \to \mathbb{R}$, $h(t,\gamma)$ is positive for almost every $t \in (0,T)$, $h(.,\gamma)$ is integrable for any γ, such that

$$H(t,\gamma,p) = h(t,\gamma)H(t_0,\gamma,p), \quad \forall (t,\gamma,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n.$$

(ii) If $0 \leq \alpha_i \leq 1, \|p_i\| = 1, i = \overline{1,m}$, and $\sum\limits_{i=1}^{m} \alpha_i = 1$, then

$$H(t,\gamma,\sum_{i=1}^{m} \alpha_i p_i) \geq \sum_{i=1}^{m} \alpha_i H(t,\gamma,p_i),$$

for all $(t,\gamma) \in [0,T] \times \mathbb{R}$.

(D) For all $(t,x) \in (0,T) \times \mathbb{R}^n$, it holds

$$\sup_{\|p\|=1} \Big(\langle p, x\rangle - \sigma^{\#}(\gamma^*,p) - \int_0^t H(\tau,\gamma^*,p)d\tau \Big) \geq 0.$$

Clearly, the above inequality is automatically satisfied if γ^* is replaced by $\gamma < \gamma^*$, since the left-hand side is equal to $+\infty$.

We define a function

$$\begin{array}{rcl} u : [0, T] \times \mathbb{R}^n & \to & \mathbb{R} \\ (t, x) & \mapsto & u(t, x) \end{array}$$

$$u(t,x) = \inf \Big\{ \gamma \in \mathbb{R} : \sup_{p \in \mathbb{R}^n} \Big(\langle p, x \rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p) d\tau \Big) \leq 0 \Big\}. \tag{3.3}$$

Remark 2. From Lemma 2.9, it is easy to see that $u(t, x) \geq \gamma^*$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

By virtue of Proposition 2.1 and the assumption (B), the function

$$p \mapsto \langle p, x \rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p) d\tau \tag{3.4}$$

is positively homogeneous of degree 1. Hence, we may restrict the points p over which the supremum is taken to the closed ball $\overline{B}(0; 1) = \{p \in \mathbb{R}^n : \|p\| \leq 1\}$, or to the spherical surface $S(0; 1) = \{p \in \mathbb{R}^n : \|p\| = 1\}$. That is,

$$\begin{aligned} u(t,x) &= \inf \Big\{ \gamma \in \mathbb{R} : \sup_{\|p\| \leq 1} \Big(\langle p, x \rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p) d\tau \Big) \leq 0 \Big\} \\ &= \inf \Big\{ \gamma \in \mathbb{R} : \sup_{\|p\| = 1} \Big(\langle p, x \rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p) d\tau \Big) \leq 0 \Big\}. \end{aligned} \tag{3.5}$$

The facts that $\gamma \geq \gamma^*$, $\sigma^{\#}(\gamma, .)$ is finite, convex and so continuous, and $\sigma^{\#}(\gamma, p) \to +\infty$ as $\gamma \to +\infty$, guarantee that $u(t, x)$ is not equal to $+\infty$ and so well-defined. Let us affirm the continuity of $u(t, x)$ as in the following lemma.

Lemma 3.1. *Let (B) hold and let $\sigma \in C(\mathbb{R}^n)$ be quasiconvex and satisfy the coercivity condition $\sigma(x) \to +\infty$ as $\|x\| \to \infty$. Then, the function $u(t, x)$ defined by (3.3) is continuous and satisfies*

$$\lim_{(t,x) \to (0, x_0)} u(t, x) = u(0, x_0) = \sigma^{\#\#}(x_0) = \sigma(x_0).$$

Proof. From Proposition 2.1, $\sigma^{\#}(\gamma, p)$ is upper semicontinuous in $\gamma \in \mathbb{R}$, and nondecreasing. Hence, it follows from Lemma 2.3 that the function

$$\gamma \mapsto \sup_{\|p\| = 1} \Big(\langle p, x \rangle - \sigma^{\#}(\gamma, p) - \int_0^t H(\tau, \gamma, p) d\tau \Big)$$

is nonincreasing and lower semicontinuous, and so continuous from the right. Using the representation (3.5), we will show that, for any $\gamma_0 \in \mathbb{R}$

$$\begin{aligned} E_{u,\gamma_0} &= \Big\{(t,x) \in [0,T] \times \mathbb{R}^n : u(t,x) \le \gamma_0\Big\} \\ &= \Big\{(t,x) : \inf\Big\{\gamma \in \mathbb{R} : \sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\Big\} \le \gamma_0\Big\} \\ &= A := \Big\{(t,x) : \sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma_0,p) - \int_0^t H(\tau,\gamma_0,p)d\tau\Big) \le 0\Big\}. \end{aligned} \tag{3.6}$$

Evidently, $A \subset E_{u,\gamma_0}$. The opposite inclusion can be obtained as follows. For $(t,x) \in E_{u,\gamma_0}$, if

$$\sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma_0+\varepsilon_0,p) - \int_0^t H(\tau,\gamma_0+\varepsilon_0,p)d\tau\Big) > 0,$$

for some $\varepsilon_0 > 0$, then, for all $\gamma < \gamma_0 + \varepsilon_0$,

$$\sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) > 0.$$

That is,

$$u(t,x) \ge \gamma_0 + \varepsilon_0,$$

a contradiction. Hence, for all $\varepsilon > 0$, we have

$$\sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma_0+\varepsilon,p) - \int_0^t H(\tau,\gamma_0+\varepsilon,p)d\tau\Big) \le 0.$$

By the continuity from the right of the function on the left-hand side mentioned above, letting $\varepsilon \to 0+$ we get

$$\sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma_0,p) - \int_0^t H(\tau,\gamma_0,p)d\tau\Big) \le 0,$$

i. e., $x \in A$, and thus $E_{u,\gamma_0} \subset A$. Hence, (3.6) holds.

If $\gamma_0 < \gamma^*$, then $E_{u,\gamma_0} = \emptyset$, a closed set. Otherwise, by virtue of Lemma 2.3, the function

$$v(t,x) = \sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma_0,p) - \int_0^t H(\tau,\gamma_0,p)d\tau\Big) \le 0$$

is lower semicontinuous and even continuous on $[0,T] \times \mathbb{R}^n$ whenever $\gamma_0 \geq \gamma^*$. Hence, from (3.6), the level set E_{u,γ_0} is closed. So, u is lower semicontinuous as well.

We have left to prove u is upper semicontinuous. Denote

$$\partial E_{\sigma,\gamma} = \{x \in \mathbb{R}^n : \sigma(x) = \gamma\},$$

which is the boundary of the level set $E_{\sigma,\gamma}$. Since $\sigma(x) \to +\infty$ as $\|x\| \to \infty$ and σ is continuous, the level sets $E_{\sigma,\gamma}$ and the boundaries $\partial E_{\sigma,\gamma}$ are compact sets in $\mathbb{R}^n$ for all $\gamma \in [\gamma^*, +\infty)$, and $E_{\sigma,\gamma} = \emptyset$ if $\gamma < \gamma^*$.

To every fixed $(t_0, x_0) \in [0,T] \times \mathbb{R}^n$, we put $\gamma_0 = u(t_0, x_0)$. Pay attention to Remark 2 that $\gamma_0 \geq \gamma^*$, so the level sets $E_{\sigma,\gamma} \neq \emptyset$ for all $\gamma \geq \gamma_0$. For $\Delta > 0$, since $E_{\sigma,\gamma_0}, \partial E_{\sigma,\gamma_0}$ are compact sets, we have

$$d = d\Big(E_{\sigma,\gamma_0}, \partial E_{\sigma,\gamma_0+\Delta}\Big) = \inf_{\substack{x \in E_{\sigma,\gamma_0} \\ y \in \partial E_{\sigma,\gamma_0+\Delta}}} \|x - y\| = \min_{\substack{x \in E_{\sigma,\gamma_0} \\ y \in \partial E_{\sigma,\gamma_0+\Delta}}} \|x - y\|.$$

If $d = 0$, then, there exists $x_0 \in E_{\sigma,\gamma_0} \cap \partial E_{\sigma,\gamma_0+\Delta}$, i. e., $\sigma(x_0) \leq \gamma_0$ and $\sigma(x_0) = \gamma_0 + \Delta$, a contradiction. Hence, $d > 0$. We will show that

$$E_{\sigma,\gamma_0} + B(0,d) \subset E_{\sigma,\gamma_0+\Delta}. \tag{3.7}$$

Actually, if $x \in E_{\sigma,\gamma_0} + B(0,d)$, then $x = y + z$, where $y \in E_{\sigma,\gamma_0}$ and $\|z\| < d$. The facts $\sigma(y + tz) \to +\infty$ as $t \to +\infty, \sigma(y) \leq \gamma_0$, and the continuity of $\sigma(x)$ imply that there exists $t' > 0$ such that

$$\sigma(y + t'z) = \gamma_0 + \Delta.$$

This means $y + t'z \in \partial E_{\sigma,\gamma_0+\Delta}$. Set $x' = y + t'z$ to get $t'd > t'\|z\| = d(y, x') \geq d$, and so $t' > 1$. Therefore, we have

$$x = y + z \in (y, y + t'z) \subset E_{\sigma,\gamma_0+\Delta},$$

where $(y, y + t'z)$ denotes the open line segment from y to $y + t'z$, since $E_{\sigma,\gamma_0+\Delta}$ is a convex set. Hence, (3.7) is verified.

Since E_{σ,γ_0} is a compact set, the supremum in the definition of $\sigma^\#(\gamma_0, p)$ is attained, i. e., for every $p \in S(0;1)$, there exists $q \in \mathbb{R}^n$ such that,

$$\begin{aligned}\sigma^\#(\gamma_0, p) &= \sup(\langle p, x\rangle : x \in E_{\sigma,\gamma_0}) \\ &= \max(\langle p, x\rangle : x \in E_{\sigma,\gamma_0}) \\ &= \langle p, q\rangle, \quad q \in E_{\sigma,\gamma_0}.\end{aligned}$$

So, from (3.7) we have

$$\begin{aligned}\sigma^\#(\gamma_0 + \Delta, p) &= \sup(\langle p, x\rangle : x \in E_{\sigma,\gamma_0+\Delta}) \\ &\geq \sup(\langle p, x\rangle : x \in E_{\sigma,\gamma_0} + B(0,d)) \\ &\geq \sup(\langle p, q\rangle + \langle p, z\rangle : \|z\| < d) \\ &\geq \sigma^\#(\gamma_0, p) + d, \quad \forall p \in S(0;1).\end{aligned} \tag{3.8}$$

Hence, from the above argument on the continuity of the function $v(t,x)$, there exists a neighbourhood V of $(t_0,x_0), V\subset[0,T]\times\mathbb{R}^n$ such that for all $(t,x)\in V$,

$$\begin{aligned}0 &\geq \sup_{\|p\|=1}\Big(\langle p,x_0\rangle-\sigma^{\#}(\gamma_0,p)-\int_0^{t_0}H(\tau,\gamma_0,p)d\tau\Big)\\ &\geq \sup_{\|p\|=1}\Big(\langle p,x\rangle-\sigma^{\#}(\gamma_0,p)-\int_0^{t}H(\tau,\gamma_0,p)d\tau\Big)-d\\ &\geq \sup_{\|p\|=1}\Big(\langle p,x\rangle-\sigma^{\#}(\gamma_0+\Delta,p)-\int_0^{t}H(\tau,\gamma_0,p)d\tau\Big)\\ &\geq \sup_{\|p\|=1}\Big(\langle p,x\rangle-\sigma^{\#}(\gamma_0+\Delta,p)-\int_0^{t}H(\tau,\gamma_0+\Delta,p)d\tau\Big).\end{aligned}\tag{3.9}$$

Now, it follows form (3.9) that

$$u(t,x)\leq\gamma_0+\Delta=u(t_0,x_0)+\Delta,\quad\forall(t,x)\in V.$$

This means that $u(t,x)$ is upper semicontinuous at (t_0,x_0). Since (t_0,x_0) is arbitrarily chosen, $u(t,x)$ is upper semicontinuous on $[0,T]\times\mathbb{R}^n$.

Finally, by Proposition 2.2, $u(0,x)=(\sigma^{\#}(\gamma,p))^{\#}(x)=\sigma^{\#\#}(x)=\sigma(x)$. □

We are now in the position to state the main result.

Theorem 3.2. *Let (A)-(D) hold. Then, the function $u(t,x)$ defined by (3.3) is a viscosity solution of Problem (3.1) - (3.2).*

Proof. We have left to prove that $u(t,x)$ in (3.3) satisfies the two inequalities. To prove the theorem, we need some auxiliary results. Assume that the point $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ is chosen arbitrarily, we set $\gamma_0=u(t_0,x_0)$. ¿From (3.6) we have

$$\begin{aligned}E_{u,\gamma_0}&=\{(t,x)\in[0,T]\times\mathbb{R}^n:u(t,x)\leq\gamma_0\}\\ &=\Big\{(t,x):\sup_{\|p\|=1}\Big(\langle p,x\rangle-\sigma^{\#}(\gamma_0,p)-\int_0^{t}H(\tau,\gamma_0,p)d\tau\Big)\leq 0\Big\}\\ &=\bigcap_{\|p\|=1}\Big\{(t,x):\langle p,x\rangle-\sigma^{\#}(\gamma_0,p)-\int_0^{t}H(\tau,\gamma_0,p)d\tau\leq 0\Big\}.\end{aligned}$$

Now we extend continuously the Hamiltonian H to the whole $n+1$-dimension space as follows:

$$H(t,\gamma_0,p)=H(0,\gamma_0,p)=0\quad\text{if}\quad t<0;$$

$$H(t, \gamma_0, p) = H(T, \gamma_0, p) \quad \text{if} \quad t > T.$$

We therefore consider the set Γ defined by constraints of the form

$$G(t, x; p) = \langle p, x\rangle - \sigma^{\#}(\gamma_0, p) - \int_0^t H(\tau, \gamma_0, p)d\tau \le 0, \quad p \in S(0; 1), (t, x) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.10}$$

Since (t_0, x_0) is an interior point of $(0, T) \times \mathbb{R}^n$, there exists a neighbourhood V' of (t_0, x_0), $V' \subset (0, T) \times \mathbb{R}^n$. Hence,

$$E_{u,\gamma_0} \cap V' = \Gamma \cap V'.$$

This means that E_{u,γ_0} and Γ have the same tangent cone and so share the same outer normal cone at (t_0, x_0).

It is easy to check that the function $G(t, x; p)$ of (t, x) and its gradients $\nabla_{(t,x)} G(t, x; p)$ are continuous in $(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $p \in S(0; 1)$.

We want to recall some concepts concerning normal cones according to Hestenes [10]. Let S_0 be the set of *active indices* at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, that is, the set of indices $p \in S(0; 1)$ such that $G(t_0, x_0; p) = 0$. Since $G(t_0, x_0; p)$ is a continuous function of p on $S(0; 1)$, the set S_0 is a compact set. In view of [Sec. 11 Chap. 4, 10], a tangent vector h of Γ at (t_0, x_0) satisfies the constraints

$$\langle \nabla G(t_0, x_0; p), h\rangle \le 0, \quad p \in S_0. \tag{3.11}$$

These constraints will be called the *tangential constraints* for Γ at (t_0, x_0). The class of vectors h satisfying these constraints will be denoted by $\mathcal{K}$. The point (t_0, x_0) will be called a *regular point* of Γ if $\mathcal{K}$ is the tangent cone of Γ at (t_0, x_0) and if, in addition, the cone $\mathcal{N}$ generated by the gradients $\nabla G(t_0, x_0; p), (p \in S_0)$ is closed. If (t_0, x_0) is a regular point of Γ, then $\mathcal{N}$ is the dual cone of the tangent cone $\mathcal{K}$ and is accordingly the outer normal cone of Γ at (t_0, x_0).

Lemma 3.3. $S_0 \ne \emptyset$.

Proof. Evidently, $\gamma_0 = u(t_0, x_0) \ge \gamma^*$.

If $\gamma_0 = \gamma^*$, by the assumption (D), there exists a point p_0 with $\|p_0\| = 1$ such that $G(t_0, x_0, p_0) \ge 0$. By the definition of $u(t_0, x_0)$, we conclude that $G(t_0, x_0, p_0) = 0$. Hence, $S_0 \ne \emptyset$.

If $\gamma_0 > \gamma^*$, assume, for the sake of contradiction, that for some $\varepsilon_0 > 0$

$$\sup_{\|p\|=1} \Big(\langle p, x_0\rangle - \sigma^{\#}(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p)d\tau \Big) < -\varepsilon_0 < 0. \tag{3.12}$$

Since H is continuous, H is uniformly continuous on the compact set $[0, T] \times [\gamma_0 - 1, \gamma_0 + 1] \times S(0; 1)$. Therefore the function $F(\gamma, p) = \int_0^{t_0} H(\tau, \gamma, p)d\tau$ is uniformly continuous on $[\gamma_0 - 1, \gamma_0 + \delta_1] \times S(0;1)$, and so the family $(F(\gamma, p))_{\|p\|=1}$ is equicontinuous in $\gamma \in [\gamma_0 - 1, \gamma_0 + 1]$. Thus, in view of Lemma 2.9, the family $(T(\gamma, p))_{\|p\|=1}, T(\gamma, p) = \langle p, x_0\rangle - \sigma^{\#}(\gamma, p) - \int_0^{t_0} H(\tau, \gamma, p)d\tau$ is equi-upper semicontinuous in $\gamma \in [\gamma_0 - 1, \gamma_0 + 1]$. It therefore follows from (3.12) that there exists $0 < \delta < 1$ such that

$$\langle p, x_0\rangle - \sigma^{\#}(\gamma, p) - \int_0^{t_0} H(\tau, \gamma, p)d\tau \leq \langle p, x_0\rangle - \sigma^{\#}(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p)d\tau + \varepsilon_0/2$$
$$< -\varepsilon_0/2 < 0, \quad \forall p \in S(0;1), \forall \gamma : \gamma_0 - \delta < \gamma < \gamma_0 + \delta.$$

That is,

$$\sup_{\|p\|=1} \Big(\langle p, x_0\rangle - \sigma^{\#}(\gamma, p) - \int_0^{t_0} H(\tau, \gamma, p)d\tau\Big) \leq -\varepsilon_0/2 < 0, \quad \forall \gamma : \gamma_0 - \delta < \gamma < \gamma_0 + \delta.$$

In particular,

$$\sup_{\|p\|=1} \Big(\langle p, x_0\rangle - \sigma^{\#}(\gamma_0 - \frac{\delta}{2}, p) - \int_0^{t_0} H(\tau, \gamma_0 - \frac{\delta}{2}, p)d\tau\Big) \leq 0,$$

where $\delta > 0$. This contradicts the definition of $u(t_0, x_0) = \gamma_0$. The proof is complete. □

Lemma 3.4. *(t_0, x_0) is a normal point and so is a regular point of Γ.*

Proof. We have $\nabla G(t_0, x_0; p) = (-H(t_0, \gamma_0, p), p) \neq 0$ for all $\|p\| = 1$. This means that for the vector $h = (H(t_0, \gamma_0, p_0), -p_0)$, for some $p_0 \in S_0$, we have

$$\langle(-H(t_0, \gamma_0, p_0), p_0), h\rangle \leq -\|p_0\|^2 = -1 < 0, \quad p_0 \in S_0,$$

which means that (t_0, x_0) is a normal point of Γ. In view of [Th. 11.2, Ch. 4, 10], (t_0, x_0) therefore is a regular point of Γ.

¿From Lemmas 3.3, 3.4, the outer normal cone of Γ and so of E_{u,γ_0} at (t_0, x_0) is given by

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) = \Big\{\sum_{i=1}^{m} \lambda_i \nabla G(t_0, x_0; p_i) : \lambda_i \geq 0, p_i \in S_0, 1 \leq i \leq m \leq n+1\Big\}. \quad (3.13)$$

We have here a similar result to the one in [5].

Lemma 3.5. *Let $S^* = \{p \in \mathbb{R}^n : G(t_0, x_0; p) = \sup_{\|q\|\leq 1} G(t_0, x_0; q)\}$ denote the set where the sup is achieved. Then $S_0 \subset S^*$, and if $(\alpha_i)_i$ is any set of m numbers with $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$, we have*

$$-\sum_{i=1}^{m} \alpha_i H(t_0, \gamma_0, p_i) \geq -H(t_0, \gamma_0, \sum_{i=1}^{m} \alpha_i p_i). \quad (3.14)$$

Proof. Obviously, (3.14) holds if H satisfies the condition (C(ii)). Assume (C(i)) and let $p' = \sum_{i=1}^{m} \alpha_i p_i$. Then $\|p'\| \le \sum_{i=1}^{m} \alpha_i \|p_i\| = \sum_{i=1}^{m} \alpha_i = 1$. Since each p_i achieves the supremum, we have for $1 \le i \le m$,

$$0 = \langle p_i, x_0 \rangle - \sigma^{\#}(\gamma_0, p_i) - \int_0^{t_0} H(\tau, \gamma_0, p_i) d\tau \ge \langle p', x_0 \rangle - \sigma^{\#}(\gamma_0, p') - \int_0^{t_0} H(\tau, \gamma_0, p') d\tau.$$

Multiply by α_i and sum on i to get

$$\langle p', x_0 \rangle - \sum_{i=1}^{m} \alpha_i \sigma^{\#}(\gamma_0, p_i) - \int_0^{t_0} \sum_{i=1}^{m} \alpha_i H(\tau, \gamma_0, p_i) d\tau \ge \langle p', x_0 \rangle - \sigma^{\#}(\gamma_0, p') - \int_0^{t_0} H(\tau, \gamma_0, p') d\tau.$$

Cancel the first equal terms and use Jensen's inequality on $\sigma^{\#}(\gamma_0, p)$, since this function is convex in p, to get

$$-\int_0^{t_0} \sum_{i=1}^{m} \alpha_i H(\tau, \gamma_0, p_i) d\tau \ge -\int_0^{t_0} H(\tau, \gamma_0, p') d\tau = -\int_0^{t_0} H(\tau, \gamma_0, \sum_{i=1}^{m} \alpha_i p_i) d\tau.$$

Therefore, by (C(i)) we have

$$-\int_0^{t_0} \sum_{i=1}^{m} \alpha_i h(\tau, \gamma_0) H(t_0, \gamma_0, p_i) d\tau \ge -\int_0^{t_0} h(\tau, \gamma_0) H(t_0, \gamma_0, p') d\tau.$$

Equivalently,

$$\begin{aligned} -\Big(\sum_{i=1}^{m} \alpha_i H(t_0, \gamma_0, p_i)\Big) \int_0^{t_0} h(\tau, \gamma_0) d\tau &\ge -H(t_0, \gamma_0, p') \int_0^{t_0} h(\tau, \gamma_0) d\tau \\ &= -H(t_0, \gamma_0, \sum_{i=1}^{m} \alpha_i p_i) \int_0^{t_0} h(\tau, \gamma_0) d\tau > 0. \end{aligned}$$

Dividing by $\int_0^{t_0} h(\tau, \gamma_0) d\tau > 0$, we obtain (3.14). The lemma is completely proved. □

In order to prove that u is a supersolution, let φ be a real smooth function on $(0, T) \times \mathbb{R}^n$, so that $u - \varphi$ achieves a zero minimum at the point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. We will prove that

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + H(t_0, u(t_0, x_0), \nabla_x \varphi(t_0, x_0)) \ge 0. \tag{3.15}$$

Since $u-\varphi$ has a zero minimum at (t_0,x_0), there exists a neighbourhood V'' of (t_0,x_0) such that $E_{u,\gamma_0}\cap V''\subset E_{\varphi,\gamma_0}\cap V''$ and so

$$\mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0})=\mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0}\cap V'')\subset \mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}\cap V'')=\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}).$$

By virtue of [Lem. 7.1, Ch. 4, 10], we have for the smooth function φ

$$\mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0})=\{\lambda\nabla\varphi(t_0,x_0),\lambda\ge 0\}.$$

Consequently,

$$\nabla\varphi(t_0,x_0)\in\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}). \tag{3.16}$$

¿From (3.16), $\nabla\varphi(t_0,x_0)$ can be expressed in the terms of $\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0})$,

$$\nabla\varphi(t_0,x_0)=\Big(\frac{\partial\varphi(t_0,x_0)}{\partial t},\nabla_x\varphi(t_0,x_0)\Big)=\sum_{i=1}^{m}\lambda_i(-H(t_0,\gamma_0,p_i),p_i), \tag{3.17}$$

where $p_i\in S_0,\lambda_i\ge 0, 1\le i\le m$. Now, we set $\alpha_i=\lambda_i/\lambda'$, where $\lambda'=\sum\limits_{i=1}^{m}\lambda_i$. Assume $\lambda'>0$. Then $1\ge\alpha_i\ge 0$, $\sum\limits_{i=1}^{m}\alpha_i=1$, and by the homogeneity of degree 1 of $H(t,\gamma,.)$ we get from (3.17)

$$\Big(\frac{\partial\varphi(t_0,x_0)}{\partial t},\nabla_x\varphi(t_0,x_0)\Big)=\sum_{i=1}^{m}\alpha_i(-H(t_0,\gamma_0,\lambda'p_i),\lambda'p_i).$$

Hence, $\nabla_x\varphi(t_0,x_0)=\sum\limits_{i=1}^{m}\lambda'\alpha_ip_i$. Applying Lemma 3.5 we get

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}=-\sum_{i=1}^{m}\alpha_iH(t_0,\gamma_0,\lambda'p_i)\ge -H(t_0,\gamma_0,\sum_{i=1}^{m}\lambda'\alpha_ip_i)=-H(t_0,\gamma_0,\nabla_x\varphi(t_0,x_0)).$$

After rearranging the terms, we obtain (3.15).

If $\lambda'=0$, then $\nabla\varphi(t_0,x_0)=0$. Since $H(t,\gamma,.)$ is positively homogeneuous, $H(t,\gamma,0)=0$ for all $(t,\gamma)\in[0,T]\times\mathbb{R}$. Hence,

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+H(t_0,u(t_0,x_0),\nabla_x\varphi(t_0,x_0))=0+H(t_0,\gamma_0,0)=0.$$

Finally, we still have to prove that u is a subsolution. Let $u-\varphi$ achieve a zero maximum at the point (t_0,x_0) with φ a smooth function. We will show that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+H(t_0,u(t_0,x_0),\nabla_x\varphi(t_0,x_0))\le 0. \tag{3.18}$$

Since $u - \varphi$ has a zero maximum at (t_0, x_0), by an argument similar to the above, we have

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) \subset \mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0}).$$

It follows from (3.13) that the outer normal cone $\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) \neq \{0\}$. On the other hand, the fact that φ is a smooth function means that the outer normal cone consists of at most one point. This implies that $\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) = \mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0})$ and $\nabla\varphi(t_0, x_0) \neq 0$. Hence, we obtain the expression of $\nabla\varphi$:

$$\nabla\varphi(t_0, x_0) = \Big(\frac{\partial\varphi(t_0, x_0)}{\partial t}, \nabla_x\varphi(t_0, x_0)\Big) = \lambda_0(-H(t_0, \gamma_0, p), p) = (-H(t_0, \gamma_0, \lambda_0 p), \lambda_0 p),$$

for some $\lambda_0 > 0$. Then,

$$\frac{\partial\varphi(t_0, x_0)}{\partial t} = -H(t_0, \gamma_0, \lambda_0 p) = -H(t_0, \gamma_0, \nabla_x\varphi(t_0, x_0)).$$

We therefore obtain (3.18). The proof is complete. □

¿From Proposition 2.6 and Theorem 3.2, we get the following.

Corollary 3.6. *Let $\sigma(x) \in C(\mathbb{R}^n)$ be quasiconvex, not attaine a local minimum in any open subset of $\mathbb{R}^n \backslash I$, and satisfy the coercivity condition, where I denotes the set on which σ attains its (global) infimum, and let (A)-(D) hold. Then, the formula (3.3) gives us a viscosity solution of Problem (3.1)-(3.2).*

Example. Consider the following Cauchy problem:

$$\frac{\partial u(t, x)}{\partial t} - (1 + t)^{-u(t,x)}\|\nabla_x u(t, x)\| = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \tag{3.19}$$

$$u(0, x) = \|x\|, \quad x \in \mathbb{R}^n. \tag{3.20}$$

Evidently, the initial function $\sigma(x) = \|x\|, x \in \mathbb{R}^n$ satisfies all conditions of Corollary 3.6, and the Hamiltonian $H(t, \gamma, p) = -(1 + t)^{-\gamma}\|p\|$ satisfies both conditions (B) and (C). We just have to check that the condition (D) holds, too. In fact, calculating directly we get

$$\sigma^{\#}(\gamma, p) = \begin{cases} -\infty & \text{if} \quad \gamma < 0, \\ \gamma\|p\| & \text{if} \quad \gamma \geq 0. \end{cases}$$

$$\gamma^* = 0, \quad \sigma^{\#}(\gamma^*, p) = \sigma^{\#}(0, p) = 0.$$

Thus, we have

$$\begin{aligned}\sup_{\|p\|=1}\Big(\langle p, x\rangle - \sigma^{\#}(\gamma^*, p) - \int_0^t H(\tau, \gamma^*, p)d\tau\Big) &= \sup_{\|p\|=1}\Big(\langle p, x\rangle + \int_0^t \|p\|d\tau\Big) \\ &= \sup_{\|p\|=1}\Big(\langle p, x\rangle + \|p\|t\Big) \geq 0,\end{aligned}$$

for all $(t, x) \in (0, T) \times \mathbb{R}^n$, i.e., the condition (D) is satisfied.

The formula (3.3) therefore gives us a viscosity solution of Problem (3.19)-(3.20) as follows:

$$\begin{aligned} u(t,x) &= \inf\Big\{\gamma \in \mathbb{R}, \gamma \geq 0 : \sup_{\|p\|=1} \big(\langle p, x\rangle - \gamma + \int_0^t (1+\tau)^{-\gamma} d\tau\big) \leq 0\Big\} \\ &= \inf\{\gamma \in \mathbb{R}, \gamma \geq 0 : \gamma - \int_0^t (1+\tau)^{-\gamma} d\tau - \|x\| \geq 0\}. \end{aligned} \tag{3.21}$$

Observe that the function

$$f(\gamma) := \gamma - \int_0^t (1+\tau)^{-\gamma} d\tau - \|x\|, \quad \gamma \in \mathbb{R},$$

is strictly increasing, $f(\gamma) \to \pm\infty$ as $\gamma \to \pm\infty$. So, the expression (3.21) yields

$$u(t, x) = \gamma_0,$$

where $\gamma_0 > 0$ is the unique solution of $\gamma - \int_0^t (1+\tau)^{-\gamma} d\tau - \|x\| = 0$.

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References

1. J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin (1984).
2. M. Bardi M. and L. C. Evans, *On Hopf's formulas for solutions of Hamilton-Jacobi equations*, Nonlinear Anal. Theory, Meth. & Appl., **8** (1984), 1373-1381.
3. E. N. Barron and W. Liu, *Calculus of variations in* L^∞, App. Math. Optim., to appear.
4. E. N. Barron, R. Jensen, and W. Liu, *Hopf-Lax- Type Formula for* $u_t + H(u, Du) = 0$, J. of Differential Equations **126** (1996), 48-61.
5. E. N. Barron, R. Jensen and W. Liu *Hopf-Type Lax-Formula for* $u_t + H(u, Du) = 0$*, II*, preprint, (1996).
6. M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277** (1983), 1-42.
7. M. G. Crandall, L. C. Evans and P. L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **282** (1984) 487-502.

8. M. G. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second-order partial differential equations*, Bull. Amer. Math. Soc., **27** (1992), 1-67.
9. K. Deimling, *Multivalued Differential Equations*, Birkhäuser, (1992).
10. M. R. Hestenes, *Optimization Theory: The finite dimensional case*, Robert E. K. Rieger Publishing Company Hungtington, New York, (1981).
11. L. Hörmander, *Notions of Convexity*, Birkhäuser, (1994).
12. E. Hopf, *Generalized solutions of nonlinear equations of first order*, J. of Math. and Mech., **14** (1965), 951-973.
13. R. Horst and H. Tuy, *Global Optimization*, Springer-Verlag, Berlin (1990).
14. H. Ishii, *Uniqueness of Unbounded Viscosity Solution of Hamilton-Jacobi Equations*, Indiana Univ. Math. J., **33**(5) (1984).
15. O. L. Mangasarian, *Nonlinear Programming*, Mc Graw-Hill, New York, (1969).
16. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, (1970).
17. A. I. Subbotin, *Minimax inequalities and Hamilton-Jacobi equations*, Nauka, Moscow, (1991).
18. A. I. Subbotin, *Generalized Solutions of First Order PDEs*, Birkhäuser, Boston, (1995).
19. Tran Duc Van, Nguyen Hoang and Nguyen Duy Thai Son, *On the explicit representation of global solutions of the Cauchy problem for Hamilton-Jacobi equations,* Acta Math. Vietnamica, **19**(2) (1994), 111-120.
20. Tran Duc Van, Nguyen Hoang and Rudolf Gorenflo, *Existence of global quasi-classical solutions of the Cauchy problem for Hamilton-Jacobi equations*, Differentsialnye Uravneniya, **31** (4) (1995), 672-676.
21. Tran Duc Van and Nguyen Hoang, *On the existence of global solutions of the Cauchy problem for Hamilton-Jacobi equations*, SEA Bull. of Math. **20** (1996), 81-88.
22. Tran Duc Van, Mai Duc Thanh and Nguyen Hoang, *On the representation of Lipschitz global solutions of the Cauchy problem for Hamilton-Jacobi equations*, Proc. of Inter. Conference on Appl. Anal. and Mech. of Cont. Media, Ho Chi Minh City, (12/1995), 428-436.
23. Tran Duc Van and Nguyen Duy Thai Son, *On a class of Lipschitz continuous functions of several variables*, Proc. of AMS, **121** (1994), 865-870.
24. Tran Duc Van, Mai Duc Thanh and Rudolf Gorenflo, *On explicit viscosity solutions for* $\partial u/\partial t + H(u, \nabla u) = 0$, preprint, (1996).