# Banach spaces with the Daugavet property

Vladimir M. Kadets[1], Roman V. Shvidkoy, Gleb G. Sirotkin and Dirk Werner

Abstract. A Banach space $X$ is said to have the Daugavet property if every operator $T\colon X \to X$ of rank 1 satisfies $\|\mathrm{Id}+T\| = 1+\|T\|$. We show that then every weakly compact operator satisfies this equation as well and that $X$ contains a copy of $\ell_1$. However, $X$ need not contain a copy of $L_1$. We also study pairs of spaces $X \subset Y$ and operators $T\colon X \to Y$ satisfying $\|J+T\| = 1+\|T\|$, where $J\colon X \to Y$ is the natural embedding. This leads to the result that a Banach space with the Daugavet property does not embed into a space with an unconditional basis. In another direction, we investigate spaces where the set of operators with $\|\mathrm{Id}+T\| = 1+\|T\|$ is as small as possible and give characterisations in terms of a smoothness condition.

## 1. Introduction

It is a remarkable result due to Daugavet [7] that the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{1.1}$$

which has become known as the Daugavet equation, holds for compact operators on $C[0,1]$; shortly afterwards the same result for compact operators on $L_1[0,1]$ was discovered by Lozanovskii [19]. During the years, (1.1) was extended to larger classes of operators on various spaces (see for instance [1], [10], [15], [21], [23], [25], [26] and the references in these papers for more information); in particular, the Daugavet equation holds for operators not fixing a copy of $C[0,1]$ defined on certain "large" subspaces of $C(K)$, where $K$ is a compact space without isolated points, and for operators not fixing a copy of $L_1[0,1]$ defined on certain "large" subspaces of $L_1[0,1]$ ([16], [24]).

[1] The work of this author was done during his visit to Freie Universität Berlin, supported by a grant from the *Deutscher Akademischer Austauschdienst*.

In Section 2 of this paper we show that the validity of (1.1) for weakly compact operators (as a matter of fact even for strong Radon-Nikodým operators) follows already from the corresponding statement for operators of rank 1 which in turn can be verified by means of a simple lemma. This has several interesting consequences and implies for instance that $X$ contains a copy of $\ell_1$ if all the rank-1 operators on $X$ satisfy the Daugavet equation. The investigation of the Daugavet equation for rank-1 operators is suggested by results in Wojtaszczyk's paper [26]. Actually, we take a broader approach and consider a Banach space $Y$ together with a closed subspace $X$ and the canonical embedding $J\colon X \to Y$, and we study the equation

$$\|J + T\| = 1 + \|T\| \tag{1.2}$$

for classes of operators from $X$ into $Y$. If (1.2) holds for operators of rank 1, we say that the pair $(X, Y)$ has the Daugavet property. We study heredity properties of Daugavet pairs and spaces and prove that the Daugavet property is inherited by $M$-ideals and by subspaces with a separable annihilator.

To explain the relevance of studying (1.2) we recall that, by an argument from [15], the well-known fact that neither $C[0,1]$ nor $L_1[0,1]$ have unconditional bases can easily be deduced from the Daugavet property for finite-rank operators on these spaces. In order to obtain the more general result that $C[0,1]$ and $L_1[0,1]$ do not even embed into spaces having an unconditional basis it is necessary to investigate (1.2). This was done in [17]. Using techniques from that paper we now prove that if $X$ is a subspace of a separable space $Y$ and $X$ has the Daugavet property, then $Y$ can be renormed so that the new norm coincides with the original one on $X$ and the pair $(X, Y)$ has the Daugavet property. This implies that no space with the Daugavet property embeds into a space with an unconditional basis.

Section 3 deals with quotient spaces of $L_1 = L_1[0,1]$ by "small" subspaces. In particular, we exhibit an example of a space with the Daugavet property not containing a copy of $L_1$; our example is $L_1/Y$ with $Y$ a space constructed by Talagrand in his work on the three space problem [22].

In Section 4 we extend results from [2] and [15] on the following question. If $T\colon X \to X$ is an operator with $\|T\| \in \sigma(T)$, then $T$ is easily seen to satisfy (1.1). We say that $X$ has the anti-Daugavet property (for a class of operators) if no other operators (in this class) satisfy (1.1). We show that the anti-Daugavet property is closely related to a smoothness property of $X$ introduced in Definition 4.1 below. In fact, we are able to characterize the anti-Daugavet property for compact operators along these lines.

We use standard notation such as $B_X$ and $S_X$ for the unit ball and the unit sphere of a Banach space $X$, and we employ the notation

$$S(x^*, \varepsilon) = \{x \in B_X : x^*(x) \geq 1 - \varepsilon\}$$

for the slice of $B_X$ determined by $x^* \in S_{X^*}$ and $\varepsilon > 0$. $\operatorname{ex} C$ stands for the set of extreme points of a set $C$. In this paper we deal with real Banach spaces although our results extend to the complex case with minor modifications.

## 2. The Daugavet property

Let $X$ be a subspace of a Banach space $Y$ and let $J\colon X \to Y$ denote the inclusion operator. We say that the pair $(X, Y)$ has the *Daugavet property* for a class $\mathcal{M}$ of operators, where $\mathcal{M} \subset L(X, Y)$, if

$$\|J + T\| = 1 + \|T\| \tag{2.1}$$

for all $T \in \mathcal{M}$. If $X = Y$, we simply say that $X$ has the Daugavet property with respect to $\mathcal{M}$, and if $\mathcal{M}$ is the class of rank-1 operators, we just say that $X$ or $(X, Y)$ has the Daugavet property.

It is well known [2] that if $T$ satisfies (2.1), then so does $\lambda T$ for every $\lambda > 0$. In particular, if $\mathcal{M}$ is a cone, it is sufficient to verify (2.1) for all $T \in \mathcal{M}$ with $\|T\| = 1$. It is obvious that $X$ has the Daugavet property once $X^*$ has it, but the converse fails (take $X = C[0,1]$); also, a 1-complemented subspace of a space with the Daugavet property might fail it (consider one-dimensional subspaces).

Here is the key lemma of our paper.

**Lemma 2.1** *Let $(X, Y)$ have the Daugavet property. Then:*

(a) *For every $y_0 \in S_Y$ and for every slice $S(x_0^*, \varepsilon_0)$ of $B_X$ there is another slice $S(x_1^*, \varepsilon_1) \subset S(x_0^*, \varepsilon_0)$ of $B_X$ such that for every $x \in S(x_1^*, \varepsilon_1)$ the inequality $\|x + y_0\| \geq 2 - \varepsilon_0$ holds.*

(b) *For every $x_0^* \in S_{X^*}$ and for every weak$^*$ slice $S(y_0, \varepsilon_0)$ of $B_{Y^*}$ (where $y_0 \in S_Y \subset S_{Y^{**}}$) there is another weak$^*$ slice $S(y_1, \varepsilon_1) \subset S(y_0, \varepsilon_0)$ of $B_{Y^*}$ such that for every $y^* \in S(y_1, \varepsilon_1)$ the inequality $\|x_0^* + y^*|_X\| \geq 2 - \varepsilon_0$ holds.*

*Proof.* Both parts are proved in a very similar fashion; so we only present the proof of (a).

Define $T\colon X \to Y$ by $Tx = x_0^*(x)y_0$. Then $\|J^* + T^*\| = \|J + T\| = 2$, so there is a functional $y^* \in S_{Y^*}$ such that $\|J^*y^* + T^*y^*\| \geq 2 - \varepsilon_0$ and $y^*(y_0) \geq 0$. Put

$$x_1^* = \frac{J^*y^* + T^*y^*}{\|J^*y^* + T^*y^*\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|J^*y^* + T^*y^*\|}.$$

Then we have, given $x \in S(x_1^*, \varepsilon_1)$,

$$\langle (J^* + T^*)y^*, x\rangle \geq (1 - \varepsilon_1)\|J^*y^* + T^*y^*\| = 2 - \varepsilon_0,$$

therefore

$$y^*(x) + y^*(y_0)x_0^*(x) \geq 2 - \varepsilon_0, \tag{2.2}$$

which implies that $x_0^*(x) \geq 1 - \varepsilon_0$, i.e., $x \in S(x_0^*, \varepsilon_0)$. Moreover, by (2.2) we have $y^*(x) + y^*(y_0) \geq 2 - \varepsilon_0$ and hence $\|x + y_0\| \geq 2 - \varepsilon_0$. □

It is evident that the converse statement is valid, too. For future reference we will record this in the following simplified version of Lemma 2.1.

**Lemma 2.2** *The following assertions are equivalent:*

(i) *The pair $(X, Y)$ has the Daugavet property.*

(ii) *For every $y \in S_Y$, $x^* \in S_{X^*}$ and $\varepsilon > 0$ there is some $x \in S_X$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$.*

(iii) *For every $y \in S_Y$, $x^* \in S_{X^*}$ and $\varepsilon > 0$ there is some $y^* \in S_{Y^*}$ such that $y^*(y) \geq 1 - \varepsilon$ and $\|x^* + y^*|_X\| \geq 2 - \varepsilon$.*

One consequence of these lemmas is that every slice of $B_X$ and every weak$^*$ slice of $B_{X^*}$ has diameter 2 if $(X, Y)$ has the Daugavet property. In particular, $X$ fails the Radon-Nikodým property, a fact originally due to Wojtaszczyk [26]. Likewise, $X^*$ fails the Radon-Nikodým property. We shall return to this circle of ideas in Theorem 2.9.

We now come to our first main result.

**Theorem 2.3** *If the pair $(X, Y)$ has the Daugavet property, then $(X, Y)$ has the Daugavet property for weakly compact operators.*

Actually, the proof will show that the Daugavet property holds for an even larger class of operators, namely the strong Radon-Nikodým operators, meaning operators $T$ for which $\overline{T(B_X)}$ is a Radon-Nikodým set. It would be interesting to decide whether the result also extends to operators not fixing a copy of $\ell_1$.

*Proof.* Let $T\colon X \to Y$ be a weakly compact operator with $\|T\| = 1$. Then $K = \overline{T(B_X)}$ is weakly compact and therefore coincides with the closed convex hull of its denting points; in fact, $K$ is the closed convex hull of its strongly exposed points [5]. So for every $\varepsilon > 0$ there is a denting point $y_0$ of $K$ with $\|y_0\| > \sup\{\|y\|\colon y \in K\} - \varepsilon = 1 - \varepsilon$, and for some $0 < \delta < \varepsilon$ there is a slice $S = \{y \in K\colon y^*(y) \geq 1 - \delta\}$ of $K$ containing $y_0$ and having diameter $< \varepsilon$; here $y^* \in Y^*$ and $\sup_{y\in K} y^*(y) = 1$. Consider $x^* = T^*y^*$. By construction $\|x^*\| = 1$ and

$$\begin{aligned} T(S(x^*,\delta)) &= \{Tx\colon x \in B_X,\ x^*(x) \geq 1-\delta\} \\ &= \{Tx\colon x \in B_X,\ y^*(Tx) \geq 1-\delta\} \ \subset\ S. \end{aligned}$$

So for every $x \in S(x^*,\delta)$ we have $\|Tx\| \geq 1 - 2\varepsilon$. Now by Lemma 2.2 select an element $x_0 \in S(x^*,\delta)$ such that $\|\, x_0 + y_0/\|y_0\|\, \| \geq 2 - \delta$ and hence $\|x_0 + y_0\| \geq 2 - 2\varepsilon$. But $Tx_0 \in S$, so $\|Tx_0 - y_0\| < \varepsilon$, and we have

$$\|J + T\| \geq \|x_0 + Tx_0\| \geq \|x_0 + y_0\| - \varepsilon \geq 2 - 3\varepsilon. \qquad \square$$

*Example.* If $X$ is a Banach space and $(\Omega, \Sigma, \mu)$ is a non-atomic measure space, then $Y := L_1(\mu, X)$ has the Daugavet property for weakly compact operators. This is a special case of a result due to Nazarenko [20]; using our preceding results we can prove this now in a few lines. Even in the case of the scalar-valued function space $L_1(\mu)$, for which other proofs have appeared for instance in [3], [13] or [19], our argument is shorter.

In fact, let $y \in S_Y$ and $y^* \in S_{Y^*}$. The functional $y^*$ can be represented by a weak$^*$ measurable function $\varphi$ taking values in $X^*$. For $\varepsilon > 0$, find a measurable subset $B$ of $\Omega$ such that $\|\chi_B y\|_{L_1} \leq \varepsilon/2$ and $\|\chi_B \varphi\|_{L_\infty} \geq 1 - \varepsilon/2$, and pick $x \in S_Y$ so that $\chi_B x = x$ and $\langle \varphi, x\rangle \geq 1 - \varepsilon$. Since clearly $\|x + y\| \geq 2 - \varepsilon$, condition (ii) of Lemma 2.2 is fulfilled.

By a similar argument, one can reprove the result from [15] that $C(K, X)$ has the Daugavet property if the compact space $K$ has no isolated points. $\square$

We now show that the Daugavet property automatically extends to certain larger range spaces.

If $K$ is a compact topological space, then we denote by $\ell_\infty(K)$ the sup-normed space of bounded real-valued functions on $K$ and by $m(K)$ the closed subspace consisting of those $f \in \ell_\infty(K)$ for which $\{t\colon f(t) \neq 0\}$ is of first category. Let us consider the quotient space

$$m_0(K) := \ell_\infty(K)/m(K).$$

It follows from the Baire category theorem that the restriction of the quotient map $Q\colon \ell_\infty(K) \to m_0(K)$ to $C(K)$ is an isometry. It is proved in [17] that the pair $(C[0,1], m_0(K))$, where $K$ is the unit ball of $C[0,1]^*$ in its weak$^*$ topology, has the Daugavet property, even for the class of so-called narrow operators, and likewise for $L_1[0,1]$.

**Proposition 2.4** *Let $(X,Y)$ have the Daugavet property and denote by $K$ the weak$^*$ closure of $\operatorname{ex} B_{Y^*}$ in $Y^*$. Then $(X, m_0(K))$ has the Daugavet property, too.*

*Proof.* The canonical map $J_1$ of $Y$ into $m_0(K)$ is the composition $QJ_0$ where $J_0\colon Y \to C(K)$ is defined by $(J_0 y)(y^*) = y^*(y)$ and $Q$ is the (restriction of the) above quotient map. As remarked above, $J_1$ is an isometry. To prove the proposition we will employ Lemma 2.2. Thus, $x^* \in S_{X^*}$, $\varepsilon > 0$ and $[f] \in S_{m_0(K)}$ are given; of course, $[f]$ denotes the equivalence class of $f \in \ell_\infty(K)$ in $m_0(K)$. We may assume, without loss of generality, that

$$A = \{t \in K\colon f(t) > 1 - \varepsilon/2\}$$

is a set of second category. Now, there is an open set $V \subset K$ such that whenever $U \subset V$ is open, then $U \cap A$ is of second category [18, p. 202].

Next we use the fact that an extreme point of a compact convex set has a neighbourhood base consisting of slices; this follows from the converse to the Krein-Milman theorem (see [6, p. 107] for details). Thus, $V$ contains a weak$^*$ slice; i.e., there are $y_0 \in S_Y$ and $\varepsilon_1 < \varepsilon/2$ such that

$$w^* \in K,\ w^*(y_0) > 1 - \varepsilon_1 \quad \Rightarrow \quad w^* \in V. \tag{2.3}$$

Since the pair $(X,Y)$ has the Daugavet property, we get from Lemma 2.2 some $x_0 \in S_X \cap S(x^*, \varepsilon/2)$ such that $\|x_0 + y_0\| > 2 - \varepsilon_1$; we wish to show that $\|[J_0 x_0 + f]\| \geq 2 - \varepsilon$. In fact, for some $w_0^* \in K$ we have $w_0^*(x_0) + w_0^*(y_0) > 2 - \varepsilon_1$ and therefore $w_0^*(y_0) > 1 - \varepsilon_1$; thus $w_0^* \in V$ by (2.3). Pick a neighbourhood $U \subset V$ of $w_0^*$ so that

$$w^*(x_0) + w^*(y_0) > 2 - \varepsilon_1 \qquad \forall w^* \in U.$$

In particular $w^*(x_0) > 1 - \varepsilon_1$, and since $U \cap A$ is of second category by construction of $V$, we have

$$\begin{aligned} \|[J_0 x_0 + f]\| &\geq \inf\{|w^*(x_0) + f(w^*)|: w^* \in U \cap A\} \\ &\geq 1 - \varepsilon_1 + 1 - \frac{\varepsilon}{2} \geq 2 - \varepsilon, \end{aligned}$$

and we are done. □

Clearly, for every space between $Y$ and $m_0(K)$ the Daugavet property holds, as well; in particular, the pair $(X, C(\overline{\mathrm{ex}\, B_{Y^*}}))$ has the Daugavet property. It remains open whether in general $(X, C(B_{Y^*}))$ has the Daugavet property; this is true for $X = Y = C[0,1]$ and $X = Y = L_1[0,1]$ [17].

Proposition 2.4 permits the following renorming theorem.

**Theorem 2.5** *Let $X \subset Y$ be separable Banach spaces, and suppose $X$ has the Daugavet property.*

(a) *If $K = \overline{\mathrm{ex}\, B_{X^*}}$ and $m_0(K)$ is as above, then there is an isomorphic embedding $S\colon Y \to m_0(K)$ with $S|_X = Q|_X$, the restriction of the quotient map from $\ell_\infty(K)$ onto $m_0(K)$.*

(b) *The space $Y$ can be renormed so that the new norm coincides with the original one on $X$ and $(X, Y)$ has the Daugavet property for the new norm.*

*Proof.* Part (a) is proved in [17, Th. 3.1] for $K = B_{X^*}$ (no matter if $X$ has the Daugavet property or not); but the proof works for all symmetric isometrically norming weak$^*$ closed subsets of $B_{X^*}$ without isolated points. So, here it is enough to check that $\mathrm{ex}\, B_{X^*}$ does not contain isolated points.

Assume that $x_0^*$ is an isolated extreme point. Since $x_0^*$ has a neighbourhood base of slices, there are some $\xi \in S_X$ and $\varepsilon > 0$ such that the only extreme point satisfying $x^*(\xi) \geq 1 - \varepsilon$ is $x^* = x_0^*$. Now the Daugavet property yields an element $x \in S_X$ with $x_0^*(x) \geq 1 - \varepsilon$ and $\|x - \xi\| \geq 2 - \varepsilon$. Pick an extreme point $x^*$ so that $x^*(x - \xi) = \|x - \xi\|$; then we have $(-x^*)(\xi) \geq 1 - \varepsilon$ and, consequently, $x^* = -x_0^*$ so that $1 - \varepsilon \leq x^*(x) = -x_0^*(x) \leq 0$: a contradiction.

(b) By part (a) and Proposition 2.4, $|||y||| = \|Sy\|$ defines a norm with the required properties. □

This theorem entails some information on the non-existence of unconditional expansions. We first formulate a lemma that gives a quantitative version of [16, Lemma 3.6].

**Lemma 2.6** *Let $X \subset Y$ be Banach spaces with $J\colon X \to Y$ the natural embedding. Suppose that the pair $(X,Y)$ has the Daugavet property with respect to a subspace $\mathcal{M} \subset L(X,Y)$ of operators. Let $T = \sum_{n=1}^{\infty} T_n$ be a pointwise unconditionally convergent series of operators $T_n \in \mathcal{M}$. Then $\|J+T\| \geq 1$. In particular, $J$ cannot be represented as a pointwise unconditionally convergent sum $J = \sum_{n=1}^{\infty} T_n$ with $T_n \in \mathcal{M}$ for all $n \in \mathbb{N}$.*

*Proof.* Denote by $\mathrm{FIN}(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. By the Banach-Steinhaus theorem, the quantity

$$\alpha = \sup\left\{ \Bigl\| \sum_{n\in A} T_n \Bigr\| \colon A \in \mathrm{FIN}(\mathbb{N}) \right\}$$

is finite, and whenever $B \subset \mathbb{N}$, then

$$\Bigl\| \sum_{n\in B} T_n \Bigr\| \leq \sup\left\{ \Bigl\| \sum_{n\in A} T_n \Bigr\| \colon A \in \mathrm{FIN}(\mathbb{N}),\ A \subset B \right\} \leq \alpha.$$

Let $\varepsilon > 0$ and pick $A_0 \in \mathrm{FIN}(\mathbb{N})$ such that $\| \sum_{n\in A_0} T_n \| \geq \alpha - \varepsilon$. Then we obtain from the Daugavet property

$$\|J+T\| \geq \Bigl\| J + \sum_{n\in A_0} T_n \Bigr\| - \Bigl\| \sum_{n\notin A_0} T_n \Bigr\| \geq 1 + \Bigl\| \sum_{n\in A_0} T_n \Bigr\| - \alpha \geq 1 - \varepsilon,$$

which proves the lemma. $\square$

We could not decide whether in general $\|J+T\| = 1 + \|T\|$ in Lemma 2.6. This is the case if the expansion is 1-unconditional, because for $\lambda_n = -1$ if $n \in A_0$ and $\lambda_n = 1$ otherwise

$$\begin{aligned} \|J+T\| &\geq 1 + 2\Bigl\| \sum_{n\in A_0} T_n \Bigr\| - \Bigl\| \sum_{n\in \mathbb{N}} \lambda_n T_n \Bigr\| \\ &\geq 1 + 2(\alpha - \varepsilon) - \|T\| \ \geq\ 1 + \|T\| - 2\varepsilon. \end{aligned}$$

**Corollary 2.7** *If $X$ is a separable Banach space with the Daugavet property, then $X$ does not embed into an unconditional sum of reflexive spaces. In particular, $X$ does not embed into a space with an unconditional basis.*

*Proof.* Assume that $X$ embeds into an unconditional sum of Banach spaces $Y = \bigoplus_n X_n$ with associated projections $P_n$ from $Y$ onto $X_n$. After replacing

$X_n$ by $P_n(X)$ we assume that the $X_n$ and $Y$ are separable. We may also assume, by renorming $Y$ according to Theorem 2.5, that the pair $(X,Y)$ has the Daugavet property so that by Theorem 2.3 every weakly compact operator from $X$ into $Y$ satisfies the Daugavet equation (1.2). Since the embedding operator $J\colon X \to Y$ has an expansion into a pointwise unconditionally convergent series $J = \sum_{n=1}^{\infty} P_n|_X$, we deduce from Lemma 2.6 that $P_{n_0}|_X$ is not weakly compact for some $n_0$, and $X_{n_0}$ is not reflexive. $\square$

Now we use Lemma 2.1 to produce $\ell_1$-copies in spaces with the Daugavet property. First, an extension of that lemma.

**Lemma 2.8** *If $(X,Y)$ has the Daugavet property, then for every finite-dimensional subspace $Y_0$ of $Y$, every $\varepsilon_0 > 0$ and every slice $S(x_0^*, \varepsilon_0)$ of $B_X$ there is a slice $S(x_1^*, \varepsilon_1)$ of $B_X$ such that*

$$\|y + tx\| \geq (1-\varepsilon_0)(\|y\| + |t|) \qquad \forall y \in Y_0,\ x \in S(x_1^*, \varepsilon_1). \tag{2.4}$$

*Proof.* Let $\delta = \varepsilon_0/2$ and pick a finite $\delta$-net $\{y_1, \dots, y_n\}$ in $S_{Y_0}$. By a repeated application of Lemma 2.1(a) we obtain a sequence of slices $S(x_0^*, \varepsilon_0) \supset S(x^{*(1)}, \varepsilon^{(1)}) \supset \dots \supset S(x^{*(n)}, \varepsilon^{(n)})$ such that one has

$$\|y_k + x\| \geq 2 - \delta \tag{2.5}$$

for all $x \in S(x^{*(k)}, \varepsilon^{(k)})$. Put $x_1^* = x^{*(n)}$ and $\varepsilon_1 = \varepsilon^{(n)}$; then (2.5) is valid for every $x \in S(x_1^*, \varepsilon_1)$ and $k = 1, \dots, n$. This implies that for every $x \in S(x_1^*, \varepsilon_1)$ and every $y \in S_{Y_0}$ the condition

$$\|y + x\| \geq 2 - 2\delta = 2 - \varepsilon_0$$

holds.

Let $0 \leq t_1, t_2 \leq 1$ with $t_1 + t_2 = 1$. If $t_1 \geq t_2$, we have for $x$ and $y$ as above

$$\begin{aligned} \|t_1 x + t_2 y\| &= \|t_1(x+y) + (t_2 - t_1)y\| \\ &\geq t_1\|x+y\| - |t_2 - t_1|\,\|y\| \\ &\geq t_1(2-\varepsilon_0) + t_2 - t_1 \\ &= t_1 + t_2 - t_1\varepsilon_0 \ \geq\ 1 - \varepsilon_0, \end{aligned}$$

and an analogous argument shows this estimate in case $t_1 < t_2$.

This implies (2.4), by the homogeneity of the norm and the symmetry of $S_{Y_0}$. $\square$

**Theorem 2.9** *If $X$ has the Daugavet property, then $X$ contains a copy of $\ell_1$.*

*Proof.* Using Lemma 2.8 inductively, it is easy to construct a sequence of vectors $e_1, e_2, \ldots$ and a sequence of slices $S(x_n^*, \varepsilon_n)$, $\varepsilon_n = 4^{-n}$, $n \in \mathbb{N}$, such that $e_{n+1} \in S(x_{n+1}^*, \varepsilon_{n+1})$ and every element of $S(x_{n+1}^*, \varepsilon_{n+1})$ is – "up to $\varepsilon_n$" – $\ell_1$-orthogonal to $\mathrm{lin}\{e_1, \ldots, e_n\}$, which means

$$\|y + x\| \geq (1 - \varepsilon_n)(\|y\| + \|x\|) \qquad \forall y \in \mathrm{lin}\{e_1, \ldots, e_n\},\ x \in S(x_{n+1}^*, \varepsilon_{n+1}).$$

The sequence $(e_n)$ is then equivalent to the unit vector basis in $\ell_1$. $\square$

The proof even shows the stronger result that $X$ contains asymptotically isometric copies of $\ell_1$ in the sense of [9].

Next, we study to what extent the Daugavet property is hereditary.

Recall that an $M$-ideal in a Banach space $X$ is a closed subspace $J$ such that $X^*$ decomposes as $X^* = V \oplus_1 J^\perp$ for some closed subspace $V$ of $X^*$, where $J^\perp = \{x^* \in X^* \colon x^*|_J = 0\}$. Then $\{x^*|_J \colon x^* \in V\}$ is linearly isometric to $J^*$, and we shall write

$$X^* = J^* \oplus_1 J^\perp. \tag{2.6}$$

**Proposition 2.10** *The Daugavet property is inherited by $M$-ideals.*

*Proof.* Suppose $J$ is an $M$-ideal in a Banach space $X$ with the Daugavet property. Let $y \in S_J$ and $\varepsilon > 0$, and let $x^* \in J^* \subset X^*$ with $\|x^*\| = 1$. Consider the slices

$$\begin{aligned} S_1 &= \{\xi \in J \colon \|\xi\| \leq 1,\ x^*(\xi) \geq 1 - \varepsilon\}, \\ S &= \{\xi \in X \colon \|\xi\| \leq 1,\ x^*(\xi) \geq 1 - \varepsilon/3\}. \end{aligned}$$

By Lemma 2.2, there is some $x \in S$ such that $\|x + y\| \geq 2 - \varepsilon/3$; hence there is some $y^* \in S_{X^*}$ with $y^*(x + y) \geq 2 - \varepsilon/3$. Decompose $y^* = y_1^* + y_2^* \in J^* \oplus_1 J^\perp$ so that $1 = \|y^*\| = \|y_1^*\| + \|y_2^*\|$. Therefore we have

$$y^*(x) + y_1^*(y) \geq 2 - \varepsilon/3$$

so that $y^*(x) \geq 1 - \varepsilon/3$ and $y_1^*(y) \geq 1 - \varepsilon/3$. Consequently, $\|y_1^*\| \geq 1 - \varepsilon/3$ and thus $\|y_2^*\| \leq \varepsilon/3$.

Let us now consider the $\sigma(X, J^*)$-topology on $X$. An application of the Hahn-Banach theorem shows that $B_J$ is $\sigma(X, J^*)$-dense in $B_X$ [12, Remark I.1.13]. We may therefore find some $\xi \in B_J$ satisfying $|y_1^*(\xi - x)| \leq \varepsilon/3$ and $|x^*(\xi - x)| \leq \varepsilon/3$, i.e., $\xi \in S_1$, and we have

$$\begin{aligned}
\|\xi + y\| &= y^*(\xi + y) \;=\; y_1^*(\xi) + y_1^*(y) \\
&\geq y_1^*(x) + y_1^*(y) - \varepsilon/3 \\
&\geq y_1^*(x) + y_2^*(x) + y_1^*(y) - 2\varepsilon/3 \\
&\qquad \text{since } \|y_2^*\| \leq \varepsilon/3 \\
&= y^*(x) + y_1^*(y) - 2\varepsilon/3 \\
&\geq 2 - \varepsilon.
\end{aligned}$$

An application of Lemma 2.2 completes the proof of the proposition. □

Obviously, if $X$ has the Daugavet property and $J \subset X$ is an $M$-ideal, then $X/J$ need not have the Daugavet property; for example, if $X = C[0, 1]$ and $J = \{f \in X : f(0) = 0\}$, then $X/J$ is one-dimensional and thus fails the Daugavet property.

We next aim at proving that subspaces of Daugavet spaces having a separable annihilator (such subspaces can be regarded as "large") have the Daugavet property as well. We shall use local $\ell_1$-complements as explained in the following lemma.

**Lemma 2.11** *If $X$ has the Daugavet property, then for every $x \in S_X$, every $\varepsilon > 0$ and every separable subspace $V$ of $X^*$ there is an element $x^* \in S_{X^*}$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x^* + v^*\| = 1 + \|v^*\|$ for all $v^* \in V$.*

*Proof.* Take a dense sequence $(v_n^*)$ in $S_V$ and a sequence of weak$^*$ compact slices $S(x, \varepsilon) \supset S(x_1, \varepsilon_1) \supset S(x_2, \varepsilon_2) \supset \dots$ such that

$$\|x^* + v_k^*\| \geq 2 - \frac{1}{n} \qquad \forall x^* \in S(x_n, \varepsilon_n),\ k = 1, \dots, n;$$

this is possible by a repeated application of Lemma 2.1(b), as in the proof of Lemma 2.8. Clearly, any $x^* \in \bigcap_{n \in \mathbb{N}} S(x_n, \varepsilon_n)$ works. □

**Corollary 2.12** *If $X$ has the Daugavet property, then $X^*$ contains an isometric copy of $\ell_1$. In particular, $X^*$ is neither strictly convex nor smooth.*

*Proof.* An isometric $\ell_1$-copy can be produced from Lemma 2.11 by an obvious inductive procedure. □

**Theorem 2.13** *Suppose $X$ has the Daugavet property and $Y \subset X$ is a subspace with a separable annihilator $Y^\perp$. Then $Y$ has the Daugavet property.*

*Proof.* Fix $y \in S_Y$ and $\varepsilon > 0$ and consider the slice

$$S = \{y^* \in Y^* = X^*/Y^\perp \colon \|y^*\| \le 1,\ y^*(y) \ge 1-\varepsilon\}.$$

Also, fix an element $[x_1^*] \in S_{X^*/Y^\perp}$. To prove the theorem it suffices, by Lemma 2.2(iii), to find some $[x_2^*] \in S$ such that $\|[x_1^* + x_2^*]\| = 2$. To achieve this, apply Lemma 2.11 with $x = y$ and $V = \operatorname{lin}(\{x_1^*\} \cup Y^\perp)$. We get a functional $x_2^* \in S_{X^*}$ such that $x_2^*(y) \ge 1-\varepsilon$ and

$$\|x_2^* + v^*\| = 1 + \|v^*\| \qquad \forall v^* \in V.$$

Then $[x_2^*] \in S$ and

$$\begin{aligned} \|[x_1^* + x_2^*]\| &= \inf\{\|x_1^* + x_2^* + z^*\| \colon z^* \in Y^\perp\} \\ &= \inf\{1 + \|x_1^* + z^*\| \colon z^* \in Y^\perp\} \\ &\qquad (\text{since } x_1^* + z^* \in V) \\ &= 1 + \|[x_1^*]\| \ = \ 2. \end{aligned}$$

This completes the proof of the theorem. □

In [15] the following property was investigated: $X$ has the hereditary Daugavet property if every finite-codimensional subspace of $X$ has the ordinary Daugavet property. The preceding theorem shows that the hereditary Daugavet property coincides with the usual one.

At the end of this section we study sums of pairs with the Daugavet property.

**Lemma 2.14** *If $(X_1, Y_1)$ and $(X_2, Y_2)$ have the Daugavet property, then so do $(X_1 \oplus_1 X_2, Y_1 \oplus_1 Y_2)$ and $(X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2)$.*

*Proof.* We first deal with $(X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2)$. Let us consider $x_j^* \in X_j^*$, $y_j \in Y_j$ $(j = 1,2)$ with $\|(y_1, y_2)\| = \max\{\|y_1\|, \|y_2\|\} = 1$, $\|(x_1^*, x_2^*)\| = \|x_1^*\| + \|x_2^*\| = 1$. Assume without loss of generality that $\|y_1\| = 1$. By Lemma 2.2 there is, given $\varepsilon > 0$, some $x_1 \in X_1$ satisfying

$$\|x_1\| = 1, \quad x_1^*(x_1) \ge \|x_1^*\|(1-\varepsilon), \quad \|x_1 + y_1\| \ge 2 - \varepsilon.$$

Also, pick $x_2 \in X_2$ such that

$$\|x_2\| = 1, \quad x_2^*(x_2) \geq \|x_2^*\|(1-\varepsilon).$$

Then $\|(x_1, x_2)\| = 1$, $\langle (x_1^*, x_2^*), (x_1, x_2)\rangle \geq 1 - \varepsilon$ and

$$\|(x_1, x_2) + (y_1, y_2)\| \geq \|x_1 + y_1\| \geq 2 - \varepsilon.$$

Thus, $(X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2)$ has the Daugavet property.

A similar calculation, based on Lemma 2.2(iii), shows that $(X_1 \oplus_1 X_2, Y_1 \oplus_1 Y_2)$ has the Daugavet property. □

We remark that the converse of the above lemma is valid, too.

**Proposition 2.15** *Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots$ are pairs of Banach spaces with the Daugavet property. Then $(c_0(X_j), c_0(Y_j))$ and $(\ell_1(X_j), \ell_1(Y_j))$ have the Daugavet property.*

*Proof.* It follows from Lemma 2.14 that $(X_1 \oplus_\infty \cdots \oplus_\infty X_n, Y_1 \oplus_\infty \cdots \oplus_\infty Y_n)$, resp. $(X_1 \oplus_1 \cdots \oplus_1 X_n, Y_1 \oplus_1 \cdots \oplus_\infty Y_n)$, have the Daugavet property for each $n \in \mathbb{N}$. Since the union of these spaces is dense in $c_0(X_j)$, $c_0(Y_j)$, $\ell_1(X_j)$ or $\ell_1(Y_j)$, respectively, the result follows. □

For $X_j = Y_j$ these results were first proved by Wojtaszczyk [26] and, in a special case, by Abramovich [1] using different approachs.

## 3. QUOTIENTS OF $L_1$

In this section, we consider the space $L_1 = L_1[0, 1]$. The Lebesgue measure is denoted by $\mu$, and $\Sigma$ stands for the Borel $\sigma$-algebra on $[0, 1]$.

The following two properties which a subspace $X$ of $L_1$ might or might not have will turn out to be relevant for the Daugavet property of $L_1/X$.

(I) For every Borel set $A \subset [0, 1]$ and every $\varepsilon > 0$ there is a positive function $f \in L_1(A)$ such that $\|f\| = 1$ and

$$(1 - \varepsilon)(|\lambda| + \|h\|) \leq \|\lambda f + h\| \leq |\lambda| + \|h\| \qquad (3.1)$$

for all $h \in X$, $\lambda \in \mathbb{R}$.

(II) For every $f_0 \in L_1$, every Borel set $A \subset [0, 1]$ and every $\varepsilon > 0$ there is a positive function $f \in L_1(A)$ such that $\|f\| = 1$ and (3.1) holds for all $h \in \operatorname{lin}(X \cup \{f_0\})$, $\lambda \in \mathbb{R}$.

**Lemma 3.1** *Let $X$ be a subspace of $L_1$.*

(a) (I) *implies* (II).

(b) *If $X$ satisfies* (II), *then $L_1/X$ has the Daugavet property.*

*Proof.* (a) Let $f_0 \notin X$, and let $P$ denote the projection from $\mathrm{lin}(X \cup \{f_0\})$ onto $\mathrm{lin}\{f_0\}$ along $X$. Given $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\mu(B) < \delta \quad \Rightarrow \quad \|\chi_B f_0\| \leq \frac{\varepsilon}{4\|P\|}.$$

For $A$ as in (II), let $B \subset A$ be a Borel set of measure $< \delta$. An application of (I) with $B$ in place of $A$ and $\varepsilon_1 = \varepsilon/(4 + 2\|P\|)$ in place of $\varepsilon$ provides us with a positive function $f \in L_1(B)$, $\|f\| = 1$, such that

$$\|\alpha f + g\| \geq (1 - \varepsilon_1)(|\alpha| + \|g\|) \qquad \forall \alpha \in \mathbb{R},\ g \in X.$$

Now let $h = \lambda f_0 + g$ be a function in $\mathrm{lin}(X \cup \{f_0\})$ with $\|h\| = 1$; i.e., $Ph = \lambda f_0$ and $|\lambda| \leq \|P\|$. To obtain (3.1), it is enough to show that $\|f + h\| \geq 2 - \varepsilon$; see the last part of the proof of Lemma 2.8. In fact, we have

$$\begin{aligned}
2 - \|f + h\| &= \|f\| + \|h\| - \|f + h\| \\
&= \int_B (|f| + |\lambda f_0 + g| - |f + \lambda f_0 + g|)\, d\mu \\
&\qquad \text{(since } f = 0 \text{ on } [0,1] \setminus B) \\
&\leq \int_B (|f| + |g| - |f + g|) + 2|\lambda| \int_B |f_0|\, d\mu \\
&\leq (\|f\| + \|g\| - \|f + g\|) + 2|\lambda| \frac{\varepsilon}{4\|P\|} \\
&\leq \varepsilon_1 (1 + \|g\|) + \frac{\varepsilon}{2} \\
&\leq (1 + \|\mathrm{Id} - P\|)\|h\| \varepsilon_1 + \frac{\varepsilon}{2} \\
&\leq \varepsilon.
\end{aligned}$$

(b) We shall verify condition (ii) of Lemma 2.2. So let $[f_0] \in S_{L_1/X}$, $F \in S_{X^\perp} \subset S_{L_\infty}$ and $\varepsilon > 0$ be given; we are going to find an equivalence class $[f_1]$ in the slice $S(F, \varepsilon)$ such that $\|[f_0 + f_1]\| \geq 2 - 2\varepsilon$.

There is a Borel set $A \subset [0,1]$ such that

$$1 - \varepsilon \leq F|_A \leq 1 \text{ a.e.} \quad \text{or} \quad -1 \leq F|_A \leq -1 + \varepsilon \text{ a.e.} \tag{3.2}$$

Let us assume the former case. If $f$ satisfies the condition in (II), then we have

$$\langle F, f\rangle = \int_A F(t)f(t)\,dt \geq 1-\varepsilon,$$

i.e., $[f] \in S(F,\varepsilon)$. If we put $f_1 = f$, we get

$$\begin{aligned}
\|[f_0+f_1]\| &= \inf\{\|f_1+f_0+g\|: g\in X\}\\
&\geq \inf\{(1-\varepsilon)(1+\|f_0+g\|): g \in X\}\\
&= (1-\varepsilon)(1+\|[f_0]\|)\\
&= 2-2\varepsilon.
\end{aligned}$$

In the latter case of (3.2) we put $f_1 = -f$ to obtain the same conclusion.

□

**Proposition 3.2** *If $X$ is a reflexive subspace of $L_1$, then $L_1/X$ has the Daugavet property.*

*Proof.* We shall use the previous lemma and stick to its notation. Now $B_X$ is uniformly integrable since it is weakly compact [4, p. 162]. Thus, given $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\mu(B) \leq \delta \qquad \Rightarrow \qquad \int_B |h(t)|\,dt \leq \varepsilon/2 \quad \forall h \in B_X.$$

For a Borel set $A$ pick any Borel subset $B \subset A$ of measure $\leq \delta$, and let $f = \chi_B/\mu(B)$. Then we have for all $h \in B_X$ and $|\lambda| + \|h\| = 1$

$$\begin{aligned}
\|\lambda f + h\| &= \int_B |\lambda f(t) + h(t)|\,dt + \int_{[0,1]\setminus B} |h(t)|\,dt\\
&\geq |\lambda| - 2\int_B |h(t)|\,dt + \int_0^1 |h(t)|\,dt\\
&\geq 1-\varepsilon,
\end{aligned}$$

therefore property (I) is fulfilled. □

Our main objective in this section is to give an example of a Banach space with the Daugavet property not containing a copy of $L_1$, thus showing that a conceivable generalization of Theorem 2.9 is not valid. Our example will be the quotient space $L_1/Y$, with $Y$ a space constructed by Talagrand [22] as a counterexample to the three space problem for $L_1$. Before we present several crucial features of $Y$, we formulate a lemma.

**Lemma 3.3** *Let $g_m = f_m + h_m \in L_1$ be a sequence of bounded functions on $[0,1]$ such that $\sup_m \|f_m\|_{L_\infty} =: M_1 < \infty$. Assume also that the supports $\Delta_m$ of $h_m$ are small in the sense that $\sum_{m=1}^\infty \mu(\Delta_m) < \infty$ and that the sequence $(g_m)$ is equivalent to the standard $\ell_1$-basis, i.e.,*

$$\Bigl\|\sum_{m=1}^{\infty} \lambda_m g_m\Bigr\| \le \sum_{m=1}^{\infty} |\lambda_m| \le M_2 \Bigl\|\sum_{m=1}^{\infty} \lambda_m g_m\Bigr\|$$

*for some $M_2 < \infty$ and all $(\lambda_m) \in \ell_1$. Then for every $\varepsilon > 0$ there is some $\delta \in (0, \varepsilon/2)$ such that whenever $A \in \Sigma$, $\mu(A) > \varepsilon$, there is a subset $B \subset A$, $\mu(B) = \delta$, satisfying*

$$\|\chi_B f\| \le \varepsilon \|f\| \qquad \forall f \in X := \overline{\mathrm{lin}}\{g_m\colon m \in \mathbb{N}\}. \tag{3.3}$$

*Proof.* Select a number $N \in \mathbb{N}$ for which $\sum_{m>N} \mu(\Delta_m) < \varepsilon/2$. Denote $M_3 := \max_{k \le N} \|h_k\|_{L_\infty}$ and put $\delta = \varepsilon/(2 + M_2(M_1 + M_3))$. Then for every $A \in \Sigma$, $\mu(A) > \varepsilon$, there is a subset $B \subset A$, $\mu(B) = \delta$, such that $B \cap (\bigcup_{m>N} \Delta_m) = \emptyset$, since $\delta \le \varepsilon/2$. Take any $f \in S_X$, $f = \sum_{m=1}^\infty \lambda_m g_m$. Then $\sum_{m=1}^\infty |\lambda_m| \le M_2$, and we obtain

$$\|\chi_B f\| = \int_B \Bigl| \sum_{m=1}^{\infty} \lambda_m f_m + \sum_{m=1}^{N} \lambda_m h_m \Bigr| \, d\mu \le \mu(B) M_2 (M_1 + M_3) < \varepsilon,$$

which proves (3.3) □

Let us now describe the structure of Talagrand's example. In [22] he constructs a space generated by a double sequence $g_{m,n} = f_{m,n} + h_{m,n}$ such that for every fixed $n$ the sequence $(g_{m,n})_m$ meets the conditions of Lemma 3.3. Moreover, if one denotes $X_n = \overline{\mathrm{lin}}\{g_{m,n}\colon m \in \mathbb{N}\} \subset L_1$, then each $X_n$ is isomorphic to $\ell_1$, with Banach-Mazur distance tending to $\infty$, though, and $\overline{\mathrm{lin}} \bigcup_n X_n$ is isomorphic to the $\ell_1$-sum of the $X_n$ with isomorphism constant $\le 20$, and not only each $X_n$ but every finite sum of these spaces meets the conditions of Lemma 3.3. The $X_n$ are constructed so as to consist only of very "peaky" functions: For every $\varepsilon > 0$ there is an $n = n(\varepsilon)$ such that [22, Th. 3.1]

$$\mu(\{|f| \ge \varepsilon\}) \le \varepsilon \qquad \forall f \in S_{X_n}. \tag{3.4}$$

Finally, whenever $(Y_n)$ is a subsequence of $(X_n)$, then $L_1/\overline{\mathrm{lin}} \bigcup_n Y_n$ fails to contain a copy of $L_1$ [22, p. 26].

**Theorem 3.4** *There is a subspace $Y$ of $L_1$ such that $L_1/Y$ has the Daugavet property, but fails to contain a copy of $L_1$.*

*Proof.* The space $Y$ will be the closed linear span of a certain subsequence $(Y_n)$ of Talagrand's sequence $(X_n)$. As reported above, it is a deep result due to Talagrand that $L_1/Y$ does not contain a copy of $L_1$.

In order to obtain the Daugavet property for $L_1/Y$ by means of Lemma 3.1, we determine the $Y_n$ as follows. First, we let $Y_1 = X_1$. Using inductively Lemma 3.3 and condition (3.4) we select a subsequence $(Y_n)$ to fulfill the following conditions:

(a) For every $n \in \mathbb{N}$ there is a $\delta_n \in (0, 2^{-n-1})$ such that for every $A \in \Sigma$, $\mu(A) > 2^{-n}$, there is some $B \subset A$, $\mu(B) = \delta_n$, such that

$$\|\chi_B f\| \leq 2^{-n}\|f\| \qquad \forall f \in Y_1 \oplus \cdots \oplus Y_n; \tag{3.5}$$

in addition we pick $\delta_n < \delta_{n-1}$.

(b) For every $g \in S_{Y_{n+1}}$ we have

$$\mu(\{|g| \geq 10^{-n}\delta_n\}) \leq 10^{-n}\delta_n. \tag{3.6}$$

Put now $Y = \overline{\text{lin}} \bigcup_n Y_n$ and let us prove the validity of condition (I) of Lemma 3.1.

Denote by $M$ the smallest constant such that

$$M\Bigl\|\sum_{n=1}^{\infty} \lambda_n f_n\Bigr\| \geq \sum_{n=1}^{\infty} |\lambda_n| \tag{3.7}$$

for every choice of $f_n \in S_{Y_n}$ $(n \in \mathbb{N})$ and for all $(\lambda_n) \in \ell_1$; in fact, $M \leq 20$ [22, Prop. 4.2]. Fix $\varepsilon > 0$ and a set $A \in \Sigma$, $\mu(A) > \varepsilon$. Pick $N \in \mathbb{N}$ such that

$$2^{-N+1}(M+2) \leq \varepsilon, \tag{3.8}$$

and apply (a) to obtain some $B \in \Sigma$ satisfying (3.5) for $n = N$. Put $g_0 = \chi_B/\mu(B)$ and consider a function $h = \sum_{n=1}^{\infty} \lambda_n f_n$ with $\|h\| = 1$ where $f_n \in S_{Y_n}$ for all $n$ and $(\lambda_n) \in \ell_1$. We need to prove that

$$\|g_0 + h\| \geq 2 - \varepsilon. \tag{3.9}$$

Denote

$$h_0 = \sum_{n=1}^{N} \lambda_n f_n, \quad h_1 = h - h_0, \quad D = \{t \in B\colon |h_1(t)| \leq 2^{-N}\}.$$

By (3.7) we know that

$$\sum_{n=1}^{\infty} |\lambda_n| \le M,$$

so for every $t \in D$ at least one of the inequalities

$$|f_n(t)| \ge 10^{-n}, \quad n = N+1, N+2, \dots$$

has to be true, hence by (3.6)

$$\mu(D) \le \sum_{n>N} \mu(\{|f_n| \ge 10^{-n}\}) \le 10^{-N}\delta_N. \tag{3.10}$$

To prove (3.9) let us note that

$$\begin{aligned} 2 - \|g_0 + h\| &= \|g_0\| + \|h\| - \|g_0 + h\| \\ &= \int_B (|g_0| + |h| - |g_0 + h|)\, d\mu \\ &= \int_{B\setminus D} (|g_0| + |h| - |g_0 + h|)\, d\mu \\ &\qquad + \int_D (|g_0| + |h| - |g_0 + h|)\, d\mu \\ &\le \int_{B\setminus D} 2|h|\, d\mu + \int_D 2|g_0|\, d\mu. \end{aligned}$$

The last two integrals can be estimated from above as follows:

$$\begin{aligned} \int_{B\setminus D} |h|\, d\mu &\le \int_B |h_0|\, d\mu + \int_{B\setminus D} |h_1|\, d\mu \\ &\le 2^{-N} \sum_{n=1}^{N} |\lambda_n| + 2^{-N}\delta_N \\ &\qquad \text{(by (3.5) and by definition of } D) \\ &\le 2^{-N}(M+1). \end{aligned}$$

Further, by definition of $g_0$ and (3.10)

$$\int_D |g_0|\, d\mu = \frac{\mu(D)}{\mu(B)} \le 10^{-N}$$

so that

$$2 - \|g_0 + h\| \le 2^{-N}(M+2) \cdot 2 \le \varepsilon$$

by (3.8), and the proof of (3.9) and thus the proof of Theorem 3.4 is completed. □

The common feature of Proposition 3.2 and Theorem 3.4 is that we factor by a subspace with the Radon-Nikodým property. Recall from [26] or [25] that $L_1/H_1$ has the Daugavet property, too (since its dual does). So a natural question that has remained open is whether a quotient $L_1/X$ by a subspace with the Radon-Nikodým property always has the Daugavet property or can at least be so renormed.

## 4. The anti-Daugavet property

In this part we consider operators $T$ from a Banach space $X$ into itself. It is an easy exercise to prove that

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{4.1}$$

if $T$ is a bounded linear operator for which $\|T\|$ belongs to the spectrum of $T$. Therefore, we say that a Banach space $X$ has the *anti-Daugavet property* for a class $\mathcal{M}$ of operators if, for $T \in \mathcal{M}$, the equivalence

$$\|\mathrm{Id} + T\| = 1 + \|T\| \quad \Longleftrightarrow \quad \|T\| \in \sigma(T)$$

holds. Again, it is enough to consider operators of norm 1. If $\mathcal{M} = L(X)$, we simply speak of the anti-Daugavet property.

We shall use some results on finite representability, ultrapowers and superreflexivity which may be found in the monograph [4]. Geometric notions such as uniform convexity and smoothness are discussed there, too; see also [8] or [11].

It was proved in [2] that uniformly convex and uniformly smooth spaces share the anti-Daugavet property, and locally uniformly convex spaces have the anti-Daugavet property for compact operators. In [15] an equivalent geometric condition (see Definition 4.1(a)) for the anti-Daugavet property in finite-dimensional spaces was presented. We shall now extend these results and introduce some geometric properties of the unit sphere.

**Definition 4.1**

(a) We say that a Banach space $X$ is *alternatively convex or smooth* (acs) if for all $x, y \in S_X$ and $x^* \in S_{X^*}$ the implication

$$x^*(x) = 1,\ \|x + y\| = 2 \quad \Rightarrow \quad x^*(y) = 1 \tag{4.2}$$

holds.

(b) We say that a Banach space $X$ is *locally uniformly alternatively convex or smooth* (luacs) if for all $x_n, y \in S_X$ and $x^* \in S_{X^*}$ the implication

$$x^*(x_n) \to 1,\ \|x_n + y\| \to 2 \quad \Rightarrow \quad x^*(y) = 1 \tag{4.3}$$

holds.

(c) We say that a Banach space $X$ is *uniformly alternatively convex or smooth* (uacs) if for all $x_n, y_n \in S_X$ and $x_n^* \in S_{X^*}$ the implication

$$x_n^*(x_n) \to 1,\ \|x_n + y_n\| \to 2 \quad \Rightarrow \quad x_n^*(y_n) \to 1 \tag{4.4}$$

holds.

Geometrically, the (acs)-property means some smoothness of the norm at points lying on a line segment in the unit sphere. Precisely, $X$ is (acs) if and only if whenever $\mathrm{co}\{x, y\} \subset S_X$, then $x$ and $y$ are smooth points of the unit ball of $\mathrm{lin}\{x, y\}$.

We remark that uniformly convex spaces and uniformly smooth spaces are (uacs), and locally uniformly convex spaces are (luacs). In [15] it was proved that in finite-dimensional spaces (where by a compactness argument (acs), (luacs) and (uacs) are equivalent) (acs) is necessary and sufficient for the anti-Daugavet property. We intend to characterize the anti-Daugavet property for the class of compact operators on infinite-dimensional spaces by means of the (luacs)-property. To this end, we shall need a lemma.

**Lemma 4.2** *Suppose $X$ is* (acs) *and $T\colon X \to X$ is a weakly compact operator with $\|T\| = 1$ and $\|\mathrm{Id} + T\| = 2$. Suppose in addition that $\|x + Tx\| = 2$ for some $x \in S_X$. Then $1$ is an eigenvalue of $T$.*

*Proof.* Consider a functional $x^* \in S_{X^*}$ such that $x^*(x) = 1$. By the (acs)-property of $X$ one has $x^*(Tx) = 1$. Therefore $x_1^* := T^*(x^*)$ attains the value 1 at $x$ and hence belongs to $S_{X^*}$. Again, using (4.2) we obtain $x_1^*(Tx) = 1$. Applying the same argument inductively shows that $x^*(T^n x) = 1$ for all $n \in \mathbb{N}$. This implies that

$$K := \overline{\mathrm{co}}\{T^n y\colon n \in \mathbb{N}\} \subset \{v \in X\colon x^*(v) = 1\} \cap B_X \subset S_X,$$

in particular $0 \notin K$. Also, $K$ is a weakly compact convex set, since $\{T^n y\colon n \geq 1\} = T(\{T^n y\colon n \geq 0\})$, which is relatively weakly compact, and $T$ maps $K$ into $K$. By the Schauder (or Markov-Kakutani) fixed point theorem, $T$ has a fixed point in $K$, which is a non-zero eigenvector for the eigenvalue 1. $\square$

**Theorem 4.3** *For a Banach space $X$, the following conditions are equivalent:*

(i) *$X$ has the anti-Daugavet property for compact operators.*

(ii) *$X$ has the anti-Daugavet property for operators of rank* 1.

(iii) *$X$ is* (luacs).

*Proof.* (i) $\Rightarrow$ (ii) is evident. For the proof of (ii) $\Rightarrow$ (iii), assume that $X$ fails to be (luacs). Then there is a functional $x^* \in S_{X^*}$ and there are elements $x_n, y \in S_X$ such that $x_n, y \in S_X$ and $\|x + y_n\| \to 2$, $x^*(x_n) \to 1$, but $x^*(y) < 1$. Consider the operator $T\colon X \to X$ defined by $Tv = x^*(v)y$. Then $\|T\| = 1$ and $\|\mathrm{Id} + T\| = 2$, since

$$\begin{aligned} \|\mathrm{Id} + T\| &\geq \limsup \|x_n + Tx_n\| \\ &= \limsup \|x_n + x^*(x_n)y\| \\ &= \limsup \|x_n + y\| \;=\; 2. \end{aligned}$$

Thus $T$ satisfies (4.1), but $1 \notin \sigma(T)$ because of $x^*(y) < 1$; so $X$ fails the anti-Daugavet property for rank 1 operators.

(iii) $\Rightarrow$ (i): Let $T$ be a compact operator with $\|T\| = 1$ and $\|\mathrm{Id} + T\| = 2$. Then there is a sequence $(x_n) \subset S_X$ for which $\|Tx_n + x_n\| \to 2$. By compactness of $T$ we may assume that $Tx_n \to y \in S_X$. Now consider $x^* \in S_{X^*}$ such that $x^*(y) = 1$; then $x^*(Tx_n) \to 1$. Put $y^* = T^*x^*$; then we have $\|y^*\| \leq 1$, and from $y^*(x_n) = x^*(Tx_n) \to 1$ we deduce that actually $\|y^*\| = 1$. So we have $\|x_n + y\| \to 2$ and $y^*(x_n) \to 1$, and from (4.3) we get that $y^*(y) = 1$. But now

$$\|y + Ty\| \geq x^*(y + Ty) = x^*(y) + y^*(y) = 2,$$

so Lemma 4.2 implies that $1 \in \sigma(T)$; hence we obtain (i). $\square$

We now turn to the relation of the (uacs)-property and the anti-Daugavet property.

**Lemma 4.4** *If $X$ is* (uacs), *then $X$ is superreflexive.*

*Proof.* The (uacs)-property provides a uniform restriction on the structure of 2-dimensional subspaces of $X$: For all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\|x + y\| > 2 - \delta$, $x, y \in S_X$, and $x^* \in S_{(\mathrm{lin}\{x,y\})^*}$ with $x^*(x) > 1 - \delta$, then $x^*(y) > 1 - \varepsilon$. Therefore, not every 2-dimensional Banach space is finitely representable in $X$, and by [14] $X$ has to be superreflexive. (In fact, $X$ is uniformly non-square, which is enough to imply superreflexivity by a theorem due to James; see [4, p. 261].) $\square$

**Theorem 4.5** *If $X$ is* (uacs)*, then $X$ has the anti-Daugavet property.*

*Proof.* Let $T\colon X \to X$ be an operator of norm 1 such that $\|\mathrm{Id} + T\| = 2$. Consider an ultrapower $X^{\mathfrak{U}}$ of $X$; $X^{\mathfrak{U}}$ is the factor space $\ell_\infty(X)/c_0^{\mathfrak{U}}(X)$ with the norm $\|[(x_n)]\| = \lim_{\mathfrak{U}} \|x_n\|$, where $c_0^{\mathfrak{U}}(X)$ consists of those sequences in $X$ that tend to zero along the ultrafilter $\mathfrak{U}$. Define $T^{\mathfrak{U}}\colon X^{\mathfrak{U}} \to X^{\mathfrak{U}}$ by $T^{\mathfrak{U}}[(x_n)] = [(Tx_n)]$.

The main advantage of considering $T^{\mathfrak{U}}$ is that $\mathrm{Id}^{\mathfrak{U}} + T^{\mathfrak{U}}$ attains its norm. Indeed, if $x_n^0 \in S_X$ are chosen so that $\|Tx_n^0\| \to 1$, then $x^0 := [(x_n^0)] \in S_{X^{\mathfrak{U}}}$ and $\|T^{\mathfrak{U}}(x^0)\| = 1$. By Lemma 4.4, $X^{\mathfrak{U}}$ is reflexive and thus $T^{\mathfrak{U}}$ is weakly compact. It is evident from the definition that if $X$ is (uacs) and $Y$ is finitely representable in $X$, then $Y$ is (uacs), i.e., (uacs) is a superproperty. Thus we conclude that $X^{\mathfrak{U}}$ is (acs), and we may apply Lemma 4.2. So there is an eigenvector $x^1 = [(x_n^1)] \in S_{X^{\mathfrak{U}}}$ with $T^{\mathfrak{U}}x^1 = x^1$. This means that $\|Tx_n^1 - x_n^1\|$ tends to zero along $\mathfrak{U}$, and 1 is an approximate eigenvalue of $T$, in particular $1 \in \sigma(T)$. □

As we said, it is evident that (uacs) is a superproperty. Also, the super anti-Daugavet property coincides with (uacs), but we doubt whether the anti-Daugavet property is a superproperty itself.

## References

[1] Y. Abramovich. *New classes of spaces on which compact operators satisfy the Daugavet equation.* J. Operator Theory **25** (1991), 331–345.

[2] Y. Abramovich, C. D. Aliprantis, and O. Burkinshaw. *The Daugavet equation in uniformly convex Banach spaces.* J. Funct. Anal. **97** (1991), 215–230.

[3] V. F. Babenko and S. A. Pichugov. *On a property of compact operators in the space of integrable functions.* Ukrainian Math. J. **33** (1981), 374–376.

[4] B. Beauzamy. *Introduction to Banach Spaces and their Geometry.* North-Holland, Amsterdam-New York-Oxford, second edition, 1985.

[5] J. Bourgain. *Strongly exposed points in weakly compact convex sets in Banach spaces.* Proc. Amer. Math. Soc. **58** (1976), 197–200.

[6] G. Choquet. *Lectures on Analysis, Vol. II.* W. A. Benjamin, New York, 1969.

[7] I. K. Daugavet. *On a property of completely continuous operators in the space $C$.* Uspekhi Mat. Nauk **18.5** (1963), 157–158 (Russian).

[8] J. Diestel. *Geometry of Banach Spaces – Selected Topics.* Lecture Notes in Math. 485. Springer, Berlin-Heidelberg-New York, 1975.

[9] P. N. Dowling, W. B. Johnson, C. J. Lennard, and B. Turett. *The optimality of James's distortion theorem.* Proc. Amer. Math. Soc. **125** (1997), 167–174.

[10] C. Foiaş and I. Singer. *Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions.* Math. Z. **87** (1965), 434–450.

[11] P. Habala, P. Hájek, and V. Zizler. *Introduction to Banach Spaces.* Matfyz Press, Prague, 1996.

[12] P. Harmand, D. Werner, and W. Werner. *M-Ideals in Banach Spaces and Banach Algebras.* Lecture Notes in Math. 1547. Springer, Berlin-Heidelberg-New York, 1993.

[13] J. R. Holub. *Daugavet's equation and operators on $L_1(\mu)$.* Proc. Amer. Math. Soc. **100** (1987), 295–300.

[14] V. M. Kadets. *Two-dimensional universal Banach spaces.* C. R. Acad. Bulgare Sci. **35** (1982), 1331–1332 (Russian).

[15] V. M. Kadets. *Some remarks concerning the Daugavet equation.* Quaestiones Math. **19** (1996), 225–235.

[16] V. M. Kadets and M. M. Popov. *The Daugavet property for narrow operators in rich subspaces of $C[0,1]$ and $L_1[0,1]$.* St. Petersburg Math. J. **8** (1996), 43–62.

[17] V. M. Kadets and R. V. Shvidkoy. *The Daugavet property for pairs of Banach spaces.* Math. Analysis, Algebra and Geometry (to appear).

[18] J. L. Kelley. *General Topology.* Van Nostrand, 1955.

[19] G. Ya. Lozanovskii. *On almost integral operators in KB-spaces.* Vestnik Leningrad Univ. Mat. Mekh. Astr. **21.7** (1966), 35–44 (Russian).

[20] I. Nazarenko. *Paper to appear.*

[21] A. M. Plichko and M. M. Popov. *Symmetric function spaces on atomless probability spaces.* Dissertationes Mathematicae **306** (1990).

[22] M. Talagrand. *The three-space problem for $L^1$.* J. Amer. Math. Soc. **3** (1990), 9–29.

[23] L. Weis. *Approximation by weakly compact operators on $L_1$.* Math. Nachr. **119** (1984), 321–326.

[24] L. Weis and D. Werner. *The Daugavet equation for operators not fixing a copy of $C[0,1]$.* J. Operator Theory (to appear).

[25] D. WERNER. *The Daugavet equation for operators on function spaces.* J. Funct. Anal. **143** (1997), 117–128.

[26] P. WOJTASZCZYK. *Some remarks on the Daugavet equation.* Proc. Amer. Math. Soc. **115** (1992), 1047–1052.

Faculty of Mechanics and Mathematics, Kharkov State University, pl. Svobody, 4, 310077 Kharkov, Ukraine

I. Mathematisches Institut, Freie Universität Berlin, Arnimallee 2–6,
D-14195 Berlin, Germany; e-mail: werner@math.fu-berlin.de