Determination of Shape and Mass Density from Gravity Anomaly Measurements : Some Uniqueness Results*

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Abstract

The ill-posed problem of determining the shape of a body immersed in a "flat earth" and the mass density within this body from gravity anomaly measurements on the surface of the earth is investigated. Under some a priori assumptions on the structure of the body's boundary a general theorem on uniqueness of the solution is established and some further theorems are derived as immediate consequences. As often is the case with inverse and ill-posed problems, this question of uniqueness is particulary important whereas because of unavoidable measurement errors, the question of existence for given date, is less so.

1991 Mathematics Subject Classification: 45G10, 86A20, 86A22, 65R30

Let V be a body of mass density ρ_1, lying below the surface of the earth represented by a half-space, and let the density of the medium surrounding V be ρ_2. Call ρ the relative density, i.e. $\rho = \rho_1 - \rho_2$. Several problems have been posed concerning V and ρ, e.g.,

Problem 1 : *Knowing ρ, find V from surface gravity anomaly measurements*

Problem 2 : *Knowing V, find ρ from surface gravity anomaly measurements*

Problem 3 : *Find ρ and V from surface gravity anomaly measurements.*

The foregoing problems are fundamental problems in Applied Geophysics and it is known that there is no uniqueness for the problems posed in such generality. Let us look at Problem 1 and clarify the meaning of the phrase "Knowing ρ". This means that a function $\overline{\rho}$ is given on all of R^3 while the density function ρ is defined only on the unknown

*The work of the first author was completed with a financial support from the National Program of Basic Reseach in the Natural Sciences of VietNam and was partially supported by the Research Group "Convolutions", Free University of Berlin, Germany. The work of the second author was completed with a financial support from the Research Group "Convolutions", Free University of Berlin, Germany. The work of the third author was carried out with a grant from the Association d'Aubonne, Culture et Education France VietNam.

domain V and $\rho = \overline{\rho}|_V$. For the case that ρ is a given constant, the problem was considered in [1] where it is proved that it admits at most one solution if V is bounded by a known horizontal base $\Omega \subset R^2$ and a surface parametrized (cf. Fig 1) by the equation

$$z = \sigma(x, y) \quad (x, y) \in \Omega$$

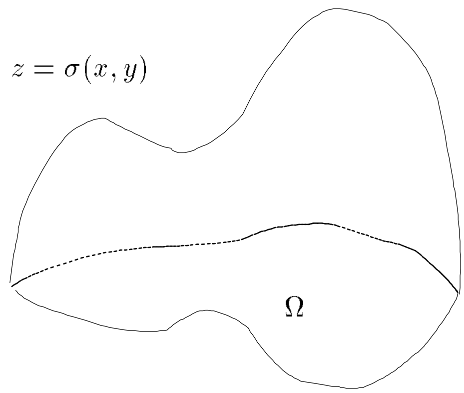

Fig. 1

The problem is to find $\sigma(x, y)$ and is reduced to a nonlinear integral equation of the first kind in σ, which is then proved to admit at most one continuous σ. We also refer to [7] for a study of a similar problem. Note that the formulation of the problem as an integral equation was given in [9] (cf. also [6]). Various aspects of the problem, including uniqueness, were considered in [8]. A bibliography of the earlier works on the subject is given in [9].

Problem 2 is considered in [3] where the authors reduce it to a moment problem which is then approximated by finite moments. Note that there is no uniqueness for this problem, in general.

The purpose of this paper is to establish a uniqueness theorem for problem 3. As pointed out in our previous work [1], the problem of uniqueness is often more important than the problem of existence. In fact, the problem of determining ρ and/or V can be formulated as a functional equation (sometimes an integral equation), the right-hand side of which is a highly regular function (at least C^∞). In practice, the right hand side is a result of experimental measurements and hence is given by a finite set of points of the graph which is patched up into an L^2–function or a continuous function. Thus we do not have existence for this approximating equation and hence we have to resort to regularizing schemes (cf. [2] for a regularization by finite dimensional approximations). Problem 1 and 2 are clearly special cases of Problem 3, however, for emphasis, we label them as distinct problems. Our version of Problem 3 reads as follows :

Find a real-valued function ρ on R^3 vanishing in the complement of a compact set $K = \overline{V}$ with the following properties :

A_1 : *V is open, simply connected in* R^3.

A_2 : $\rho > 0$ *is Hölder continuous on* V.

A_3 : $\rho = \rho(x, y)$, (x, y, z) *in* V.

A_4 : ∂V *is piecewise* C^1.

The assumption $\rho > 0$ in A_2 means that the material of V is strictly heavier than that of the surrounding medium. Assumption A_4 on the piecewise smoothness of the boundary ∂V is needed for technical reasons since we shall have to use the Divergence Theorem. Note that by its definition $\overline{V} = supp\ \rho$.

For definiteness, let the problem be referred to a system of cartesian coordinates x, y, z with V lying in the half-space $z < H$ and at a positive distance from the plane $z = H > 0$ as shown in Fig. 2 (We take the origin of the axes not on the surface of the earth but at some point below).

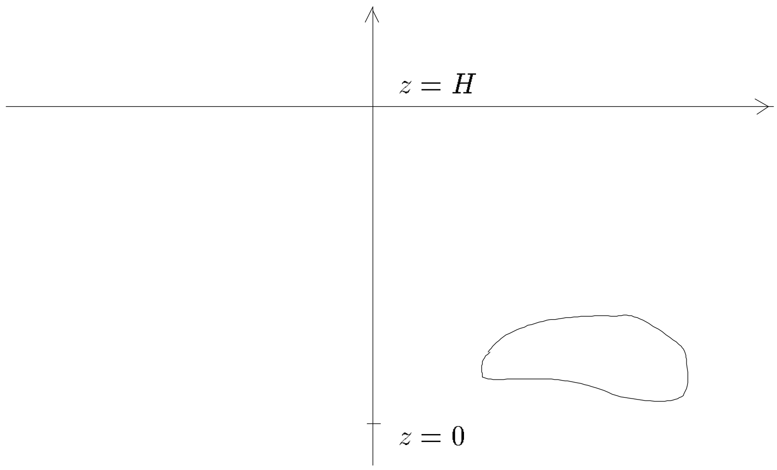

Fig. 2

The gravity potential created by V is

$$U(x, y, z) = \frac{1}{4\pi} \int_V \frac{\rho dv}{((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{1/2}} \tag{1}$$

where $\rho = \rho(\xi, \eta)$, $dv = d\xi d\eta d\zeta$.

The corresponding gravity anomaly ([4], [6], [9]) is given by

$$-\frac{\partial U}{\partial z}(x, y, z) = \frac{1}{4\pi} \int_V \frac{(z-\zeta)\rho dv}{((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{3/2}}.$$

Denote the gravity anomaly on $z = H$ by δg [4].

Then the problem reduces to finding ρ and $V = supp\ \rho$ from the relation

$$\begin{aligned} \int_V \frac{(H-\zeta)\rho\ dv}{((x-\xi)^2 + (y-\eta)^2 + (H-\zeta)^2)^{3/2}} &= 4\pi\delta\ g(x, y) \\ &\equiv f(x, y). \end{aligned} \tag{2}$$

We now state our

Theorem 1. *(General Uniqueness Theorem)*

Let $\mathcal{F}$ be a family of real-valued functions on R^3 satisfying $A_1 - A_4$. Suppose in addition that $\mathcal{F}$ satisfies the following assumptions :

A_5. The boundary of the support of each member of $\mathcal{F}$ contains a subsurface S_0, the same for all members of $\mathcal{F}$, such that

$$\int_{S_0} \frac{n_3 \rho d\sigma}{((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{1/2}} = \int_{S_0} \frac{n_3 \rho' d\sigma}{((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2)^{1/2}} \tag{3}$$

for any two ρ, ρ' in $\mathcal{F}$, where $n_3 = n_3(\xi, \eta, \zeta)$ is the $z-$component of the exterior unit normal to S_0 at (ξ, η, ζ)

A_6. (Boundary property) With S_0 as in A_5, for any two ρ, ρ' in $\mathcal{F}$, letting $\partial V, \partial V'$ be respectively the boundaries of $V = supp\ \rho, V' = supp\ \rho'$, the union $S \cup S'$ of the subsurfaces $S = \partial V \backslash S_0, S' = \partial V' \backslash S_0$ is the boundary of the unbounded connected component of $R^3 \backslash (S \cup S')$

Then Eq. 2 has at most one solution ρ in $\mathcal{F}$.

Before giving the proof of this theorem, a few remarks are in order.

Remark 1. *For Problem 1, from the point of view of our analysis, the knowledge of ρ is useful only if S_0 in A_5 exists, in which case relation (3) in Theorem 1 holds since both sides of this relation are equal.*

Remark 2. *Concerning A_5, it is only assumed that a certain S_0 exists that is a subsurface of supp ρ for each ρ in $\mathcal{F}$ such that A_5 holds. It is not necessary to know S_0. For the case of Problem 1, as it is formulated in* [1] *(cf. the preceding introductory paragraphs) S_0 is known, in fact, it is the known base Ω of body V.*

The Boundary Property A_6 in Theorem 1 is satisfied in many situations of practical interest. The three results to follow are important corollaries to Theorem 1 and, for emphasis, are stated as distinct theorems.

Theorem 2. *Let $\mathcal{F}$ be a family of real-valued functions on R^3 satisfying $A_1 - A_4$ and the boundary property A_6. Suppose in addition that for each ρ in $\mathcal{F}$, the boundary of supp ρ contains a vertical subsurface S_0 (i.e. at each point of S_0 the unit normal to S_0 at the point is orthogonal to the $z-$axis), the same for all ρ in $\mathcal{F}$. Then Eq. (2) has at most one solution.*

Proof. Since at each point (x, y, z) of S_0, we have $n_3(x, y, z) = 0$, property A_5 is trivially satisfied (each member of (3) is zero for any ρ, ρ' in $\mathcal{F}$). Thus Theorem 2 follows directly from Theorem 1. ■

Theorem 3. *Let $\mathcal{F}$ be a family of real-valued functions on R^3 satisfying $A_1 - A_5$. Suppose in addition that there exists a point O such that,*

(A_7). For each ρ in $\mathcal{F}$, each ray from O cuts $\partial V \setminus S_0, (V \equiv supp\ \rho)$ at one point at most.

Then Eq. (2) admits at most one solution.

Proof. In order to apply Theorem 1, it is sufficient to show that $S \cup S'$ is the boundary of the unbounded component of $R^3 \setminus S \cup S'$ (i.e., the Boundary Property holds). Let a be a point of $S \cup S'$, a in S, say. We shall show that if ω is an open ball of R^3 centered at a, then ω contains a point in the unbounded component of $R^3 \setminus S \cup S'$. Indeed, let L be a line through a. If L does not cut S' then clearly each point of L other than a can be connected by a segment to any point outside B where B is a ball containing $S \cup S'$ in its interior. If L cuts S' at the (unique) point b, then taking ω_0 be a ball centered at a of sufficiently small radius, each point of ω_0 is in the unbounded connected component of $R^3 \setminus (S \cup S')$. Thus the Boundary Property holds and the theorem is proved.

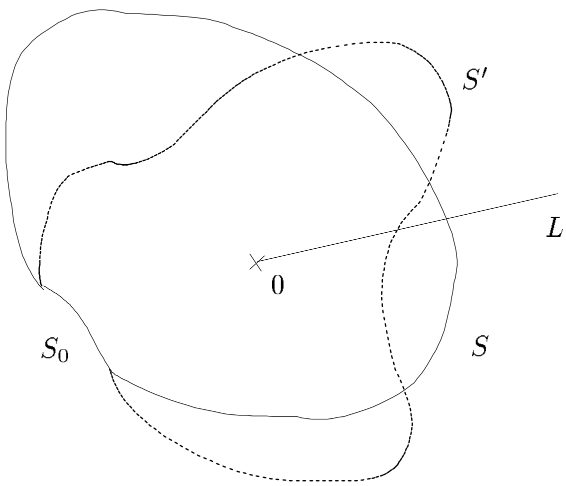

Fig. 3

■

Theorem 4. *Let $\mathcal{F}$ be a family of functions on R^3 satisfying $A_1 - A_5$. Suppose in addition that there is a direction $\vec{e}$ such that for each ρ in $\mathcal{F}$, a ray in the direction $\vec{e}$ cuts $S = \partial(supp\ \rho) \setminus S_0$ (S_0 as in A_5) at one point at most. Then Eq. (2) has at most one solution in $\mathcal{F}$. In particular if $\mathcal{F}$ satisfies $A_1 - A_5$ and if for any ρ in $\mathcal{F}$, the surface $S = \partial(supp\ \rho) \setminus S_0$ can be represented by*

$$z = \sigma(x, y)$$

Then Eq. (2) has at most one solution σ in $\mathcal{F}$.

In order to apply Theorem 1, it is sufficient to show that $\mathcal{F}$ satisfies the boundary property A_6, i.e., for any two ρ, ρ' in $\mathcal{F}$, the set $S \cup S'$ with $S = \partial(supp\ \rho) \setminus S_0$, $S' = \partial(supp\ \rho') \setminus S_0$ is the boundary of the unbounhed connected component of $R^3 \setminus (S \cup S')$. The proof of this is almost identical with the proof of the boundary property in the case A_7 of theorem 3.

Remark 3. *Theorem 4 is an extension of Theorem 1 of* [1] *where* $S = \partial(supp\ \rho) \setminus S_0$ *can be represented by*

$$z = \sigma(x, y)$$

Fig. 4 shows a surface S and a direction $\vec{e}$ along which a line in the direction $\vec{e}$ cuts S at one point at most. The surface S is not seen to be representable by any function of the form

$$z = \sigma(x, y)$$

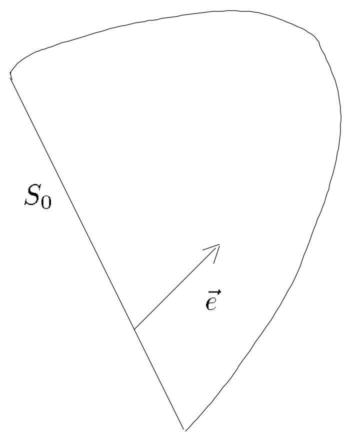

Fig. 4

Remark 4. *Theorem 4 extends but does not contain Theorem 1 of* [1] *since there are no regularity requirements other than continuity in the latter.*

We now turn to the

Proof of Theorem 1.

Consider Eq. (2), which we rewrite as

$$\int_V \frac{\rho(\xi, \eta)(H - \zeta)d\zeta d\eta d\xi}{((x - \xi)^2 + (y - \eta)^2 + (H - \zeta)^2)^{3/2}} = h(x, y) \tag{4}$$

where h differs from f by a multiplicative constant. Letting ρ, ρ' be two solutions of (2), we want to prove $\rho = \rho'$. Indeed, we have for all $(x, y) \in R^2$,

$$\int_V \frac{\rho(\xi, \eta)(H - \zeta)d\zeta d\eta d\xi}{((x - \xi)^2 + (y - \eta)^2 + (H - \zeta)^2)^{3/2}} - \int_{V'} \frac{\rho'(\xi, \eta)(H - \zeta)d\zeta d\eta d\xi}{((x - \xi)^2 + (y - \eta)^2 + (H - \zeta)^2)^{3/2}} = 0$$

where V, V' are the supports of ρ, ρ' respectively.

Consider the function

$$\begin{aligned} F(x,y,z) &= \int_V \frac{\rho(\xi,\eta)(z-\zeta)d\zeta d\eta d\xi}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{3/2}} - \\ &- \int_{V'} \frac{\rho'(\xi,\eta)(z-\zeta)d\zeta d\eta d\xi}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{3/2}}. \end{aligned} \tag{5}$$

We shall show that

$$F(x,y,z) = 0 \quad on \quad R^3. \tag{6}$$

Noting that

$$\frac{\rho(\xi,\eta)(z-\zeta)}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{3/2}} = \frac{\partial}{\partial\zeta}\left(\frac{\rho(\xi,\eta)}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}}\right) \tag{7}$$

we have by the Divergence Theorem

$$\begin{aligned} F(x,y,z) &= \int_{\partial V} \frac{n_3\rho(\xi,\eta)d\sigma}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} - \\ &- \int_{\partial V'} \frac{n_3\rho'(\xi,\eta)d\sigma}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} \end{aligned} \tag{8}$$

where n_3 is the $z-$component of the outer unit normal to ∂V (or $\partial V'$). Letting $S = \partial V \setminus S_0$, $S' = \partial V' \setminus S_0$ with S_0 as in A_5, we have by (8) and A_5,

$$\begin{aligned} F(x,y,z) &= \int_S \frac{n_3\rho(\xi,\eta)d\sigma}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} - \\ &- \int_{S'} \frac{n_3\rho'(\xi,\eta)d\sigma}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}}. \end{aligned} \tag{9}$$

Now, $F(x,y,z)$ is harmonic in $R^3\setminus(S\cup S')$, vanishes at infinity and at $z = H$. Hence $F(x,y,z)$ vanishes in the unbounded component of $R^3 \setminus (S \cup S')$ and by continuity vanishes on $S \cup S'$. Note that the boundary of each connected component of $R^3 \setminus (S \cup S')$ is contained in $S \cup S'$ and that $F(x,y,z) = 0$ on $S \cup S'$. Hence by the maximum principle for harmonic functions, F vanishes on each connected component of $R^3 \setminus (S \cup S')$ and thus $F \equiv 0$ on R^3 and, by (5)

$$\int_V \frac{\rho(\xi,\eta)(z-\zeta)d\zeta d\eta d\xi}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{3/2}} - \int_{V'} \frac{\rho'(\xi,\eta)(z-\zeta)d\zeta d\eta d\xi}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{3/2}} = 0. \tag{10}$$

It follows that

$$\int_V \frac{\rho(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} - \int_{V'} \frac{\rho'(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} = v(x,y) \tag{11}$$

for some $v = v(x,y)$, and since the LHS $\to 0$ for $z \to \infty$, we have

$$v(x,y) = 0$$

that is

$$\int_V \frac{\rho(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} - \int_{V'} \frac{\rho'(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} = 0$$

which we rewrite as

$$\int_{R^3} \frac{(\rho-\rho')(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} = 0. \tag{12}$$

Applying the operator $\triangle$ to the LHS of (12), noting that $\rho - \rho'$ is Hölder continuous, one can show [5] that

$$\triangle \left(\int_{R^3} \frac{(\rho-\rho')(\xi,\eta)d\xi d\eta d\zeta}{((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)^{1/2}} \right) = 4\pi(\rho-\rho') \tag{13}$$

which by (12) gives

$$\rho = \rho'.$$

This completes the proof of Theorem 1. ■

Before closing, we remark that assumption A_5 in Theorem 1 is crucial for our analysis. Indeed with a subsurface S_0 as in A_5, the potential is a harmonic function on the complement of a set of measure zero and the analysis becomes considerably simpler. To conclude, we point to a rather unexpected result that is a direct consequence of Theorem 1, emphasizing the role of A_5 essential for uniqueness. The result reads as follows

Let V be a given simply connected bounded domain in R^3 with a vertical wall S_0 and with $\partial V \setminus S_0$ being a C^1-smooth but otherwise arbitrary (cf. Fig. 5). Then the problem of determining $\rho > 0, \rho = \rho(x,y)$, Holder continuous on V from boundary gravity measurements (i.e. Problem 2), admits at most one solution.

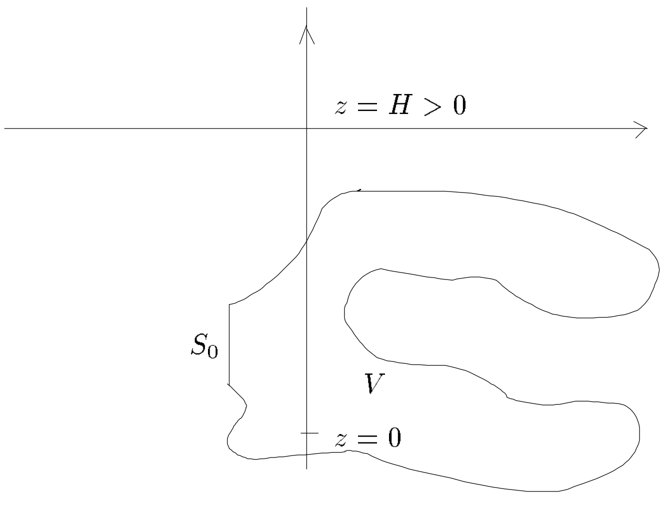

Fig. 5

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