A Multidimensional Hausdorff Moment Problem: Regularization by Finite Moments and Error Estimates [1]

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We propose to present a regularization of a Hausdorff moment problem, with error estimates given. The literature on the moment problem is impressive, we list only a few of them in the bibliography (cf [2,3,4,5,6,7,8,10]). Of particular relevance to the present work are the papers by Ang-Vy-Gorenflo [2] and by Talenti [8] (see also Inglese [6]). In [8], the author considered the one dimensional case and proved the stability of the finite moment approximation with stability estimates given. But, the relation of the finite moment approximation to the original moment problem was not considered. In the present paper, we shall consider the multidimensional case, in fact, the three dimensional case. Specifically, we shall consider the problem (MP) of finding a function u in $L^2(I)$ satisfying the sequence of equations

$$\int_I u(x,y,z)x^i y^j z^k dxdydz = \mu_{ijk}, \quad i,j,k = 0,1,2.. \tag{1}$$

where $I = [0,1]^3 \subset R^3$ and $\mu = (\mu_{ijk})$ is a given bounded triple sequence of real numbers. It can be proved that (MP) is an ill-posed problem. Note also that (MP) does not always have a solution for an arbitrary $\mu \in l_\infty$ in the right hand side of (1). Hence it is realistic to deal also with cases where (MP) does not have a solution. We refer to the paper [2] of which the present work can be seen as a continuation. However while [2] deals with a general moment problem, the Hausdorff moment has it own peculiarities that we have taken into account, in departure from the general case, especially in the crucial choice of the function $f(t)$, $[f^{-1}(\epsilon^{1/2})]$ being the dimension of our finite approximation.

In our construction of finite dimensional approximations, we obtain an orthonormalization of the basis functions $\{x^i y^j z^k\}$, $i,j,k = 0,1..$, not through the Gram-Schmidt process but by using products of shifted Legendre polynomials:

$$L_{ijk}(x,y,z) = L_i(x)L_j(y)L_k(z) \quad i,j,k = 0,1,.. \tag{2}$$

where

$$L_m(x) = (2m+1)^{1/2}\frac{1}{m!}\frac{d^m}{dx^m}(x^m(1-x)^m), \quad m = i,j,k \tag{3}$$

Note that the $L_m(x)$'s, except for the alternating signs, are obtained from $1, x, x^2, ..$ by the Gram-Schmidt orthonormalization and that the triple sequence $L_iL_jL_k$, $i,j,k = 0,1,..$ forms a complete orthonormal set in $L^2(I)$.

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Before constructing minimal norm solutions of (1), we first derive some useful formulae. From (3), we get

$$L_m(x) = \sum_{j-0}^{m} C_{mj} x^j, \qquad m = 0, 1, .. \tag{4}$$

where

$$C_{mj} = (2m+1)^{1/2}(-1)^j C_m^j C_{m+j}^j \tag{5}$$

with

$$C_m^j = \frac{m!}{j!(m-j)!}$$

¿From (4), we have

$$L_{ijk} = \sum_{p=0}^{i} \sum_{q=0}^{j} \sum_{r=0}^{k} C_{ip} C_{jq} C_{kr} x^p y^q z^r, \quad i, j, k = 0, 1, .. \tag{6}$$

If $\mu = (\mu_{ijk})$ is a real sequence, we define the sequence

$$\lambda = \lambda(\mu) = (\lambda_{ijk})$$

as follows

$$\lambda_{ijk} = \lambda_{ijk}(\mu) = \sum_{p=0}^{i} \sum_{q=0}^{j} \sum_{r=0}^{k} C_{ip} C_{jq} C_{kr} \mu_{pqr}$$

Now put:

$$p^n = p^n(\mu) = \sum_{i,j,k=0}^{n} \lambda_{ijk}(\mu) L_{ijk} \tag{7}$$

Then p^n is a minimal norm solution of (1)

The main resuls of this paper are Theorem 1 and Theorem 2 below.

Theorem 1. *Let $\mu = (\mu_{ijk})$,i,j,k=0,1,2,.. be a given sequence of real numbers. Then a necessary and sufficient condition for (1) to have a solution is that*

$$\sum_{i,j,k=0}^{\infty} \left(\sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} \mu_{pqr} \right)^2 < \infty \tag{8}$$

where C_{ij} is defined by (4) if $j \leq i$ and $C_{ij} = 0$ if $j > i$.

If u is the unique solution of (1) then

$$p^n(\mu) \to u \ \text{ in } L^2(I) \text{ as } n \to \infty \tag{9}$$

Moreover, if the solution u is in $H^1(I)$ then

$$\|p^n(\mu) - u\| \leq \frac{1}{2(n+1)} \|u\|_{H^1(I)} \quad n = 1, 2, .. \tag{10}$$

where $\| \,.\, \|$ *is the* $L^2(I)-norm$

Remark 1. It can be shown that if $u \in H^m(I)$ then we would have an estimate of the form

$$\|p^n(\mu) - u\| \leq C_m \|u\|_{H^m(I)} (n+1)^{-m}$$

It is of importance to give estimates for C_m. For m=1, we have shown that we can take

$$C_1 = \frac{1}{2}$$

Similarly, for m=2, we can show that

$$C_2 = \frac{1}{2\sqrt{2}}$$

For the case m=1, we use Lemma 1 instead of the inequality derived by Talenti from the equation for the Legendre equation and valid for $H^1(0,1)$. Our estimate of C_2 is based on an appropriate modified versions of our Lemma 1. We do not go into details.

In Theorem 1 above, it is assumed that a solution exists. In the case a solution may not exist, Theorem 2 below will be useful.

Theorem 2. *Let* $u_0 \in L^2(I)$ *be the unique solution of (1) corresponding to* $\mu^0 = (\mu^0_{ijk})$*, i,j,k=0,1,2,.. in the right hand side of (1). Let*

$$f(t) = \left(\frac{3}{2}\right)^3 \left(\frac{81}{80}\right)^{3/2} (2t+1)^{3/2} 3^{6t} \tag{11}$$

and for $0 < \epsilon < 1$*, put*

$$n(\epsilon) = \left[f^{-1}(\epsilon^{-1/2}) \right] \tag{12}$$

where $[x]$ *is the largest integer* $\leq x$*.*

Then there exists a function $\eta(\epsilon)$*,* $0 < \epsilon < 1$*, such that* $\eta(\epsilon) \to 0$ *as* $\epsilon \to 0$ *and that for all sequences* μ *satisfying*

$$\|\mu - \mu^0\|_\infty \equiv \sup_{i,j,k \in N} |\mu_{ijk} - \mu^0_{ijk}| < \epsilon$$

we have

$$\|p^{n(\epsilon)}(\mu) - u_0\| \leq \eta(\epsilon)$$

Moreover, if $u_0 \in H^1(I)$*, then*

$$\|p^{n(\epsilon)}(\mu) - u_0\| \leq \epsilon^{1/2} + \frac{\|u_0\|_{H^1(I)}}{C(\epsilon)}$$

where

$$C(\epsilon) = 2(2\ln 3 + \frac{1}{2}\ln 2)^{-1} \ln \left(\frac{8\sqrt{5}}{27\sqrt{2}\sqrt[6]{\epsilon}} \right)$$

Remark 2. If $u_0 \in H^2(I)$, then using Remark 1, we can show that

$$\|p^{n(\epsilon)}(\mu) - u_0\| \le \epsilon^{1/2} + \frac{\|u_0\|_{H^2(I)}}{C^1(\epsilon)}$$

where

$$C^1(\epsilon) = \frac{1}{\sqrt{2}}(C(\epsilon))^2$$

Proof of Theorem 1

We easily get the first result of Theorem 1 using the orthonormality property of the L_{ijk}'s and considering the following series representation:

$$u = \sum_{i,j,k=0}^{\infty} \lambda_{ijk}(\mu) L_{ijk} \tag{13}$$

¿From (13) we also get (9)

For the proof of (10), we rely on the following lemma:

Lemma 1. *For every $v \in H^1(0,1)$, we have:*

$$6\alpha_1^2 + \sum_{k=2}^{\infty}(2k+1)^2\alpha_k^2 \le \int_0^1 |v'|^2 dx \tag{14}$$

where:

$$\alpha_k = \int_0^1 v(x) L_k(x) dx$$

The proof of this lemma is deferred until a later section. Continuing with our proof of Theorem 1, we note that (14) gives

$$\sum_{i,j,k=1}^{\infty} 4i^2\lambda_{ijk}^2 \le \int_I \left|\frac{\partial u}{\partial x}(x,y,z)\right|^2 dxdydz \tag{15}$$

$$\sum_{i,j,k=1}^{\infty} 4j^2\lambda_{ijk}^2 \le \int_I \left|\frac{\partial u}{\partial y}(x,y,z)\right|^2 dxdydz \tag{16}$$

$$\sum_{i,j,k=1}^{\infty} 4k^2\lambda_{ijk}^2 \le \int_I \left|\frac{\partial u}{\partial z}(x,y,z)\right|^2 dxdydz \tag{17}$$

Adding (15),(16),(17) together we get:

$$\sum_{i,j,k=1}^{\infty} 4\{i^2 + j^2 + k^2\}\lambda_{ijk}^2 \le \|u\|_{H^1(I)}^2 \tag{18}$$

Now, we have:

$$p^n(\mu) - u = - \sum_{max(i,j,k)>n} \lambda_{ijk} L_{ijk}$$

Hence:

$$\|p^n(\mu) - u\|^2 = \sum_{max(i,j,k)>n} \lambda_{ijk}^2 \tag{19}$$

Note that:

$$4\{i^2 + j^2 + k^2\} \geq 4(n+1)^2 \quad as\ max(i,j,k) > n \tag{20}$$

Combining (18)-(20), we get:

$$\begin{aligned} (n+1)^2 \|p^n(\mu) - u\|^2 &\leq \sum_{max(i,j,k)>n} (n+1)^2 \lambda_{ijk}^2(\mu) \\ &\leq \frac{1}{4} \|u\|_{H^1(I)}^2 \end{aligned}$$

which gives the desired inequality.

To complete the proof of the theorem 1, we now turn to the

Proof of Lemma 1

We first consider the case v is a polynomial. We can write:

$$v = \sum_{j=0}^{n} \lambda_j L_j, \quad \lambda_j = \int_0^1 L_j(x) v(x) dx \tag{21}$$

Now (21) implies

$$v' = \sum_{j=0}^{n} \lambda_j L_j' \tag{22}$$

We have

$$\int_0^1 L_j'(x) L_k(x) dx = 0 \qquad for\ 0 \leq j \leq k \tag{23}$$

Hence:

$$\int_0^1 v'(x) L_k(x) dx = \sum_{j=k+1}^{n} \lambda_j \int_0^1 L_j'(x) L_k(x) dx \tag{24}$$

for k=0,1,2,..,n-1.

We have for $k < j$:

$$\begin{aligned} \int_0^1 L_j'(x) L_k(x) dx &= L_j(x) L_k(x)|_0^1 - \int_0^1 L_j(x) L_k'(x) dx \\ &= L_j(x) L_k(x)|_0^1 \end{aligned} \tag{25}$$

Here we have used (23).

But:

$$L_j(x) = (2j+1)^{1/2} \frac{1}{j!} \frac{d^j}{dx^j} [x(1-x)]^j$$

Hence:

$$L_j(0) = (2j+1)^{1/2}, \quad L_j(1) = (-1)^j (2j+1)^{1/2} \tag{26}$$

In view of (24)-(26), equality (24) gives:

$$\int_0^1 v'(x) L_k(x) dx = \sum_{j=k+1}^{n} \lambda_j (2k+1)^{1/2} (2j+1)^{1/2} ((-1)^{j+k} - 1) \tag{27}$$

By (27), the orthonormality of the family (L_j) gives:

$$\|v'\|^2 = \sum_{k=0}^{n-1} (2k+1) S_k^2 \tag{28}$$

with

$$S_k = \sum_{j=k+1}^{n} \lambda_j (2j+1)^{1/2} (1 - (-1)^{j+k}) \tag{29}$$

By (28) we have:

$$\begin{aligned} 2\|v'\|^2 &= \sum_{k=0}^{n-3} (2k+1)(S_k^2 + S_{k+2}^2) + S_0^2 + 3S_1^2 \\ &+ 4 \sum_{k=2}^{n-1} S_k^2 + (2n-3) S_{n-2}^2 + (2n-1) S_{n-1}^2 \\ &= \sum_{k=2}^{n-3} (2k+3)(S_k^2 + S_{k+2}^2) + 2(S_0^2 + S_2^2) + \\ &+ 5(S_1^2 + S_3^2) + S_1^2 + S_2^2 + \\ &+ (2n-1) S_{n-2}^2 + (2n+1) S_{n-1}^2 \end{aligned} \tag{30}$$

¿From (29) one has:

$$S_k - S_{k+2} = 2\lambda_{k+1} (2k+3)^{1/2} \quad for\ k = 0, 1, 2, .., n-2 \tag{31}$$

Hence, it follows from (31) that:

$$\begin{aligned} 2(S_k^2 + S_{k+2}^2) &\geq |S_k - S_{k+2}|^2 \\ &\geq 4(2k+3)\lambda_{k+1}^2 \end{aligned} \tag{32}$$

On the other hand, we have in view of (29)

$$S_{n-2} = 2\lambda_{n-1} (2n-1)^{1/2} \tag{33}$$

$$S_{n-1} = 2\lambda_n (2n+1)^{1/2} \tag{34}$$

Substituting (32)-(34) into (30) one has:

$$2\|v'\|^2 \geq \sum_{k=2}^{n-1} 2(2k+3)^2 \lambda_{k+1}^2 + 12\lambda_1^2 + 50\lambda_2^2$$

This gives:

$$\|v'\|^2 \geq 6\lambda_1^2 + \sum_{k=2}^{n} (2k+1)^2 \lambda_k^2 \tag{35}$$

Now, we want to prove that (35) holds for every $v \in H^1(0,1)$. Let (P_m) be a sequence of polynomials such that:

$$P_m \to v \quad in\ H^1(0,1) \tag{36}$$

By (35) one has for every p=2,3,..

$$\|P_m'\|^2 \geq 6\lambda_{1m}^2 + \sum_{k=2}^{p} (2k+1)^2 \lambda_{km}^2 \tag{37}$$

where:

$$\lambda_{km} = \int_0^1 P_m(x) L_k(x) dx, \qquad k = 0, 1, 2, ..$$

Letting $m \to \infty$ in (37), we get in view of (36):

$$\|v'\|^2 \geq 6\alpha_1^2 + \sum_{k=2}^{p} (2k+1)^2 \alpha_k^2 \tag{38}$$

where:

$$\alpha_k = \int_0^1 v(x) L_k(x) dx$$

Since p is any positive integer, one has:

$$\|v'\|^2 \geq 6\alpha_1^2 + \sum_{k=2}^{\infty} (2k+1)^2 \alpha_k^2$$

This proves Lemma 1 and hence completes the proof of Theorem 1.

We finally give a

Proof of Theorem 2

We have:

$$\|p^n(\mu) - u_0\| \leq \|p^n(\mu) - p^n(\mu^0)\| + \|p^n(\mu^0) - u_0\|, \quad n = 1, 2, .. \tag{39}$$

Using (7) we get:

$$p^n(\mu) - p^n(\mu^0) = \sum_{i,j,k=0}^{n} \sum_{p,q,r=0}^{\infty} C_{ip} C_{jq} C_{kr} (\mu_{pqr} - \mu_{pqr}^0) L_{ijk}$$

Hence:

$$\begin{aligned}\|p^n(\mu)-p^n(\mu^0)\|^2 &= \sum_{i,j,k=0}^{n}\left(\sum_{p,q,r=0}^{\infty} C_{ip}C_{jq}C_{kr}(\mu_{pqr}-\mu^0_{pqr})\right)^2 \\ &\leq \|\mu-\mu^0\|^2_\infty \sum_{i,j,k=0}^{n}\left(\sum_{p,q,r=0}^{\infty}|C_{ip}C_{jq}C_{kr}|\right)^2 \qquad (40)\end{aligned}$$

¿From (5) we have:

$$C_{mj}=(2m+1)^{1/2}(-1)^j\frac{(m+j)!}{(j!)^2(m-j)!}$$

But

$$\frac{(m+j)!}{(j!)^2(m-j)!}\leq(1+1+1)^{m+j}=3^{m+j}$$

Hence

$$\sum_{j=0}^{m}|C_{mj}|\leq(2m+1)^{1/2}\sum_{j=0}^{m}3^{m+j} \qquad (41)$$

Now

$$\sum_{j=0}^{m}3^{m+j}=3^m\frac{3^{m+1}-1}{3-1}<\frac{3}{2}.3^{2m}$$

Hence, we get in view of (41):

$$\sum_{j=0}^{m}|C_{mj}|\leq\frac{3}{2}(2m+1)^{1/2}3^{2m} \qquad (42)$$

Noting that $C_{mj}=0$ for $j>m$, we have:

$$\sum_{p,q,r=0}^{\infty}|C_{ip}C_{jq}C_{kr}|=\sum_{p=0}^{i}|C_{ip}|\sum_{q=0}^{j}|C_{jq}|\sum_{r=0}^{k}|C_{kr}| \qquad (43)$$

In view of (42)-(43), the following holds:

$$\left(\sum_{p,q,r=0}^{\infty}|C_{ip}C_{jq}C_{kr}|\right)^2\leq\left(\frac{3}{2}\right)^6(2i+1)(2j+1)(2k+1)3^{4(i+j+k)} \qquad (44)$$

¿From (44) one has:

$$\begin{aligned}\sum_{i,j,k=0}^{n}\left(\sum_{p,q,r=0}^{\infty}|C_{ip}C_{jq}C_{kr}|\right)^2 &\leq \left(\frac{3}{2}\right)^6\left(\sum_{i=0}^{n}(2i+1)3^{4i}\right)^3 \\ &\leq \left(\frac{3}{2}\right)^6(2n+1)^3\left(\sum_{i=0}^{n}3^{4i}\right)^3\end{aligned}$$

Hence

$$\sum_{i,j,k=0}^{n}\left(\sum_{p,q,r=0}^{\infty}|C_{ip}C_{jq}C_{kr}|\right)^{2} \leq \left(\frac{3}{2}\right)^{6}\left(\frac{81}{80}\right)^{3}(2n+1)^{3}3^{12n} \tag{45}$$

In view of (45), inequality (40) gives:

$$\|p^{n}(\mu)-p^{n}(\mu^{0})\| \leq \|\mu-\mu^{0}\|_{\infty}\left(\frac{3}{2}\right)^{3}\left(\frac{81}{80}\right)^{3/2}(2n+1)^{3/2}3^{6n} \tag{46}$$

Put

$$\begin{aligned} f(t) &= \left(\frac{3}{2}\right)^{3}\left(\frac{81}{80}\right)^{3/2}(2t+1)^{3/2}3^{6t} \\ n(\epsilon) &= [f^{-1}(\epsilon^{-1/2})] \end{aligned}$$

Then we have in view of (46):

$$\|p^{n(\epsilon)}(\mu)-p^{n(\epsilon)}(\mu^{0})\| \leq \epsilon^{1/2} \tag{47}$$

If we put:

$$\eta(\epsilon) = \epsilon^{1/2} + \left(\sum_{max(i,j,k)>n(\epsilon)}\left(\sum_{p,q,r=0}^{\infty}C_{ip}C_{jq}C_{kr}\mu^{0}_{pqr}\right)^{2}\right)^{1/2}$$

then by (39),(47),(13) we have:

$$\|p^{n(\epsilon)}(\mu)-u_0\| \leq \eta(\epsilon)$$

As $\epsilon \to 0$, we have $f^{-1}(\epsilon^{-1/2}) \to \infty$ and $n(\epsilon) \to \infty$. By (8)

$$\sum_{max(i,j,k)>n(\epsilon)}\left(\sum_{p,q,r=0}^{\infty}C_{ip}C_{jq}C_{kr}\mu^{0}_{pqr}\right)^{2} \to 0 \quad as\ n(\epsilon) \to 0$$

Hence $\eta(\epsilon) \to 0$ as $\epsilon \to 0$.

Finally, let $u_0 \in H^1(I)$. By Theorem 1 one has:

$$\|p^{n(\epsilon)}(\mu^{0})-u_0\| \leq \frac{1}{2(n(\epsilon)+1)}\|u_0\|_{H^1(I)} \tag{48}$$

By the definition of $n(\epsilon)$, one has :

$$n(\epsilon)+1 > f^{-1}(\epsilon^{-1/2}) \geq n(\epsilon)$$

Since f is a monotone increasing function, the foregoing inequalities give:

$$f(n(\epsilon)+1) > \epsilon^{-1/2}$$

or

$$\left(\frac{3}{2}\right)^3\left(\frac{81}{80}\right)^{3/2}(2t_\epsilon+1)^{3/2}3^{6t_\epsilon} > \epsilon^{-1/2}$$

where $t_\epsilon = n(\epsilon)+1$.

This implies:

$$\frac{3}{2}\left(\frac{81}{80}\right)^{1/2}(2t_\epsilon+1)^{1/2}3^{2t_\epsilon} > \epsilon^{-1/6} \tag{49}$$

Note that

$$n+1 \le 2^n, \quad \forall n = 1,2,..$$

Hence, (49) implies :

$$\frac{3}{2}\left(\frac{81}{80}\right)^{1/2}\sqrt{2}.2^{t_\epsilon/2}.3^{2t_\epsilon} > \epsilon^{-1/6} \tag{50}$$

It follows that :

$$e^{(2\ln 3+\frac{1}{2}\ln 2)t_\epsilon} > \frac{8\sqrt{5}}{27\sqrt{2}\sqrt[6]{\epsilon}}$$

Thus:

$$t_\epsilon > \left(2\ln 3+\frac{1}{2}\ln 2\right)^{-1}\ln\left(\frac{8\sqrt{5}}{27\sqrt{2}\sqrt[6]{\epsilon}}\right) \tag{51}$$

In view of (51), inequality (48) gives:

$$\|p^{n(\epsilon)}(\mu^0)-u_0\| \le \frac{1}{C(\epsilon)}\|u_0\|_{H^1(I)} \tag{52}$$

where

$$C(\epsilon) = 2\left(2\ln 3+\frac{1}{2}\ln 2\right)^{-1}\ln\left(\frac{8\sqrt{5}}{27\sqrt{2}\sqrt[6]{\epsilon}}\right)$$

By (47),(52), the proof of Theorem 2 is completed.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York (1972).

[2] D. D. Ang, L. K. Vy and R. Gorenflo, A Regularization Method for the Moment Problem, in *Inverse Problems: Principles and Applications in Geophysics, Technology and Medicine,* Math. Research vol. 74, Akademie Verlag, Berlin (1993), 37-45.

[3] D. D. Ang, R. Gorenflo and L. K. Vy, Backus-Gilbert Regularization of a Moment Problem. *Preprint, Institute of Math.,* Freie Universitat Berlin (1993).

[4] R. Askey, I. J. Schoenberg and A. Sharma, Hausdorff's Moment Problem and Expansion in Legendre Polynomials. *J. Math. Anal. Appl.* **86** (1982).

[5] M. Bertero, C. De Mol and E. Pike, Linear Inverse Problems with Discrete Data. I: General Formulation and Singular System Analysis, *Inverse Problems* **1** (1985), 301.
[6] G. Inglese, Recent Results in the Study of the Moment Problem, in *Geophysical Data Interpretation by Inverse Modeling, Proc. 9th Intern. Sem. on Model Optim. in Exploration Geophysics*, Berlin (1991).
[7] A. Kirsch, B. Schomburg and G. Berendt, the Backus-Gilbert method, *Inverse Problems* **4** (1988), 771-783.
[8] G. Talenti, Recovering a Function from a Finite Number of Moments, *Inverse Problems* **3** (1987), 501-517.
[9] G.Sansone, *Orthogonal Functions*, Interscience, N.Y. (1959).
[10] J. A. Shohat and J. D. Tamarkin, the Problem of Moment, *AMS Math. Surveys*, Providence R I (1943).

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