ON GENERALIZED MITTAG-LEFFLER TYPE FUNCTIONS

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Abstract

The paper is devoted to the study of the properties of the special functions

$$E_{\alpha.m.l}(z) = \sum_{k=0}^{\infty} c_k z^k, \ z \in \mathbf{C},$$

$$c_0 = 1, \ c_k = \prod_{i=0}^{k-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)} \ (k = 1, 2, \cdots)$$

with $\alpha > 0$, $m > 0$, real l and $\alpha(im+l)+1 \neq 0, -1, -2, \cdots$ for $i = 0, 1, 2, \cdots$. For $m = 1$, $E_{\alpha,1,l}(z)$ coincides with the Mittag-Leffler type function

$$E_{\alpha,\alpha l+1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha l + 1)}$$

up to the constant $\Gamma(\alpha l + 1)$. The order and type of the entire function $E_{\alpha.m.l}(z)$ are evaluated and recurrence relations are given. For the functions $E_{n,m,l}(z)$ $(n = 1, 2, \cdots)$ connections with functions of hypergeometric type are studied and differentiation formulae are proved.

1. Introduction.

In recent years more and more attention has been paid to the well-known function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \ (\alpha \in \mathbf{C}, \ Re(\alpha) > 0), \tag{1.1}$$

introduced by Mitttag-Leffler [29] and named after him [9]. The properties of this entire function investigated by many mathematicians, can be found in [1],[6]-[7], [9], [12]-[14], [30], [38]-[39], etc. In applications various problems have been found for which the knowledge of properties of $E_{\alpha}(z)$ is of great importance. Some of these problems were studied in [3]-[5], [10]-[11], [15], [25], [27], [31],

[35], [37]. Mainly these applications are due to connections of the Mittag-Leffler function (1.1) with fractional calculus [6], [14], [17], [20]-[24], [28], [33], and in particular with solutions of integral and differential equations of fractional order [10], [16]-[17], [22]-[24], [28], [32], [35].

At the beginning of the sixth decade of our century a certain generalization of (1.1), namely

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \ (\alpha,\ \beta \in \mathbf{C}, Re(\alpha) > 0,\ Re(\beta) > 0), \tag{1.2}$$

called Mittag-Leffler type function [6]-[7], was introduced by Agarwal and Humbert [2], [18], [19]. The properties of (1.2) (in particular asymptotic behavior and zero distributions) were studied in [6], [7], [9], [12], [34], [36].

A new generalization of (1.2) was given in [20] in connection with solutions of Abel-Volterra integral equations. Such a function is defined by

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k \ (z \in \mathbf{C}),\ c_k = \prod_{i=1}^{k-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)} \ (k = 0, 1, 2, \cdots) \tag{1.3}$$

where an empty product is defined as equal to one, and $\alpha,\ m$ and l are real numbers such that

$$\alpha > 0,\ m > 0,\ \alpha(im + l) + 1 \neq -1, -2, -3, \cdots \ (i = 0, 1, 2, \cdots). \tag{1.4}$$

In particular, if $m = 1$, the conditions in (1.4) take the form

$$\alpha > 0,\ \alpha(i + l) + 1 \neq -1, -2, -3, \cdots \ (i = 0, 1, 2, \cdots). \tag{1.5}$$

and (1.3) is reduced to the Mittag-Leffler type function given in (1.2):

$$E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1) E_{\alpha,\alpha l+1}(z). \tag{1.5}$$

Therefore we shall call $E_{\alpha,m,l}(z)$ the generalized Mittag-Leffler type function. This function has been shown to be useful for solving in closed form new classes of integral and differential equations of fractional order [21]-[24], [32].

When $\alpha = n \in \mathbf{N} = \{1, 2, \cdots\}$, $E_{n,m,l}(z)$ takes the form

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \left(\frac{1}{[n(im+l)+1] \cdots [n(im+l)+n]} \right) z^k, \tag{1.6}$$

where $n,\ m$ and l are real numbers such that

$$n \in \mathbf{N},\ m > 0,\ n(im + l) \neq -1, -2, \cdots, -n \ (i = 0, 1, 2, \cdots). \tag{1.7}$$

Our paper is devoted to the study of the properties of the generalized Mittag-Leffler type functions $E_{\alpha,m,l}(z)$ and $E_{n,m,l}(z)$ given by (1.3) and (1.6). Section 2 deals with evaluating the order and type of an entire function $E_{\alpha,m,l}(z)$ $(\alpha > 0)$. Section 3 is devoted to recurrence formulas for such a function. In Section 4 we show that the functions $E_{n,m,l}(z)$ $(n = 1, 2, \cdots)$ are functions of hypergeometric type. Differentiation properties of $E_{n.m.l}(z^{nm})$ and $E_{n.m.l}(z^{-nm})$ are studied in Section 5.

2. Order and type of the entire function $E_{\alpha,m,l}(z)$.

In this section we find some characteristics of $E_{\alpha.m.l}(z)$. First of all we show that the generalized Mittag-Leffler type function (1.3) is an entire function.

Lemma 1. *If α, m and l are real numbers such that the condition (1.4) is satisfied, then $E_{\alpha,m,l}(z)$ is an entire function of the variable z.*

Proof. According to (1.3) and the relation [8, 1.18(4)]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^2}\right)\right] \ (z \to \infty), \qquad (2.1)$$

we have the asymptotics

$$\frac{c_{n+1}}{c_n} = \frac{\Gamma[\alpha(nm+l)+1]}{\Gamma[\alpha(nm+l+1)+1]} \sim (\alpha mn)^{-\alpha} \to 0 \ \ (n \to \infty).$$

Therefore by the D'Alembert criterion the power series in (1.2) converges in the whole complex plane, and hence $E_{\alpha,m,l}(z)$ is an entire function.

Corollary 1. *For $\alpha > 0$, $m > 0$ and $l > -1/\alpha$ the function $E_{\alpha,m,l}(z)$ is an entire function of z.*

Corollary 2. *If $n \in \mathbf{N}$, $m > 0$ and l are real numbers such that the condition (1.7) is satisfied, then the generalized Mittag-Leffler type function $E_{n,m,l}(z)$ given by (1.6) is an entire function of z.*

The main characteristics of an entire function are its order and type.

Definition 1. [26] *An entire function $F(z)$ is said be to be a function of finite order if there exists a positive constant k such that the inequality*

$$\max_{|z|=r} |F(z)| < \exp(r^k) \qquad (2.2)$$

is valid for all sufficiently large values of r $(r > r_0(k))$. The greatest lower bound of such numbers k is called the order of the entire function $F(z)$:

$$\rho = \inf\{k > 0 : (2.2) \textit{ takes place } \}. \qquad (2.3)$$

For a function of a given order a more precise characterization of the growth is given by its type.

Definition 2. [26] *By the type σ of an entire function $F(z)$ of order ρ we denote the greatest lower bound of positive numbers A for which, if r is large enough,*

$$\max_{|z|=r} |F(z)| < \exp(Ar^\rho) : \tag{2.4}$$

$$\sigma = \inf\{A > 0 : \ (2.4) \ \textit{takes place} \ \}. \tag{2.5}$$

If the entire function $F(z)$ can be represented in the form of a power series

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \ z \in \mathbf{C}, \tag{2.6}$$

then its order and type are evaluated by [26]

$$\rho = \limsup_{n\to\infty} \frac{n\log(n)}{\log(1/|c_n|)}, \tag{2.7}$$

$$(\sigma e \rho)^{1/\rho} = \limsup_{n\to\infty} \left(n^{1/\rho} (|c_n|)^{1/n} \right). \tag{2.8}$$

The order and type of the generalized Mittag-Leffler type function (1.3) are given by the following statement.

Theorem 1. *If α, m and l are real numbers such that the condition (1.4) is satisfied, then $E_{\alpha,m,l}(z)$ is an entire function of order $\rho = 1/\alpha$ and type $\sigma = 1/m$. Moreover, for every positive ε the estimate*

$$|E_{\alpha,m,l}(z)| < \exp\left(\left[\frac{1}{m} + \epsilon \right] |z|^{1/\alpha} \right), \tag{2.9}$$

holds for $|z| \geq r_0(\varepsilon), r_0(\varepsilon)$ sufficiently large.

Proof. Applying (2.7) we first find the order ρ of $E_{\alpha,m,l}(z)$. According to (1.3) we have

$$c_n = \frac{\Gamma(\alpha l + 1)\Gamma(\alpha l + \alpha m + 1)\cdots\Gamma(\alpha l + \alpha m[n-1] + 1)}{\Gamma(\alpha l + \alpha + 1)\Gamma(\alpha l + \alpha + \alpha m + 1)\cdots\Gamma(\alpha l + \alpha + \alpha m[n-1] + 1)}.$$

Let

$$z_k = \alpha l + \alpha m k + 1 \ (k \in \mathbf{N} = \{1, 2, \cdots\}). \tag{2.10}$$

Using (2.1) with $z = z_k$, $a = 0$ and $b = \alpha$ we obtain that for any $d > 0$ there exists $k_0 \in \mathbf{N}$ such that

$$(1-d) z_k^\alpha \leq \left| \frac{\Gamma(z_k + \alpha)}{\Gamma(z_k)} \right| \leq (1+d) z_k^\alpha \quad \forall k > k_0. \tag{2.11}$$

Therefore, for $n > k_0$ we have

$$\log\left(\frac{1}{|c_n|}\right) = \sum_{k=0}^{n-1} \log\left|\frac{\Gamma(z_k+\alpha)}{\Gamma(z_k)}\right|$$

$$= \sum_{k=0}^{k_0} \log\left|\frac{\Gamma(z_k+\alpha)}{\Gamma(z_k)}\right| + \sum_{k=k_0+1}^{n-1} \log\left|\frac{\Gamma(z_k+\alpha)}{\Gamma(z_k)}\right|$$

$$\leq d_1 + \sum_{k=k_0+1}^{n-1} \log(1+d) + + \sum_{k=k_0+1}^{n-1} \log(z_k^{\alpha})$$

$$= d_1 + (n-k_0+1)\log(1+d) + \alpha \sum_{k=k_0+1}^{n-1} \left[\log(k) + \log(\alpha m) + \log\left(1+\frac{\alpha l+1}{klm}\right)\right]$$

and hence

$$\log\left(\frac{1}{|c_n|}\right) \leq d_3 + nd_4 + \alpha n\log(n), \tag{2.12}$$

where d_3 and d_4 are certain positive constants. Similarly

$$\log\left(\frac{1}{|c_n|}\right) \geq d_4 + (n-k_0+1)\log(1-d)$$

$$+\alpha \sum_{k=k_0+1}^{n-1} \left[\log(k) + \log(\alpha m) + \log\left(1+\frac{\alpha l+1}{klm}\right)\right]$$

and

$$\log\left(\frac{1}{|c_n|}\right) \geq d_5 + nd_6 + \alpha n\log(n). \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\lim_{n\to\infty} \frac{n\log(n)}{\log(1/|c_n|)} = \frac{1}{\alpha}, \tag{2.14}$$

and hence in accordance with (2.7) the order ρ of $E_{\alpha,m,l}(z)$ is given by

$$\rho = \frac{1}{\alpha}. \tag{2.15}$$

Next we use (2.8) to find the type of $E_{\alpha,m,l}(z)$. Applying (2.11), (2.10) and (1.3) we have

$$\prod_{k=0}^{k_0} \left|\frac{\Gamma(z_k+\alpha)}{\Gamma(z_k)}\right| \left(\frac{1}{1+c}\right)^{n-k_0-1} \prod_{k=k_0+1}^{n-1} z_k^{\alpha} \leq |c_n|$$

$$\leq \prod_{k=0}^{k_0} \left| \frac{\Gamma(z_k+\alpha)}{\Gamma(z_k)} \right| \left(\frac{1}{1-c} \right)^{n-k_0-1} \prod_{k=k_0+1}^{n-1} z_k^{\alpha}. \tag{2.16}$$

Using this formula and the asymptotic relation

$$\prod_{k=k_0}^{n-1} \left(\frac{1}{\alpha k m} \right)^{\alpha} \sim \left(\frac{1}{n!} \right)^{\alpha} \left(\frac{1}{\alpha m} \right)^{\alpha n} \sim \left(\frac{e}{\alpha n m} \right)^{\alpha n} \quad (n \to \infty),$$

we obtain from (2.8) $\sigma^{\alpha} = m^{-\alpha}$, and hence the type σ of $E_{\alpha,m,l}(z)$ is given by

$$\sigma = \frac{1}{m}. \tag{2.17}$$

The asymptotic estimate (2.9) follows from (2.7)-(2.8) and Definitions 1 and 2. This completes the proof of theorem.

Corollary 1.1. *If $n \in \mathbf{N}$, $m > 0$ and l are real numbers such that the condition (1.7) is satisfied, then the generalized Mittag-Leffler type function $E_{n,m,l}(z)$ given by (1.6) is an entire function of z with order $\rho = 1/n$ and type $\sigma = 1/m$.*

Corollary 1.2. *The Mittag-Leffler function $E_{\alpha}(z)$ and the Mittag-Leffler type function $E_{\alpha,\beta}(z)$ given in (1.1) and (1.2) have the same order and type:*

$$\rho = \frac{1}{\alpha}, \quad \sigma = 1. \tag{2.18}$$

Remark 1. The assertions of Corollary 2 are well known - see [6], [7], [26].

Remark 2. Theorem 1 shows that the generalized Mittag-Leffler type function (1.3) has the same order as the Mittag-Leffler function (1.1) and the Mittag-Leffler type function (1.2). But the type of $E_{\alpha,m,l}(z)$ depends on m.

3. Recurrence relations for $E_{\alpha,m,l}(z)$.

In this section we give recurrence relations for $E_{\alpha,m,l}(z)$.

Theorem 2. *Let α, m and l be real numbers such that the condition (1.4) is satisfied and let $n \in \mathbf{N}$. Then for the generalized Mittag-Leffler type function (1.3) the recurence relation*

$$z^n \left[E_{\alpha,m,l+nm}(z) - 1 \right] = \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm+l+1)+1]}{\Gamma[\alpha(jm+l)+1]}$$

$$\left[E_{\alpha,m,l}(z) - 1 - \sum_{i=1}^{n} \left(\prod_{j=0}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^i \right] \tag{3.1}$$

holds.

Proof. By (1.3) we have

$$E_{\alpha,m,l+nm}(z) = 1 + \sum_{k=1}^{\infty} \left(\prod_{i=0}^{k-1} \frac{\Gamma[\alpha(im+nm+l)+1]}{\Gamma[\alpha(im+nm+l+1)+1]} \right) z^k.$$

Substituting summation indices first by $j = i + n$, then by $i = k + n$, and executing a few other manipulations, we obtain

$$E_{\alpha,m,l+nm}(z) = 1 + \sum_{k=1}^{\infty} \left(\prod_{j=n}^{n+k-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^k$$

$$= 1 + \sum_{i=n+1}^{\infty} \left(\prod_{j=n}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^{i-n}$$

$$= 1 + \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm+l+1)+1]}{\Gamma[\alpha(jm+l)+1]} \sum_{i=n+1}^{\infty} \left(\prod_{j=0}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^{i-n}$$

$$= 1 + \frac{1}{z^n} \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm+l+1)+1]}{\Gamma[\alpha(jm+l)+1]} \times$$

$$\times \left[1 + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^i \right.$$

$$\left. - \sum_{i=1}^{n} \left(\prod_{j=0}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^i - 1 \right]$$

$$= 1 + \frac{1}{z^n} \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]}$$

$$\left[E_{\alpha,m,l}(z) - \sum_{i=1}^{n} \left(\prod_{j=0}^{i-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]} \right) z^i - 1 \right],$$

and (3.1) is proved.

Corollary 2.1. *If the conditions of Theorem 2 are fulfilled, then*

$$zE_{\alpha,m,l+m}(z) = \frac{\Gamma(\alpha l + \alpha + 1)}{\Gamma(\alpha l + 1)} \left[E_{\alpha,m,l}(z) - 1 \right] \tag{3.2}$$

and

$$z^n E_{\alpha,m,l+nm}(z) = \prod_{j=0}^{n-1} \frac{\Gamma[\alpha(jm+l+1)+1]}{\Gamma[\alpha(jm+l)+1]}$$

$$\left[E_{\alpha,m,l}(z) - 1 - \sum_{i=1}^{n-1}\left(\prod_{j=0}^{i-1}\frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]}\right) z^i\right] \tag{3.3}$$

for $n = 2, 3, \cdots$.

Corollary 2.2. *If* $\alpha > 0$, $\beta > 0$ *and* $n \in \mathbf{N}$, *then for the Mittag-Leffler type function* $E_{\alpha,\alpha n+\beta}(z)$ *the recurrence relations*

$$zE_{\alpha,\alpha+\beta}(z) = E_{\alpha,\beta}(z) - \frac{1}{\Gamma(\beta)} \tag{3.4}$$

and

$$z^n\Gamma(\alpha n+\beta)E_{\alpha,\alpha n+\beta}(z) =$$

$$= \prod_{j=0}^{n-1}\frac{\Gamma(\alpha j+\alpha+\beta)}{\Gamma(\alpha j+\beta)}\left[\Gamma(\beta)E_{\alpha,\beta}(z) - 1 - \sum_{i=1}^{n-1}\left(\prod_{j=0}^{i-1}\frac{\Gamma(\alpha j+\beta)}{\Gamma(\alpha j+\alpha+\beta)}\right)z^i\right] \tag{3.5}$$

hold for $n = 2, 3, \cdots$.

Corollary 2.3. *If* $\alpha > 0$ *and* $n \in \mathbf{N}$, *then the Mittag-Leffler type function* $E_{\alpha,\alpha n+1}(z)$ *is expressed via the Mittag-Leffler function (1.1) by*

$$zE_{\alpha,\alpha+1}(z) = E_\alpha(z) - 1 \tag{3.6}$$

and

$$z^n E_{\alpha,\alpha n+1}(z) = E_\alpha(z) - \sum_{i=1}^{n}\frac{z^i}{\Gamma(\alpha i+1)} \tag{3.7}$$

for $n = 2, 3, \cdots$.

4. Connection of $E_{n,m,l}(z)$ with functions of hypergeometric type.

As we have mentioned in the Introduction, $E_{\alpha,m,l}(z)$ is a generalization of the Mittag-Leffler type function $E_{\alpha,\beta}(z)$ in (1.2), in particular of the Mittag-Leffler function $E_\alpha(z)$ in (1.1). When $\alpha = n \in \mathbf{N}$, $E_{n,m,l}(z)$ in (1.6) becomes a function of hypergeometric type. Such a function ${}_pF_q\left(a_1, a_2.\cdots, a_p; b_1, b_2.\cdots, b_q; z\right)$ for $p \in \mathbf{N}_0 = \mathbf{N}\bigcup\{0\}, q \in \mathbf{N}_0$, $a_1, a_2.\cdots, a_p \in \mathbf{C}$, $b_1, b_2, \cdots, b_q \in \mathbf{C}$ and $z \in \mathbf{C}$, $|z| < 1$, is defined by the hypergeometric series [8]

$${}_pF_q\left(a_1, a_2.\cdots, a_p; b_1, b_2.\cdots, b_q; z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(a_q)_k}\frac{z^k}{k!}, \tag{4.1}$$

where

$$(a)_0 = 1,\ (a)_k = a(a+1)\cdots(a+k-1)\ (k = 1, 2, \cdots). \tag{4.2}$$

Theorem 3. *Let $n \in N, m > 0$ be real numbers such that the condition (1.7) be satisfied. Then $E_{n,m,l}(z)$ is given by*

$$E_{n,m,l}(z) = {}_1F_n\left(1; \frac{nl+1}{nm}, \frac{nl+2}{nm} \cdots \frac{nl+n}{nm}; \frac{z}{(nm)^n}\right) \tag{4.3}$$

and is an entire function of z with order $\rho = 1/n$ and type $\sigma = 1/m$.

Proof. According to (1.3), (4.1) and (4.2) we have

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)} \right) z^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{k!}{(nl+1)\cdots(n[(k-1)m+l])\cdots(nl+n)\cdots(n[(k-1)m+l]+n)} \frac{z^k}{k!}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(1)_k}{([nl+1])/[nm])_k \cdots ([nl+n]/[nm])_k} \frac{1}{(nm)^{nk}} \frac{z^k}{k!}$$

$$= {}_1F_n\left(1; \frac{nl+1}{nm}, \frac{nl+2}{nm} \cdots \frac{nl+n}{nm}; \frac{z}{(nm)^n}\right)$$

and (3.3) is proved. The last assertion follows from Theorem 1. This completes the proof.

Corollary 3.1. *If $n \in \mathbf{N}$ and $\beta > 0$, then the Mittag-Leffler type function $E_{n,\beta}(z)$ is given by*

$$E_{n,1,(\beta-1)/n}(z) = \Gamma(\beta) E_{n,\beta}(z)$$

$$= {}_1F_n\left(1; \frac{nl+1}{n}, \frac{nl+2}{n} \cdots \frac{nl+n}{n}; \frac{z}{n^n}\right) \tag{4.4}$$

and is an entire function of order $\rho = 1/n$ and type $\sigma = 1$.

Corollary 3.2. *If*

$$m > 0,\ l \in \mathbf{R},\ im + l \neq -1, -2, -3, \cdots\ (i = 0, 1, 2, \cdots), \tag{4.5}$$

then $E_{1,m,l}(z)$ is given by

$$E_{1,m,l}(z) = {}_1F_1\left(1; \frac{l+1}{m}; \frac{z}{m}\right) = \Gamma\left(\frac{l+1}{m}\right) E_{1,(l+1)/m}\left(\frac{z}{m}\right) \tag{4.6}$$

and is an entire function of z with order $\rho = 1$ and type $\sigma = 1/m$.

Corollary 3.3. *If $l \in \mathbf{R}, l \neq -1, -2, -3, \cdots$, then $E_{1,1,l}(z)$ is given by*

$$E_{1,1,l}(z) = {}_1F_1(1; l+1; z) = \Gamma(l+1) E_{1,l+1}(z) \tag{4.7}$$

and is an entire function of z with order $\rho = 1$ and type $\sigma = 1$. In particular, if $l \in \mathbf{N}$,

$$E_{1,1,l}(z) = l! E_{1,l+1}(z) = \frac{l!}{z^l}\left[e^z - \sum_{k=0}^{l-1} \frac{z^k}{k!}\right] \tag{4.8}$$

and

$$E_{1,1,0}(z) = E_1(z) = e^z. \tag{4.9}$$

5. Differentiation properties of $E_{n,m,l}(z)$.

In this section we give two differentiation formulae for the generalized Mittag-Leffler type function (1.3). The first one is given by the following statement.

Theorem 4. *Let $n \in \mathbf{N}$, $m > 0$ and $l \in \mathbf{R}$ be numbers such that the conditions in (1.7) are satisfied and let $a \in \mathbf{C}$. Then for $E_{n,m,l}(az^{nm})$ the differentiation formula*

$$\left(\frac{d}{dz}\right)^n \left[z^{n(l-m+1)} E_{n,m,l}(az^{nm})\right] = \prod_{j=1}^{n} [n(l-m)+j] z^{n(l-m)} + az^{nl} E_{n,m,l}(az^{nm}) \tag{5.1}$$

holds. In particular, if

$$n(l-m) = -j \ \text{for some} \ j = 1, 2, \cdots, n, \tag{5.2}$$

then

$$\left(\frac{d}{dz}\right)^n \left[z^{n(l-m+1)} E_{n,m,l}(az^{nm})\right] = az^{nl} E_{n,m,l}(az^{nm}). \tag{5.3}$$

Proof. By (1.3) we have

$$\left(\frac{d}{dz}\right)^n \left[z^{n(l-m+1)} E_{n,m,l}(az^{nm})\right] = \left(\frac{d}{dz}\right)^n \left[z^{n(l-m+1)} + \sum_{j=1}^{\infty} c_j a^j z^{n(l-m+1)+nmj}\right]$$

$$= [n(l-m+1)][n(l-m+1)-1] \cdots [n(l-m+1)-n+1] z^{n(l-m)}$$

$$+ \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j-1} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right) \frac{\Gamma[n\{(j-1)m+l+1\}+1]}{\Gamma[n(\{(j-1)m+l\}+1]} a^j z^{n(l-m)+nmj}$$

$$= \prod_{j=1}^{n} [n(l-m)+j] z^{n(l-m)} + az^{nl}$$

$$+ \sum_{j=2}^{\infty} \left(\prod_{i=1}^{j-2} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right) a^j z^{n(l-m)+nmj}.$$

Making the change of indices $k = j - 1$, we obtain

$$\left(\frac{d}{dz}\right)^n \left[z^{n(l-m+1)}E_{n,m,l}(z^{nm})\right]$$

$$= \prod_{j=1}^{n}[n(l-m)+j]z^{n(l-m)} + az^{nl}$$

$$+\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right)a^{k+1}z^{nl+nmk}$$

$$= \prod_{j=1}^{n}[n(l-m)+j]z^{n(l-m)}+az^{nl}\left[1+\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k-1}\frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right)(az^{nm})^k\right],$$

and (5.1) is proved in accordance with (1.3). (5.3) follows from (5.1). The theorem is proved.

Corollary 4.1. *If $n = 1, 2, \cdots$, $\beta > 0$ and $a \in \mathbf{C}$, then for the Mittag-Leffler type function $E_{n,\beta}(az^n)$ in (1.2)*

$$\left(\frac{d}{dz}\right)^n \left[z^{\beta-1}E_{n,\beta}(az^n)\right] = \frac{1}{\Gamma(\beta-n)}z^{\beta-n-1} + az^{\beta-1}E_{n,\beta}(az^n). \tag{5.4}$$

Corollary 4.2. *If $n = 1, 2, \cdots$, $k \in \mathbf{N}$ $(1 \leq k \leq n)$ and $a \in \mathbf{C}$, then for the Mittag-Leffler type function $E_{n,k}(az^n)$*

$$\left(\frac{d}{dz}\right)^n \left[z^{k-1}E_{n,k}(az^n)\right] = az^{k-1}E_{n,k}(az^n). \tag{5.5}$$

In particular when $k = 1$, for the Mittag-Leffler function $E_n(az^n)$ in (1,1)

$$\left(\frac{d}{dz}\right)^n [E_n(az^n)] = aE_n(az^n). \tag{5.6}$$

Remark 3. By (1.2), the relation (5.4) can be rewritten in the form

$$\left(\frac{d}{dz}\right)^n \left[z^{\beta-1}E_{n,\beta}(az^n)\right] = z^{\beta-n-1}E_{n,\beta-n}(az^n), \tag{5.7}$$

which is known for $a = 1$ [6, (1.6)].

Remark 4. When $a = 1$, the relation (5.6) is well known - see [9, 18.1(15)].

Another differentiation relation for the generalized Mittag-Leffler type function (1.3) is given by

Theorem 5. *Let $n \in \mathbf{N}$, $m > 0$ and $l \in \mathbf{R}$ be numbers such that the conditions in (1.7) are satisfied and let $a \in \mathbf{C}$. Then for $E_{n,m,l}(az^{-nm})$ the differentiation formula*

$$\left(\frac{d}{dz}\right)^n \left[z^{n(m-l)-1} E_{n,m,l}(az^{-nm})\right]$$

$$= \prod_{j=1}^{n} [n(m-l)-j] z^{n(m-l-1)-1} + (-1)^n a z^{-n(l+1)-1} E_{n,m,l}(az^{-nm}) \qquad (5.8)$$

holds. In particular, if the conditions in (5,2) are satisfied, then

$$\left(\frac{d}{dz}\right)^n \left[z^{n(m-l)-1} E_{n,m,l}(az^{-nm})\right] = (-1)^n a z^{-n(l+1)-1} E_{n,m,l}(z^{-nm}). \qquad (5.9)$$

Proof. By (1.3) we have

$$\left(\frac{d}{dz}\right)^n \left[z^{n(m-l)-1)} E_{n,m,l}(az^{-nm})\right]$$

$$= \left(\frac{d}{dz}\right)^n \left[z^{n(m-l)-1} 1 + \sum_{j=1}^{\infty} c_j a^j z^{n(m-l)-nmj-1}\right]$$

$$= [n(m-l)-1] \cdots [n(m-l)-n] z^{n(m-l)-1-n}$$

$$+ \sum_{j=1}^{\infty} a^j \left(\prod_{i=1}^{j-1} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right) \times$$

$$\times [n(m-l)-nmj-1] \cdots [n(m-l)-nmj-n] z^{n(m-l)-nmj-n-1}$$

$$= \prod_{j=1}^{n} [n(m-l)-j] z^{n(m-l-1)-1}$$

$$+ (-1)^n \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j-1} \frac{\Gamma(\alpha[im+l]+1)}{\Gamma(\alpha[im+l+1]+1)}\right) \times$$

$$\times \frac{\Gamma(n[\{j-1\}m+l+1]+1)}{\Gamma(n[\{j-1\}m+l]+1)} a^j z^{n(m-l)-nmj-n-1}$$

$$= \prod_{j=1}^{n} [n(m-l)-j] z^{n(m-l-1)-1}$$

$$+ (-1)^n a z^{-n(l+1)-1} \left[1 + \sum_{j=2}^{\infty} a^j \left(\prod_{i=1}^{j-2} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right) a^j z^{-nm(j-1)}\right].$$

Changing indices by $k = j - 1$, we obtain

$$\left(\frac{d}{dz}\right)^n \left[z^{n(m-l)-1} E_{n,m,l}(az^{-nm})\right]$$

$$= \prod_{j=1}^{n} [n(m-l) - j] z^{n(m-l-1)-1} +$$

$$+(-1)^n a z^{-n(l+1)-1} \left[1 + \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k-1} \frac{\Gamma(n[im+l]+1)}{\Gamma(n[im+l+1]+1)}\right) (az^{-nm})^k\right]$$

and (5.8) is proved in accordance with (1.3). (5.9) follows from (5.8). The theorem is proved.

Corollary 5.1. *If $n = 1, 2, \cdots$, $\beta > 0$ and $a \in \mathbf{C}$, then for the Mittag-Leffler type function $E_{n,\beta}(az^{-n})$ in (1.2)*

$$\left(\frac{d}{dz}\right)^n \left[z^{n-\beta} E_{n,\beta}(az^{-n})\right] = \frac{(-1)^n}{\Gamma(\beta - n)} z^{-\beta} + (-1)^n a z^{-n-\beta} E_{n,\beta}(az^{-n}). \quad (5.10)$$

Corollary 5.2. *If $n = 1, 2, \cdots$, $k \in \mathbf{N}$ $(1 \leq k \leq n)$ and $a \in \mathbf{C}$, then for the Mittag-Leffler type function $E_{n,k}(az^{-n})$*

$$\left(\frac{d}{dz}\right)^n \left[z^{n-k} E_{n,k}(az^{-n})\right] = (-1)^n a z^{-n-k} E_{n,k}(az^{-n}). \quad (5.11)$$

In particular when $k = 1$, for the Mittag-Leffler function $E_n(az^{-n})$ in (1,1)

$$\left(\frac{d}{dz}\right)^n \left[z^{n-1} E_n(az^{-n})\right] = (-1)^n a z^{-n-1} E_n(az^{-n}). \quad (5.12)$$

Remark 5. The relations (5.1), (5.8) and (5.3), (5.9) can be considered as nonhomogeneous and homogeneous differential equations of order n for the functions $z^{n(l-m+1)} E_{n,m,l}(az^{nm})$ and $z^{n(m-l)-1} E_{n,m,l}(az^{-nm})$, respectively. In this way explicit solutions of new classes of differential equations were obtained in [24], [32]. In particular, (5.4), (5.10) and (5.5), (5.11) are nonhomogeneous and homogeneous differential equations for the functions $z^{\beta-1} E_{n,\beta}(az^n)$ and $[z^{n-j} E_{n,\beta}(az^{-n})$. The function $z^{j-1} E_{n,\beta}(az^n)$ as explicit solution of a differential equation was found earlier by reduction of the differential equation to the corresponding Volterra integral equation [33, Section 42.1].

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