Periodic and Solitary Traveling Wave Solutions for the Generalized Kadomtsev-Petviashvili Equation

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Summary.
This paper is concerned with travelling waves for the generalized Kadomtsev–Petviashvili equation $(w_t + w_{\xi\xi\xi} + f(w)_\xi)_\xi = w_{yy}$, $(\xi, y) \in I\!R^2$, $t \in I\!R$, i. e. solutions of the form $w(t, \xi, y) = u(\xi - ct, y)$. We study both, solutions periodic in $x = \xi - ct$ and solitary waves, which are decaying in x, and their interrelations. In particular, we prove the existence of a sequence of k-periodic solutions, $k \in I\!N$, which is uniformly bounded in norm and converges to a solitary wave in a suitable toplogy. This result also holds for the corresponding ground states, i. e. solutions with minimal energy.

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0 Introduction

We study traveling wave solutions, i. e. solutions of the form

$$w(t,\xi,y) = u(\xi - ct, y), \quad c > 0,$$

of the generalized Kadomtsev–Petviashvili (KP) equation

$$(w_t + w_{\xi\xi\xi} + f(w)_\xi)_\xi = w_{yy}\,, \quad (\xi, y) \in I\!R^2,\ t \in I\!R. \tag{1}$$

with a superquadratic nonlinearity f. For solutions of such kind the equation reduces to the following one

$$(-cu_x + u_{xxx} + f(u)_x)_x = u_{yy}\,, \quad (x,y) \in I\!R^2. \tag{2}$$

In the case $f(u) = 3u^2$ we get the original KP equation which is of great importance in physics (cf. [5]). This last equation (among other completely integrable systems) was studied extensively by means of algebro-geometric techniques (see, e.g., [6]).

Solitary waves for equation (1), i.e. solutions whose profile function u is a decaying solution of (2), were studied by M. J. Ablowitz, H. Segur and X. P. Wang [1], A. de Bouard and J.-P. Saut [3], [4], Y. Liu and X. P. Wang [8], and M. Willem [14], [15]. In [3] the existence of nontrivial solitary waves is proved in the case

$$f(u) = u^{p-1}, \quad p = 3, 4, 5\,.$$

Rational powers $2 < p < 6$ were considered in [1], but the proof seems to be incomplete. A more general case was treated in [14]. The techniques of [1] and [3] are based on an appropriate minimization procedure, while in [14] the Mountain Pass Theorem of A. Ambrosetti and P. H. Rabinowitz is used. However, it must be pointed out that the proof of Lemma 7.4 in [14] is incorrect. The correct proof may be carried out along the same lines as in Lemma 4 below.

In contrast to those papers we consider both solitary and periodic in x traveling waves. Our main goal is to clarify the interconnection between these two classes of solutions. The main assumptions are essentially the same as in [14]. Denote by $F(u) = \int_0^u f(s)\,ds$ the primitive function of f. We make the following assumptions:

1^o $f \in C^1(I\!R, I\!R)$, $f(0) = 0$.

2^o $|f(u)| \le C(1 + |u|^{p-1})$, $2 < p < 6$ and $f(u) = o(|u|)$ as $|u| \to 0$.

3^o There exists $\varphi \in C_0^\infty(I\!R^2)$ such that

$$\frac{1}{\lambda^2} \int_{I\!R^2} F(\lambda D_x \varphi)\,dxdy \to +\infty \quad \text{as} \quad \lambda \to +\infty.$$

4^o There exists $\mu > 2$ such that $\mu F(u) \le u f(u)$ for every $u \in I\!R$.

Remark that if $F(u) > 0$ for all $u > 0$, then Assumption 3^o follows from 1^o, 2^o and 4^o. Also we note that making a simple change of variables one can choose the function φ in Assumption 3^o having arbitrarily small support.

First of all, in Section 2, we prove that, given $k > 0$, there exists a nontrivial k-periodic in x solution of equation (2) (see Theorem 1 below). Moreover, these solutions are uniformly bounded and bounded away from zero with respect to an appropriate norm. To do this we use a variational approach based on the Mountain Pass Theorem together with a simple concentration compactness argument.

In the rest of the paper we study the limit behaviour of such periodic solutions as $k \to \infty$. Here we exploit a version of the techniques developed in our previous paper [10] dealing with a nonlinear Schrödinger equation. In Section 3 we prove that, modulo spatial translations and passage to a subsequence, such solutions converge locally in an appropriate topology to a nontrivial solitary wave. The next step is to look at the behaviour of these solutions non-locally, on the entire strip of periodicity $Q_k = (-k/2, k/2) \times I\!R$. In Section 4 we impose an additional restriction on f and prove a result which may be described, informally, as follows. For periodic solutions u_k like in Theorem 1, there exists a finite number of solitary wave solutions u^i, $i = 1, \dots, m$, and a set of vectors $\zeta_k^i \in I\!R^2$ such that the difference

$$u_k(\cdot) - \sum_i u^i(\cdot + \zeta_k^i)$$

tends to zero on Q_k with respect to an appropriate norm, i.e. the shape of u_k on Q_k looks like a sum of translations of a finite number of solitary solutions (this is true for a subsequence of u_k). In fact, Theorem 3 contains a slightly more general result.

Then, in Section 5, we consider "ground state" solutions provided the nonlinearity f satisfies some further assumption. Here we use the so-called Nehari variational principle. For any $k > 0$, we prove the existence of a nontrivial k-periodic solution u_k having lowest k-periodic energy among all nontrivial solutions, i.e. a k-periodic ground state. Also we prove the existence of a solitary ground state. Moreover, the solitary ground state u may be choosen in such a way that $u_k(\cdot) - u(\cdot + \zeta_k)$ tends to zero on Q_k, modulo passage to a subsequence. This means that the shapes of periodic and solitary ground states on Q_k are almost the same provided the period is big enough.

Finally, we want to point out that it is easy to state the existence of nontrivial double-periodic solutions of (2) having arbitrary prescribed periods. It is very natural to look what may happen if the periods are big enough. One can expect that the situation is similar to that considered in the present paper. Nevertheless, we have no rigorous results in this direction, since the cut-off arguments, which are used in our proofs, break down in this new setting.

1 Preliminaries. Periodic traveling waves

First we consider the case of solitary waves. Following [2], [3], and [14], we rewrite equation (2) as follows

$$\Big(-u_{xx} + D_x^{-2} u_{yy} + cu - f(u)\Big)_x = 0, \quad (x, y) \in I\!R^2, \tag{3}$$

where

$$D_x^{-1} u(x, y) = \int_{-\infty}^{x} u(s, y)\, ds.$$

Now we present the variational approach to the solitary wave problem for the KP equation. We define the Hilbert space X as the completion of $Y = \{D_x\varphi \mid \varphi \in C_0^\infty(I\!R^2)\}$ in the norm $\|u\| = (u,u)_X^{1/2}$, where the the scalar product is defined by

$$(u,v)_X = \int_{I\!R^2} \left(u_x v_x + D_x^{-1}u_y D_x^{-1}v_y + cuv\right)\, dxdy\,.$$

Then the functional

$$J(u) = \int_{I\!R^2} \left[\frac{1}{2}\left(u_x^2 + (D_x^{-1}u_y)^2 + cu^2\right) - F(u)\right]\, dxdy$$

is in $C^2(X, I\!R)$ with the derivative

$$\langle J'(u), v\rangle = (u,v)_X - \int_{I\!R^2} f(u)v\, dxdy,$$

and the weak solutions of (3) are the critical points of J. In [14] the following is shown:

Lemma 1 *The embedding $X \hookrightarrow L^p(I\!R^2)$ is continuous for $2 \le p \le 6$ and $X \hookrightarrow L^p_{loc}(I\!R^2)$ is compact for $2 \le p < 6$.*

In the case of k-periodic traveling waves we consider instead of (3) the following equation on the infinite strip $Q_k = (-k/2, k/2) \times I\!R$ of $I\!R^2$ of width $k \in I\!N$:

$$\left(-u_{xx} + D_{x,k}^{-2}u_{yy} + cu - f(u)\right)_x = 0 \quad \text{in } \; Q_k\,, \quad u_k \text{ is } k-\text{periodic in } x, \tag{4}$$

where

$$D_{x,k}^{-1}u(x,y) = \int_{-k/2}^{x} u(s,y)ds.$$

The fact that we consider here k as an integer is not important, but is only for simplicity of notations. In fact we may consider any sequence of stripes Q_{r_k}, where $r_k \in I\!R_+$ is a sequence which tends to ∞ as $k \to \infty$.

Denote by $C_k^\infty(I\!R^2)$ the space of smooth functions which are $k-$periodic in x and have compact support in y and define

$$Y_k = \{D_x\varphi_{|Q_k} \mid \varphi \in C_k^\infty(I\!R^2)\}\,.$$

Then any function v from Y_k satisfies $\int\limits_{-k/2}^{k/2} v(x,y)dx = 0$. We define X_k as the completion of Y_k in the norm $\|u\|_k = (u,u)_k^{1/2}$, where

$$(u,v)_k = \int_{Q_k} \left(u_x v_x + D_{x,k}^{-1}u_y D_{x,k}^{-1}v_y + cuv\right)\, dxdy\,.$$

We introduce the energy functional

$$J_k(u) = \int_{Q_k} \left[\frac{1}{2}\left(u_x^2 + (D_{x,k}^{-1}u_y)^2 + cu^2\right) - F(u)\right]\, dxdy.$$

In the following constructions we often need a cut-off operator. Let χ_k be a $C_0^\infty(I\!R)$ cut-off function satisfying $\chi_k(x) = 1$ for $x \in [-k/2, k/2]$, $\chi(x) = 0$ for $|x| \geq (k+1)/2$ and $|\chi_k'|, |\chi_k''| \leq C_0$. We define the cut-off operator $P_k : X_k \to X$ as follows:

$$P_k u(x,y) = \Big[\chi_k(x) D_{x,k}^{-1} u(x,y)\Big]_x .$$

Lemma 2 *P_k is a uniformly bounded (with respect to k) linear operator from X_k into X and $P_k u(x,y) = u(x,y)$ for $(x,y) \in Q_k$.*

Proof. For $u \in X_k$ we have

$$\int_{I\!R^2} |P_k u(x,y)|^2 dxdy \leq 2 \int_{I\!R^2} \Big[|\chi_k(x) u(x,y)|^2 + |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 \Big] \, dxdy \, . \qquad (5)$$

The first integral on the right can easily be estimated by $2\|u\|_{L^2(Q_k)}^2$. To estimate the second one, we denote $I_1 = [-(k+1)/2, -k/2]$, $I_2 = [k/2, (k+1)/2]$. For $x \in I_1$, we have

$$\begin{aligned} |D_{x,k}^{-1} u(x,y)|^2 &\leq \Big| \int_{-k/2}^{x} |u(s,y)| \, ds \Big|^2 \\ &\leq \int_{I_1} |u(s,y)|^2 \, ds \end{aligned}$$

and for $x \in I_2$

$$\begin{aligned} |D_{x,k}^{-1} u(x,y)|^2 &= \Big| \int_{k/2}^{x} u(s,y) \, ds \Big|^2 \\ &\leq \int_{I_2} |u(s,y)|^2 \, ds \, . \end{aligned}$$

The second integral in (5) can now be estimated by

$$\begin{aligned} \int_{I\!R^2} |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 dxdy &= \int\limits_{I_1 \cup I_2} \int\limits_{I\!R} |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 dydx \\ &\leq \int\limits_{I_1 \cup I_2} \int\limits_{I\!R} |\chi_k'(x)|^2 \int\limits_{-k/2}^{k/2} |u(s,y)|^2 \, dsdydx \\ &\leq C_0^2 \int\limits_{I_1 \cup I_2} \|u\|_{L^2(Q_k)}^2 dx \ \leq \ C_0^2 \|u\|_{L^2(Q_k)}^2 \, . \end{aligned}$$

This shows that

$$\int_{I\!R^2} |P_k u(x,y)|^2 dxdy \leq C_1 \|u\|_{L^2(Q_k)}^2 \, .$$

In a similar manner we can estimate

$$\begin{aligned} \int_{I\!R^2} \Big| [P_k u(x,y)]_x \Big|^2 dxdy &\leq 4 \int_{I\!R^2} \Big[|\chi_k u_x|^2 + 2|\chi_k' u|^2 + |\chi_k'' D_{x,k}^{-1} u|^2 \Big] \, dxdy \\ &\leq C_2 \Big(\|u_x\|_{L^2(Q_k)}^2 + \|u\|_{L^2(Q_k)}^2 \Big) \, . \end{aligned}$$

Finally we get

$$\begin{aligned}\int_{\mathbb{R}^2} \left|[D_x^{-1} P_k u]_y\right|^2 dx dy &= \int_{\mathbb{R}^2} |\chi_k D_{x,k}^{-1} u_y|^2 \, dx dy \\ &\leq 2\, \|D_{x,k}^{-1} u_y\|^2_{L^2(Q_k)} .\end{aligned}$$

This shows that $P_k : X_k \to X$ is uniformly bounded. The second statement of the lemma is obvious. $\square$

Remark. Note that, for $u \in X_k$, $P_k u$ is zero on the boundary of Q_{k+1}. Extending this function perodically, we have $P_k u \in X_{k+1}$. Therefore, P_k may either be regarded as a continuous operator from X_k to X, or from X_k to X_{k+1}.

Now Lemma 1 and Lemma 2 easily imply

Lemma 3 *The embeddings $X_k \hookrightarrow L^p(Q_k)$ are continuous for $2 \leq p \leq 6$, where the embedding constants are uniformly bounded with respect to k, and $X_k \hookrightarrow L^p_{loc}(Q_k)$ is compact for $2 \leq p < 6$.*

To prove the existence of a nontrivial k-periodic solution, we need the following lemma, where $Q_r(\zeta)$ denotes the cube with edge length r, centered at the point $\zeta \in \mathbb{R}^2$.

Lemma 4 *If $u_n \in X_k$, $n = 1, 2, \ldots$ is a bounded sequence and there exists an $r > 0$ such that*

$$\lim_{n\to\infty} \sup_{\zeta \in Q_k} \int_{Q_r(\zeta)} |u_n|^2 dx dy = 0,$$

then $\|u_n\|_{L^s(Q_k)} \to 0$ for $s \in (2, 6)$.

Proof. By Hölder's inequality we have for $2 < s < 6$ and $\sigma = (s-2)/4$

$$\int_{Q_r(\zeta)} |u_n|^s \leq \left(\int_{Q_r(\zeta)} |u_n|^2\right)^{1-\sigma} \left(\int_{Q_r(\zeta)} |u_n|^6\right)^{\sigma} . \tag{6}$$

To proceed, we need a local version of the embedding theorem (Lemma 3), i. e. an estimate of the form

$$\|u\|_{L^6(Q_r(\zeta))} \leq C \|u\|_{k, Q_{r'}(\zeta)} . \tag{7}$$

Here $r' = r + \alpha$, where $\alpha \in (0, r/2)$ is arbitrarily small but fixed and $\|\cdot\|_{k,\Omega}$ is the norm in X_k restricted to the set Ω. Therefore, let $I_\alpha = [\xi - r - \alpha, \xi + r + \alpha] \subset [-k/2, k/2]$. First, given $u \in X_k$, we construct a function $u_1 \in X_k$ such that

$$u_1 = u \text{ on } [\xi - r, \xi + r] \times \mathbb{R}, \quad \mathrm{supp} u_1 \subset S := I_\alpha \times \mathbb{R}, \quad \|u_1\|_{k,S} \leq C \|u\|_{k,S} \tag{8}$$

with a constant $C > 0$ depending only on r and α. To this end assume without loss of generality that $u = \varphi_x$, with $\varphi \in C_k^\infty(\mathbb{R}^2)$, and denote $\psi = \varphi_y$. The continuous function

$$G(x) = \int_{-\infty}^{\infty} |\psi(x, y)|^2 dy, \quad x \in I_\alpha$$

attains its minimum at some point $x_0 \in I_\alpha$. We set

$$u_1 = [\chi(x) D^{-1}_{x,x_0} u]_x, \quad \text{where} \quad D^{-1}_{x,x_0} u(x,y) = \int_{x_0}^{x} u(s,y)ds,$$

and

$$\chi(x) = \begin{cases} 1 & \text{on} \quad [\xi - r, \xi + r] \\ 0 & \text{if} \quad |x-\xi| \geq r + \alpha \end{cases}$$

is a cut-off function, $|\chi| \leq 1$, $|\chi'|, |\chi''| \leq C_1$. For this function u_1 we obtain the following estimates. First,

$$\int_S |u_1|^2 dxdy \leq 2 \int_S \left(|\chi u|^2 + |\chi' D^{-1}_{x,x_0} u|^2 \right) dxdy.$$

The first term can be estimated by the L^2-norm of u on S and to estimate the second one, we denote $I_1 = [\xi - r - \alpha, \xi - r]$, $I_2 = [\xi + r, \xi + r + \alpha]$. We get

$$\begin{aligned} \int_S |\chi' D^{-1}_{x,x_0} u|^2 &\leq C_1 \int_{I_1 \cup I_2} \int_{\mathbb{R}} |\int_{x_0}^{x} u(s,y)ds|^2 dydx \\ &\leq C_1 2(r+\alpha) \int_{I_1 \cup I_2} \int_{\mathbb{R}} \int_{I_\alpha} |u(s,y)|^2 dsdydx \\ &\leq C_1 2(r+\alpha) 2\alpha \int_S |u(s,y)|^2 dsdy. \end{aligned}$$

In a similar way we can treat

$$\int_S |u_{1,x}|^2 dxdy = \int_S |\chi u + 2\chi' u + \chi'' D^{-1}_{x,x_0} u|^2 dxdy.$$

To estimate

$$\int_S |D^{-1}_{x,k} u_{1,y}|^2 dxdy = \int_S |\chi D^{-1}_{x,x_0} u_y|^2 dxdy\,,$$

we set $\psi_1 = D^{-1}_{x,x_0} u_y$. This function is related to $\psi = D^{-1}_{x,k} u_y$ by

$$\psi_1(x,y) = \psi(x,y) - \psi(x_0,y),$$

and we obtain

$$\begin{aligned} \int_S |\chi D^{-1}_{x,x_0} u_y|^2 dxdy &\leq \int_S |\psi_1|^2 dxdy \\ &\leq \int_S \left(|\psi(x,y)|^2 + 2|\psi(x,y)||\psi(x_0,y)| + |\psi(x_0,y)|^2 \right) dxdy. \end{aligned}$$

By our choice of x_0 we have

$$\int_S |\psi(x,y)|^2 dydx = \int_{I_\alpha} G(x)dx \geq \int_{I_\alpha} G(x_0)dx = \int_S |\psi(x_0,y)|^2 dxdy.$$

This shows that

$$\int_S |\chi D^{-1}_{x,x_0} u_y|^2 dxdy \leq 4 \int_S |\psi(x,y)|^2 dxdy = 4 \int_S |D^{-1}_{x,k} u_y|^2 dxdy\,.$$

Thus, we have proved that $\|u_1\|_{k,S} \leq C\|u\|_{k,S}$.

To finish the proof of (7), we also need a cut-off procedure in the $y-$direction. Given $Q_r(\zeta)$, with $\zeta = (\xi, z)$, we construct u_1 as above and define a smooth function $\bar{\chi}(y) \geq 0$ such that $\bar{\chi} = 1$ on $[z-r, z+r]$, $\bar{\chi} = 0$ outside of $[z-r-\alpha, z+r+\alpha]$ and $|\bar{\chi}'| \leq C_2$. Set

$$u_2(x, y) = \bar{\chi}(y) u_1(x, y).$$

Then

$$\int_{Q_{r'}(\zeta)} |D_{x,k}^{-1} u_{2,y}|^2 \leq 2 \int_{Q_{r'}(\zeta)} \left[|\bar{\chi}(y) D_{x,k}^{-1} u_{1,y}|^2 + |\bar{\chi}'(y) D_{x,k}^{-1} u_1|^2 \right] dxdy\,.$$

Here, the first term on the right is clear and, since u_1 has compact support in x, the second one can be estimated by

$$\begin{aligned} \int_{Q_{r'}(\zeta)} |\bar{\chi}'(y) D_{x,k}^{-1} u_1|^2 dxdy &\leq C_2 k \int_{Q_{r'}(\zeta)} \int_{\zeta-r-\alpha}^{\zeta+r+\alpha} |u_1(s, y)|^2 \, dsdxdy \\ &\leq C_2 k \, 2(r+\alpha) \, \|u_1\|^2_{L^2(Q_{r'}(\zeta))} \end{aligned}$$

Since the estimates $\|u_2\|^2_{L^2(Q_{r'}(\zeta))} \leq \|u_1\|^2_{L^2(Q_{r'}(\zeta))}$ and $\|u_{2,x}\|^2_{L^2(Q_{r'}(\zeta))} \leq \|u_{1,x}\|^2_{L^2(Q_{r'}(\zeta))}$ are obvious, we claim that $\|u_2\|^2_{k,Q_{r'}(\zeta)} \leq C_3 \|u_1\|^2_{k,Q_{r'}(\zeta)}$. By construction, we have $u_2 = u$ on $Q_{r'}(\zeta)$ and $u_2 \in X_k$. Thus, Lemma 3 and the estimates above imply

$$\|u\|_{L^6(Q_r)} = \|u_2\|_{L^6(Q_r)} \leq C' \|u_2\|_{k,Q_r'(\zeta)} \leq C \|u\|_{k,Q_r'(\zeta)}\,.$$

Now we can fix α and cover Q_k by a countable number of cubes $Q_r(\zeta)$ such that each point of Q_k is contained in at most three of the cubes $Q_{r'}(\zeta)$, $r' = r + \alpha$. Using (6) and (7), we get

$$\int_{Q_r(\zeta)} |u_n|^s \leq C \left(\int_{Q_r(\zeta)} |u_n|^2 \right)^{1-\sigma} \left(\int_{Q_{r'}(\zeta)} u_{n,x}^2 + (D_{x,k}^{-1} u_{n,y})^2 + c u_n^2 \right)^{3\sigma}.$$

For $s \geq 10/3 =: s_1$ we have $3\sigma - 1 \geq 0$. Hence,

$$\left(\int_{Q_{r'}(\zeta)} u_{n,x}^2 + (D_{x,k}^{-1} u_{n,y})^2 + c u_n^2 \right)^{3\sigma} \leq \|u_n\|_k^{2(3\sigma-1)} \|u_n\|^2_{k,Q_{r'}(\zeta)}.$$

This implies

$$\|u_n\|^s_{L^s(Q_r(\zeta))} \leq C \|u_n\|^{2(1-\sigma)}_{L^2(Q_r(\zeta))} \|u_n\|_k^{2(3\sigma-1)} \|u_n\|^2_{k,Q_{r'}(\zeta)}, \quad \text{if} \quad s \geq s_1.$$

Now we can take the sum over all these cubes and get

$$\begin{aligned} \|u_n\|^s_{L^s(Q_k)} &\leq C \, \|u_n\|_k^{2(3\sigma-1)} \sum_{\zeta} \left(\|u_n\|^{2(1-\sigma)}_{L^2(Q_r(\zeta))} \, \|u_n\|^2_{k,Q_{r'}(\zeta)} \right) \\ &\leq 3C \, \|u_n\|_k^{6\sigma} \, \sup_{\zeta} \|u_n\|^{2(1-\sigma)}_{L^2(Q_r(\zeta))}\,. \end{aligned}$$

By our assumptions we claim that $\|u_n\|_{L^s(Q_k)} \to 0$ for $s \in [s_1, 6)$.
For $s \in (2, s_1)$, we set $s = 2\lambda + s_1(1-\lambda)$. Then by Hölder's inequality we have

$$\|u\|^s_{L^s(Q_k)} \leq \|u\|^{2\lambda}_{L^2(Q_k)} \|u\|^{(1-\lambda)s_1}_{L^{s_1}(Q_k)}.$$

Since the statement of Lemma 4 is true for s_1, we conclude. □

Remark. This lemma and its proof remain true if we replace X_k by X, Q_k by $I\!R^2$ and $D^{-1}_{x.k}$ by D^{-1}_x. Then we obtain the assertion of Lemma 7.4 in [14].

Theorem 1 *For each k there exists a nontrivial weak solution u_k of (4) in X_k and the sequence $c_k = J(u_k)$ is bounded from below and above by positive constants not depending on k.*

Proof. From Assumption 2^o we find constants C_1, C_2 such that

$$\begin{aligned} J_k(u) &\geq \frac{1}{2}\|u\|_k^2 - \int_{Q_k} \frac{1}{4}\,|u|^2 + C_1|u|^p\,dxdy \\ &\geq \frac{1}{4}\|u\|_k^2 - C_2\|u\|_k^p \end{aligned}$$

holds for every $u \in X_k$. Since $p > 2$, we claim that $J_k(u) \geq \delta > 0$ if $\|u\|_k = \rho$ is small enough and δ does not depend on k.
Without loss of generality we may now assume that φ from Assumption 3^o has compact support in Q_1. We claim that $\eta = D_x\varphi$ satisfies

$$J_1(\lambda\eta) = \frac{\lambda^2}{2}\,\|\eta\|_k^2 - \int_{Q_1} F(\lambda\eta)dxdy \to -\infty \qquad \text{as} \quad \lambda \to \infty$$

and we fix λ_0 such that $e_1 = \lambda_0\eta$ satisfies $\|e_1\|_k \geq \rho$ and $J_1(e_1) \leq 0$. For $k > 1$ we define

$$\varphi_k(x,y) = \begin{cases} \varphi & , \quad (x,y) \in Q_1 \\ 0 & , \quad (x,y) \in Q_k \setminus Q_1 \end{cases} .$$

Extending φ_k periodically, we have $\eta_k = D_x\varphi_k \in X_k$ and

$$\|\lambda_0\eta_k\|_k = \|e_1\|_k > \rho \qquad \text{and} \qquad J_k(\lambda_0\eta_k) = J_1(e_1) \leq 0\,.$$

We set $e_k = \lambda_0\eta_k$ and define

$$c_k = \inf_{\gamma\in\Gamma_k}\max_{t\in[0,1]} J_k(\gamma(t))\,, \quad \Gamma_k = \{\gamma \in C([0,1], X_k) \mid \gamma(0) = 0, \gamma(1) = e_k\}\,.$$

Now we want to show that c_k is a critical value of J_k. Note that, according to our choice of e_k, the set $\{te_k \mid t \in [0,1]\}$ belongs to Γ_k and

$$0 < \delta \leq c_k \leq \max_{t\in[0,1]} J_k(te_k) = \max_{t\in[0,1]} J_k(te_1),$$

which shows that c_k are uniformly bounded from above and below by positive constants.

The Mountain Pass Lemma without Palais-Smale condition (cf. [14], Thm 2.8) now shows that there is a sequence $u_{k,n} \in X_k$, $n \in I\!N$, such that

$$J_k'(u_{k,n}) \to 0, \quad J_k(u_{k,n}) \to c_k \qquad \text{as} \quad n \to \infty \,.$$

For n large enough we obtain from Assumption 4^o that

$$(\frac{1}{2} - \frac{1}{\mu})\|u_{k,n}\|_k^2 \le J_k(u_{k,n}) - \frac{1}{\mu}\langle J_k'(u_{k,n}), u_{k,n}\rangle \le c_k + 1 + \|u_{k,n}\|_k$$

and we claim that the sequence $u_{k,n}$ is bounded. Consequently, it has a weakly convergent subsequence. Now we use Lemma 4 to show that this sequence has a nontrivial limit.

Assume on the contrary that there exists an $r > 0$ such that

$$\sup_{\zeta \in Q_k} \int_{Q_r(\zeta)} |u_{k,n}|^2 dxdy \to 0\,.$$

Then $\|u_{k,n}\|_{L^s(Q_k)} \to 0$ for $s \in (2,6)$ and there is a sequence $\varepsilon_n \to 0$ such that

$$\begin{aligned} c_k &= J_k(u_{k,n}) - \frac{1}{2}\langle J_k'(u_{k,n}), u_{k,n}\rangle + \varepsilon_n \\ &= \int_{Q_k} \left[\frac{1}{2} f(u_{k,n})u_{k,n} - F(u_{k,n})\right] dxdy + \varepsilon_n \\ &\le \varepsilon\|u_{k,n}\|_{L^2(Q_k)}^2 + C_\varepsilon \|u_{k,n}\|_{L^p(Q_k)}^p + \varepsilon_n\,. \end{aligned}$$

Since $c_k \ge \delta > 0$ and the right hand side can be made arbitrary small as $n \to \infty$, this is a contradiction.

Consequently, there is a sequence $(x_n, y_n) \in I\!R^2$ such that, along a subsequence,

$$\int_{Q_1(x_n,y_n)} |u_{k,n}|^2 dxdy \ge d > 0 \qquad \text{for all} \quad n.$$

Then the shifted functions $\tilde{u}_n(x,y) = u_{k,n}(x + x_n, y + y_n)$ satisfy $\int_{Q_1(0)} |\tilde{u}_n|^2 dxdy \ge d$ and this sequence converges to $u_k \in X_k$ weakly in X_k and strongly in $L^p_{loc}(Q_k)$. Clearly, $u_k \ne 0$. Now for every function $\eta \in Y_k$ we have

$$\langle J_k'(u_k), \eta\rangle = \lim_{n\to\infty} \langle J_k'(\tilde{u}_n), \eta\rangle = 0,$$

which shows that u_k is a nontrivial solution of (4). □

2 Passage to the Limit

First we prove some a priori estimates which are valid for all nontrivial critical points of J_k.

Lemma 5 *Any critical point u_k of J_k satisfies the estimate $\|u_k\|_k \le C_k$ with a constant $C_k > 0$ only depending on the critical value.*

Proof. From $c_k = J_k(u_k)$ and $J_k'(u_k) = 0$ we obtain

$$\begin{aligned} c_k &= J_k(u_k) - \frac{1}{2}\langle J_k'(u_k), u_k\rangle \\ &\ge (\frac{\mu}{2} - 1)\int_{Q_k} F(u_k)dxdy\,. \end{aligned} \tag{9}$$

This implies

$$\begin{aligned} \|u_k\|_k^2 &= 2\left(J_k(u_k) + \int_{Q_k} F(u_k)dxdy\right) \\ &\le 2\left(c_k + \frac{2}{\mu-2}c_k\right) = \frac{2\mu}{\mu-2}c_k\,. \end{aligned}$$

□

Lemma 6 *For every nontrivial $u_k \in X_k$ and $u \in X$ which satisfy $\langle J_k' u_k, u_k\rangle = 0$ and $\langle J'u, u\rangle = 0$, respectively, the following holds: There exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ not depending on k such that $\|u_k\|_k \ge \varepsilon_1$, $\|u\| \ge \varepsilon_1$, $J_k(u_k) \ge \varepsilon_2$ and $J(u) \ge \varepsilon_2$.*

Proof. By Assumption 2^o, given $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned} \|u\|_k^2 &= \int_{Q_k} f(u_k)u_k\,dxdy \\ &\le \varepsilon\|u\|_k^2 + C_\varepsilon\|u\|_k^p\,, \end{aligned}$$

which shows $\|u_k\|_k \ge [(1-\varepsilon)/C_\varepsilon]^{1/(p-2)}$ (where C_ε depends on the embedding constants for $X_k \to L^p(Q_k)$).

For $J_k(u_k)$ we get, according to Assumption 4^o,

$$\begin{aligned} J_k(u_k) &\ge \frac{1}{2}\|u_k\|_k^2 - \frac{1}{\mu}\int_{Q_k} f(u_k)u_k\,dxdy \\ &= \frac{\mu-2}{2\mu}\|u_k\|_k^2\,. \end{aligned}$$

This, together with the first estimate, gives the desired lower bound. Clearly, the arguments for $u \in X$ are the same. □

Now to show that u_k converges to a nontrivial solution we need the following:

Lemma 7 *Let $u_k \in X_k$ be a sequence of k-periodic functions such that $\|u_k\|_k \le C$. Let $Q_r(\zeta)$ be the closed r-cube centered at the point ζ and assume that there is an $r > 0$ such that*

$$\lim_{k\to\infty}\left(\sup_{\zeta\in\mathbb{R}^2}\int_{K_r(\zeta)} |u_k|^2 dxdy\right) = 0.$$

Then $\|u_k\|_{L^s(Q_k)} \to 0$ for $s \in (2,6)$.

Proof. Without any restriction we may assume that $r < 1$. First, we want to show that

$$\sup_\zeta \int_{Q_r(\zeta)} |P_k u_k|^2 \leq C \sup_\zeta \int_{Q_r(\zeta)} |u_k|^2, \tag{10}$$

where P_k is the cut-off operator from Lemma 2. By periodicity, we may assume that the supremum is taken over all ζ such that either $Q_r(\zeta) \subset Q_k$, or $Q_r(\zeta) \subset Q_{k+1} \setminus Q_k$. If $Q_r(\zeta) \subset Q_k$, there is nothing to do, so assume $Q_r(\zeta) \subset Q_{k+1} \setminus Q_k$. We have

$$\int_{Q_r(\zeta)} |P_k u(x,y)|^2 dxdy \leq 2 \int_{Q_r(\zeta)} \left[|\chi_k(x) u(x,y)|^2 + |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 \right] dxdy\,. \tag{11}$$

As in Lemma 2, let $I_1 = [-(k+1)/2, -k/2]$, $I_2 = [k/2, (k+1)/2]$. Choose $\zeta = (\zeta_1, \zeta_2)$ such that $\zeta_1 \in I_1$ and $[\zeta_1 - r/2, \zeta_1 + r/2] \subset I_1$. The first term on the right in (11) is easily estimated, since

$$\int_{Q_r(\zeta)} |\chi_k(x) u(x,y)|^2 dxdy \leq \int_{Q_r(\zeta)} |u(x,y)|^2 dxdy = \int_{Q_r(\zeta_1 + k, \zeta_2)} |u(x,y)|^2 dxdy$$

and $Q_r(\zeta_1 + k, \zeta_2) \subset Q_k$. For the second term in (11) we have $|D_{x,k}^{-1} u(x,y)|^2 \leq \int_{I_1} |u(s,y)|^2\, ds$ for $x \in I_1$. Therefore,

$$\begin{aligned}\int_{Q_r(\zeta)} |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 dxdy &\leq \int_{\zeta_1 - r/2}^{\zeta_1 + r/2} \int_{\zeta_2 - r/2}^{\zeta_2 + r/2} |\chi_k'(x)|^2 \int_{I_1} |u(s,y)|^2\, dsdydx \\ &\leq C_0 r \int_{\zeta_2 - r/2}^{\zeta_2 + r/2} \int_{-(k+1)/2}^{-k/2} |u(s,y)|^2\, dsdy\,.\end{aligned}$$

Since r is fixed, there is a finite number of points $\zeta_1^i, i = 1, \ldots, l$, such that I_1 is the union of l intervals $[\zeta_1^i - r/2, \zeta_1^i + r/2]$ and l depends only on r. This finally shows (set $\zeta^i = (\zeta_1^i, \zeta_2)$) that

$$\int_{Q_r(\zeta)} |\chi_k'(x) D_{x,k}^{-1} u(x,y)|^2 dxdy \leq C_0 r \sum_i \int_{Q_r(\zeta^i)} |u|^2 dxdy = C_0 r \sum_i \int_{Q_r(\zeta_1^i + k, \zeta_2)} |u|^2 dxdy,$$

where $Q_r(\zeta_1^i + k, \zeta_2) \subset Q_k$. Clearly, the arguments for $\zeta_1 \in I_2$ are the same and we can estimate the term $\int_{Q_r(\zeta)} |P_k u_k|^2$ in (10) by a finite sum (the number of summands does not depend on k) of integrals of the type $\int_{Q_r(\tilde\zeta)} |u_k|^2$, where $Q_r(\tilde\zeta) \subset Q_k$.

From the remark following Lemma 4 we claim that $P_k u_k \to 0$ in $L^s(I\!R^2)$ for $2 < s < 6$. Since $u_k = P_k u_k$ on Q_k, the assertion follows. □

Lemma 8 *If a sequence $u_k \in X_k$ is uniformly bounded in the $\|\cdot\|_k$-norm and satisfies $J_k'(u_k) \to 0$, then the following alternative holds: Either*

(i) $\|u_k\|_k \to 0$ *as* $k \to \infty$ *; or*

(ii) there exist $r, \eta > 0$ *and a sequence of points* $\zeta_k \in I\!R^2$ *such that, along a subsequence,*

$$\lim_{k \to \infty} \left(\int_{K_r(\zeta_k)} |u_k|^2 dxdy \right) \geq \eta\,.$$

Proof. Assume that (ii) does not hold. Then from Lemma 7 we get $\|u_k\|_{L^s(Q_k)} \to 0$ for $2 < s < 6$. Since $\langle J_k' u_k, u_k\rangle = \varepsilon_k \|u_k\|_k$ with $\varepsilon_k \to 0$, we obtain from Assumption 2^o that

$$\|u_k\|_k^2 \leq \varepsilon \|u_k\|_k^2 + C_\varepsilon \|u_k\|_{L^p(Q_k)}^p + \varepsilon_k \|u_k\|_k \,.$$

Choosing $\varepsilon < 1$, we get $\|u_k\|_k \to 0$. □

Theorem 2 *Let $u_k \in X_k$ be a sequence of nontrivial solutions of (4) such that the sequence $J_k(u_k)$ is bounded from above. Then there exists a sequence $\zeta_k \in I\!R^2$ and a function $u \in X$, such that $P_k u_k(\cdot + \zeta_k)$ converges weakly to u along a subsequence, and u is a nontrivial solution of (3).*

Proof. From Lemmas 5 and 6 we have $\|u_k\|_k \leq C_1$ and $\|u_k\|_k \geq \varepsilon_1$ with constants not depending on k. Thus, case (i) of Lemma 8 is not possible and from (ii) we obtain a sequence $\zeta_k \in I\!R^2$ such that the shifted functions $\tilde{u}_k = u_k(\cdot + \zeta_k)$ satisfy (for large k)

$$\int_{Q_r(0)} |\tilde{u}_k|^2 dx dy \geq \eta/2 \,,$$

with appropriate $r, \eta > 0$, and $\tilde{u}_k$ are critical points of J_k. Since the sequence $P_k \tilde{u}_k$ is bounded in X, there exists a subsequence which converges weakly in X to a nontrivial function $u \in X$. Since the embedding $X \to L^p(\Omega)$ is compact for any bounded domain $\Omega \subset I\!R^2$, we claim that $f(\tilde{u}_k) \to f(u)$ strongly in $L^{p/(p-1)}(\Omega)$. Now for every test function $\psi = \varphi_x \in Y$ with $\operatorname{supp} \varphi \subset \Omega$ we claim that, for k so large that $\Omega \subset Q_k$,

$$\begin{aligned} (P_k\tilde{u}_k, \psi)_X &= \int_\Omega (P_k\tilde{u}_k)_x \psi_x + D_x^{-1}(P_k\tilde{u}_k)_y D_x^{-1}\psi_y + c(P_k\tilde{u}_k)\psi \\ &= \int_{Q_k} \tilde{u}_{k,x}\psi_x + D_{x,k}^{-1}\tilde{u}_{k,y} D_{x,k}^{-1}\psi_y + c\tilde{u}_k\psi \;=\; (\tilde{u}_k, \psi)_k \,. \end{aligned} \tag{12}$$

This shows that $(\tilde{u}_k, \psi)_k \to (u, \psi)_X$ and, since for large k, ψ is also a member of Y_k, we see that

$$\begin{aligned} \langle J'(u), \psi\rangle &= (u,\psi)_X - \int_\Omega f(u)\psi \\ &= \lim_{k\to\infty} \left((\tilde{u}_k, \psi)_k - \int_\Omega f(\tilde{u}_k)\psi \right) \\ &= \lim_{k\to\infty} \langle J_k'(\tilde{u}_k), \psi\rangle \;=\; 0 \end{aligned}$$

for all $\psi \in Y$, i. e. u is a nontrivial weak solution of (3). □

3 A general convergence result

The following theorem gives a more general connection between periodic and finite energy solutions of (3). For this we have to assume a little bit more regularity for f, namely

$$|f(u+v) - f(u)| \leq C(1 + |u|^{p-2} + |v|^{p-2})|v| \quad \text{for all } v. \tag{13}$$

Theorem 3 *Let $u_k \in X_k$ be a sequence such that $J_k'(u_k) \to 0$, $c_k = J_k(u_k) \to c > 0$ as $k \to \infty$ and $\|u_k\|_k \le M$ (independent of k). Then there exist critical points u^i, $i = 1, \dots, \ell$, of J and squences $\zeta_k^i \in I\!R^2$, such that*

$$\|u_k - \sum_{i=1}^{\ell} u^i(\cdot + \zeta_k^i)\|_k \to 0 \qquad \textit{and} \qquad \sum_{i=1}^{\ell} J(u^i) = c\,.$$

Proof. We fix the following notations. Lemma 8 and the proof of Theorem 2 show that there exist $\zeta_k \in I\!R^2$ such that $\tilde{u}_k = u_k(\cdot + \zeta_k)$ satisfies $P_k\tilde{u}_k \rightharpoonup u$ weakly in X and strongly in $L^p_{loc}(I\!R^2)$, and u is a nontrivial solution of (3). Since Y is dense in X, we find a sequence $w_k = \varphi_{k,x} \in Y$, $\varphi_k \in C_0^\infty(I\!R^2)$ such that $w_k \to u$ strongly in X and we may assume that $\operatorname{supp} w_k \subset Q_k$. Again w_k may also be viewed as a member of Y_k. Then we define

$$v_k = \tilde{u}_k - w_k \in X_k\,.$$

Obviously,

$$\|\tilde{u}_k - u\|_k \to 0 \quad \text{iff} \quad \|v_k\|_k \to 0\,. \tag{14}$$

Furthermore, for any $\varepsilon > 0$ there exists $\Omega \subset I\!R^2$ and $k_\varepsilon \in I\!N$ such that, for $k \ge k_\varepsilon$, we have

$$\|w_k\|_{X(I\!R^2\setminus\Omega)} := \int_{I\!R^2\setminus\Omega} w_{k,x}^2 + (D_x^{-1}w_{k,y})^2 + cw_k^2\,dxdy \quad \le \quad \varepsilon\,, \tag{15}$$

$$\int_{I\!R^2\setminus\Omega} F(w_k)\,dxdy \quad \le \quad \varepsilon\,. \tag{16}$$

First we shall prove the following:

(i) $J_k'(v_k) \to 0$ as $k \to \infty$.

(ii) $J_k(v_k) \to c - J(u)$ as $k \to \infty$.

(i) Since $J \in C^1$, we have $J'(w_k) \to 0$ and $J_k(w_k) = J(w_k) \to J(u)$. Consequently, for every function $\psi_k \in X_k$ we obtain

$$\langle J_k'(w_k), \psi_k\rangle = \langle J'(w_k), P_k\psi_k\rangle \le \varepsilon_k\|P_k\psi_k\| \le \varepsilon_k C\|\psi_k\|_k\,,$$

where C is independent of k and $\varepsilon_k \to 0$. This shows that $J_k'(w_k) \to 0$. For v_k we have

$$\begin{aligned}\langle J_k'(v_k), \psi_k\rangle \quad &= \quad \langle J_k'(\tilde{u}_k), \psi_k\rangle - \langle J_k'(w_k), \psi_k\rangle \\ &\quad + \int_{Q_k} [f(\tilde{u}_k) - f(w_k) - f(\tilde{u}_k - w_k)]\,\psi_k\,dxdy\,.\end{aligned} \tag{17}$$

Since $\tilde{u}_k \to u$ locally in L^p, we get, for any bounded domain $\Omega \subset I\!R^2$ and k large enough,

$$\left|\int_\Omega [f(\tilde{u}_k) - f(w_k) - f(\tilde{u}_k - w_k)]\,\psi_k\,dxdy\right| \le \varepsilon\|\psi_k\|_k\,.$$

From (15) we also have

$$\left| \int_{Q_k \setminus \Omega} f(w_k)\psi_k \, dxdy \right| \leq \varepsilon \|\psi_k\|_k$$

for large k and appropriate Ω. From (13) and Hölder's inequality we obtain

$$\int_{Q_k \setminus \Omega} |f(v_k - w_k) - f(v_k)| \, |\psi_k| \, dxdy \leq C \int_{Q_k \setminus \Omega} (1 + |v_k|^{p-2} + |w_k|^{p-2}) |w_k| \, |\psi_k| dxdy$$

$$\begin{aligned} &\leq C \left(\|\psi_k\|_k \left[\|w_k\|_{X(I\!R^2 \setminus \Omega)} + \|w_k\|^{p-1}_{X(I\!R^2 \setminus \Omega)} \right] + \|\psi_k\|_k^{2/p} \|w_k\|^{2/p}_{X(I\!R^2 \setminus \Omega)} \|v_k\|^{p-2}_{X(I\!R^2 \setminus \Omega)} \right) \\ &\leq \varepsilon \left(\|\psi_k\|_k + \|\psi_k\|_k^{2/p} \right) . \end{aligned}$$

Thus (17) implies (i).

(ii) We have

$$\begin{aligned} J_k(\tilde{u}_k) &= J_k(v_k) + J_k(w_k) + (v_k, w_k)_k \\ &\quad - \int_{Q_k} [F(v_k + w_k) - F(v_k) - F(w_k)] \, dxdy \, . \end{aligned} \tag{18}$$

For $\Omega \subset I\!R^2$ and k large enough, we get as in (12) (with $\tilde{u}_k$ replaced by v_k and ψ replaced by w_k)

$$\begin{aligned} |(v_k, w_k)_k| &\leq \left| \int_\Omega \left[(P_k v_k)_x w_{k,x} + D_x^{-1}(P_k v_k)_y D_x^{-1} w_{k,y} + c(P_k v_k) w_k \right] dxdy \right| \\ &\quad + C \, \|v_k\|_k \, \|w_k\|_{X(I\!R^2 \setminus \Omega)} \, . \end{aligned}$$

Since $P_k v_k = P_k \tilde{u}_k - w_k$ tends weakly to 0 and $w_k \to u$ stongly in X, we claim from (15) that the right hand side tends to zero as $k \to \infty$. For the integral in (18), we get

$$\left| \int_\Omega [F(v_k + w_k) - F(v_k) - F(w_k)] \, dxdy \right| \leq \varepsilon$$

if k is sufficiently large, while by the mean value theorem and Hölder's inequality we have

$$\begin{aligned} &\left| \int_{Q_k \setminus \Omega} [F(v_k + w_k) - F(v_k)] \, dxdy \right| \\ &\leq \int_{Q_k \setminus \Omega} \left[|v_k| + |w_k| + C(|v_k| + |w_k|)^{p-1} \right] \, |w_k| \, dxdy \\ &\leq C \left[\|v_k\|_k + \|w_k\|_k + \|v_k\|_k^{p-1} + \|w_k\|_k^{p-1} \right] \|w_k\|_{X(I\!R^N \setminus \Omega)} \\ &\leq \varepsilon \, . \end{aligned}$$

This, together with (16) and (18), implies

$$c = \lim_{k \to \infty} J_k(\tilde{u}_k) = \lim_{k \to \infty} J_k(v_k) + J(u) \, .$$

Now if $\|v_k\|_k \to 0$, we also have $J_k(v_k) \to 0$, while if $\|v_k\|_k \geq \varepsilon > 0$, we claim from $J'_k(v_k) \to 0$, as in Lemma 6, that $J_k(v_k) \geq \varepsilon' > 0$. In both cases we obtain

$$J_k(v_k) \to c - J(u) \geq 0, \quad \text{i. e.} \quad J(u) \leq c\,.$$

If $J(u) = c$, we claim that $\|v_k\|_k \to 0$, since otherwise we can use the same arguments as above for v_k instead of u_k and obtain a nontrivial solution v of (3) with $J(v) \leq 0$, which is not possible in view of Lemma 6. If $J(u) < c$, then Lemma 6 shows $J(u) \geq \varepsilon_2 > 0$ and we can repeat the arguments above with u_k replaced by v_k and c replaced by $c - J(u) \leq c - \varepsilon_2$. After a finite number of steps we are done. □

4 Ground States

In the following we use another variational approach to construct periodic traveling waves. Instead of 4^o we now assume that f satisfies

5^o There exists $\theta \in (0,1)$ such that $0 \leq u^{-1} f(u) \leq \theta f_u(u)$ for all $u \neq 0$.

Let

$$I_k(u) = \langle J'_k(u), u\rangle = \|u\|_k^2 - \int_{Q_k} f(u)u\,dxdy.$$

Then a solution of (4) with least energy is a solution of the following Nehari minimization problem:

$$m_k = \inf\{J_k(u) \mid u \in S_k\}, \quad S_k = \{u \in X_k \mid u \neq 0,\ I_k(u) = 0\}. \tag{19}$$

Note that in this setting we have

$$J_k(u) = \int_{Q_k} \frac{1}{2} f(u)u - F(u)\,dxdy \qquad \text{for all} \qquad u \in S_k\,. \tag{20}$$

In particular, Lemma 6 shows that $m_k \geq \varepsilon_2$ for all k. From Assumption 5^o we derive the following estimates:

$$0 \ \leq\ \frac{1+\theta}{\theta} F(u) \ \leq\ f(u)u \quad \text{with} \quad \frac{1+\theta}{\theta} > 2, \tag{21}$$

$$f(\tau u) \begin{cases} \geq & \tau^{1/\theta} f(u)u \quad \text{if } \tau \geq 1 \\ \leq & \tau^{1/\theta} f(u)u \quad \text{if } \tau \leq 1 \end{cases}. \tag{22}$$

Theorem 4 *Assume that f satisfies $1^0 - 3^0$, 5^0. Then there exists a nontrivial minimizer for (19) which is a solution of (4).*

Proof. Let $u_{k,n} \in S_k$ be a minimizing sequence. From Lemma 6 we obtain $\|u_{k,n}\|_k \geq \varepsilon_1$ and from (20) and (21) we get

$$J_k(u_{k,n}) \geq \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right) \int_{Q_k} f(u_{k,n})u_{k,n}\,dxdy = \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right) \|u_{k,n}\|_k^2,$$

which shows that $\|u_{k,n}\|_k$ is bounded. To show that there is a convergent subsequence, with the limit $u_k \in S_k$, we use the concentration-compactness lemma of Lions [7], applied to the sequence $\rho_n = (D_x u_{k,n})^2 + (D_{x,k}^{-1} D_y u_{k,n})^2 + u_{k,n}^2$. Since the arguments here are similar to the case of semilinear elliptic equations which were treated in [7], p. 266 and in [9], we only sketch the rest of the proof. First, by using Lemma 4, the "vanishing" case is excluded. To exclude "dichotomy", one shows that

$$m_k < m_{k,\alpha} := \inf\{J_k(u) - \frac{1}{2} I_k(u) \mid I_k(u) = \alpha\} \qquad \text{for all} \quad \alpha < 0\,.$$

Now if dichotomy would occur, i. e. $u_{k,n}$ splits into a sum $u_{k,n}^1 + u_{k,n}^2$ and the distance of the supports of these functions tends to ∞, then one shows that $I_k(u_{k,n}^1) \to \alpha$, $I_k(u_{k,n}^2) \to -\alpha$, $\alpha \in I\!R$ and $m_k \geq m_{k,\alpha} + m_{k,-\alpha} > m_k$, which is a contradiction.

Therefore, the sequence $u_{k,n}$ concentrates and the limit satisfies $I_k(u_k) \leq 0$. The case $I_k(u_k) < 0$ can be excluded by the same reasoning as above and we obtain $u_k \in S_k$, a minimizer for (19).

Now since u_k is a minimizer for (19), there exists a Lagrange multiplier λ, such that $\lambda I_k'(u_k) = J_k'(u_k)$. Since $\langle J_k'(u_k), u_k\rangle = 0$ and

$$\begin{aligned}
\langle I_k'(u_k), u_k\rangle &= \|u_k\|_k^2 - \int_{Q_k} f_u(u_k) u_k \, dxdy \\
&\leq \|u_k\|_k^2 - \frac{1}{\theta}\int_{Q_k} f(u_k) u_k \, dxdy \\
&= \frac{\theta - 1}{\theta}\|u_k\|_k^2 \; < \; 0,
\end{aligned}$$

we claim that $\lambda = 0$, i. e. u_k is a solution of (4). □

Theorem 5 *Let u_k be the sequence of periodic minimizers of Theorem 4. Then there exists a nontrivial minimal energy solution $u \in X$ of (3) and a sequence of vectors $\zeta_k \in I\!R^2$ such that a subsequence of $P_k u_k(\cdot + \zeta_k)$ converges weakly to u.*
If in addition f satisfies (13), then we have

$$\lim_{k\to\infty} \|u_k - u(\cdot + \zeta_k)\|_k = 0\,. \tag{23}$$

Proof. The existence of a nontrivial solution can be proved with the same arguments as in the proof of Theorem 2. We only have to find a uniform bound for $\|u_k\|_k$. Let $\eta = D_x\varphi$, where $\varphi \in C_0^\infty(Q_1)$ and $\varphi \neq 0$. In view of (22) we can find $\tau > 0$ such that $v_1 := \tau\eta \in S_1$. For $k \geq 2$ we set

$$v_k(x,y) = \begin{cases} v_1(x,y) & , \quad |x| < 1/2 \\ 0 & , \quad |x| \geq 1/2 \end{cases}.$$

Then $v_k \in S_k$ and we obtain

$$J_k(u_k) \leq J_k(v_k) = J_1(v_1) \qquad \text{for all} \qquad k \geq 1\,.$$

Now the uniform bound for u_k follows from Lemma 5 (note, that we have only used the fact that $\langle J_k'(u_k), u_k\rangle = 0$ there). As in Theorem 2 we claim that a subsequence of $P_k u_k(\cdot + \zeta_k)$ converges weakly to a nontrivial solution u of (3).

Now we want to show that this solution is actually a ground state, i. e. it is a minimizer for

$$m = \inf\{J(u) \mid u \in S\}, \quad S = \{u \in X \mid u \neq 0,\ I(u) = 0\}. \tag{24}$$

To show this, we first prove the following: For every $v \in S$ and every $\varepsilon > 0$ there exist k_ε and $v_k \in S_k$ such that

$$J_k(v_k) \leq J(v) + \varepsilon \qquad \text{for all} \qquad k \geq k_\varepsilon. \tag{25}$$

Fix $v \in S$. Since J and I are continuous on X, there is a sequence $\eta_k = D_x\varphi_k$, where $\varphi_k \in C_0^\infty(Q_k)$, such that $\eta_k \to v$ in the norm $\|\cdot\|_X$ and

$$J(\eta_k) \to J(v), \qquad I(\eta_k) \to I(v) = 0\,.$$

By (22) we claim that there exists $\tau_k > 0$ such that $I(\tau_k\eta_k) = 0$. Moreover, we have the following estimates for τ_k:

$$\text{If} \qquad I(\eta_k) = \delta_k \leq 0, \qquad \text{then} \qquad 1 \geq \tau_k \geq \left(1 + \frac{\delta_k}{\int_{\mathbb{R}^2} f(\eta_k)\eta_k\,dxdy}\right)^{\frac{\theta}{1-\theta}},$$

$$\text{If} \qquad I(\eta_k) = \delta_k \geq 0, \qquad \text{then} \qquad 1 \leq \tau_k \leq \left(1 + \frac{\delta_k}{\int_{\mathbb{R}^2} f(\eta_k)\eta_k\,dxdy}\right)^{\frac{\theta}{1-\theta}}.$$

From Lemma 6 we see that $v \in S$ satisfies the estimate

$$\int_{\mathbb{R}^2} f(v)v\,dxdy = \|v\|_X \geq \varepsilon_1 > 0\,.$$

Consequently, for k large enough we claim that

$$\int_{\mathbb{R}^2} f(\eta_k)\eta_k\,dxdy \geq \varepsilon_1/2\,.$$

Since $\delta_k \to 0$, we obtain

$$\tau_k \to 1 \qquad \text{and} \qquad J(\tau_k\eta_k) \to J(v) \qquad \text{as} \quad k \to \infty\,.$$

Now given $\varepsilon > 0$ we can choose k_ε large enough so that $v_k := \tau_k\eta_k \in S_k$ and $J_k(v_k) = J(v_k) \leq J(u) + \varepsilon$ for all $k \geq k_\varepsilon$, which proves (25).

In particular, this implies that $\limsup\, m_k \leq J(v) + \varepsilon$ for any $v \in S$ and any $\varepsilon > 0$. Hence,

$$\limsup\, m_k \leq m\,.$$

Now (21) implies that the function $g(v) := [\frac{1}{2}f(v)v - F(v)]$ is nonnegative for every v. Moreover we have

$$J_k(v_k) = \int_{Q_k} g(v_k)\,dxdy \quad \text{for every} \quad v_k \in S_k$$

and, consequently,

$$m_k = J_k(\tilde{u}_k) = \int_{Q_k} g(\tilde{u}_k)\, dxdy\,.$$

Since the integrand is nonnegative, we claim that for any bounded $\Omega \subset I\!R^2$ and k large enough

$$m_k \geq \int_\Omega g(\tilde{u}_k)\, dxdy\ .$$

Passing to the limit, we obtain

$$\liminf m_k \geq \int_\Omega g(u)\, dx\ .$$

Since Ω is an arbitrary domain and u is a nontrivial solution, this gives rise to the inequality

$$\liminf m_k \geq J(u) \geq m\ .$$

Therefore, $\lim m_k = m = J(u)$ and u is a ground state.

Finally, to prove (23), we observe that, since $J_k(u_k) \to J(u) = m$, the assertion of Theorem 3 holds with $c = m$, $\ell = 1$ and $u^1 = u$. □

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