# On the Regularization of Linear Abel-Type Integral Equations of First Kind with Hölder-Continuous Kernels

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We consider the integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K(t,s)u(s)ds = f(t),\ 0 \le t \le T,\ \alpha \in (0,1),$$

with a continuous kernel $K(t,s)$ satisfying a Hölder condition with respect to $t$. We derive stability estimates in exponentially weighted $L_p$-norms ($1 \le p \le \infty$) and discuss their consequences for the problem of regularization.

KEY WORDS: Abel integral equation, fractional differentiation and integration, ill-posed problems, regularization, stability of solution.

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## §1. Introduction

Let $p \in [1,\infty]$, $\alpha \in (0,1)$, $T \in \mathbf{R}^+ = (0,\infty)$, $V = \{(t,s)|\ 0 \le s \le t \le T\}$, and let $K : V \to \mathbf{C}$ be a continuous function, defined on the triangle $V$ and satisfying there, with an exponent $\lambda \in (\alpha, 1]$ and a positive number $m$ a Hölder condition

$$|K(t_2,s) - K(t_1,s)| \le m|t_2 - t_1|^\lambda,\ 0 \le s \le \min(t_1,t_2) \le T.$$

Furthermore, we assume $K(t,t) = 1$ for $0 \leq t \leq T$. Then the integral operator $A_\alpha : L_p \to L_p$, defined by

$$(A_\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(t,s) y(s) ds, \; y \in L_p, \; 0 \leq t \leq T,$$

is linear and compact, hence the problem of solving the generalized Abel integral equation of first kind

$$A_\alpha u = f, \; f \text{ given}, \; u \text{ unknown}, \tag{1.1}$$

is illposed in $L_p$. We simply write $L_p$ for $L_p(0,T)$.

For the basic theory concerning existence and uniqueness of the solution $u$ of this equation we refer the reader to §31.3 and §31.4 of [SKM]. There it is shown that there exists exactly one solution $u$ in $L_p$ if $f \in J^\alpha(L_p)$, the operator $J^\beta$ being the operator of fractional integration, defined for all $\beta > 0$, by

$$(J^\beta y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds. \tag{1.2}$$

It is also shown there that $A_\alpha(L_p) = J^\alpha(L_p)$.

For discussing the problem of regularization, however, we do not need a sharp characterization of the range of $A_\alpha$. The above mentioned equality of the ranges of $A_\alpha$ and $J^\alpha$ well suffices for our purposes. Our model assumption for regularization is that there exists a "true" element $u$ in $L_p$ that we want to "recover" (better to "approximate") from inaccurately given data $\phi(t)$, $0 \leq t \leq T$, deviating from $f = A_\alpha u$ by an error at most $\varepsilon$,

$$\|A_\alpha u - \phi\|_p \leq \varepsilon. \tag{1.3}$$

¿From this and from a known bound $E$ on some nonnegative functional of $u$ an approximation $\tilde{u}$ in $L_p$ should be constructed and an error estimate

$$\|u - \tilde{u}\|_p \leq \tau(E, \varepsilon)$$

should be provided, the wish being that $\tau(E,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Of course, under these model assumptions a "true" right hand side $f$ lying in the range of the operator $A_\alpha$ does exist, but we need not worry about the fine structure of this range.

Various methods of regularizing the equation (1.1) under differentiability assumptions on the kernel $K$ or even under the very special assumption $K(s,t) \equiv 1$ (then one has Abel's classical equation) have been described in the literature, see e.g., [AGH], but the study of this regularization problem under our weaker assumption of only Hölder continuity seems to have been neglected.

For convenience we introduce a few more operators for whose theory and interconnections one may also consult [SKM] or [GV]. We use the operator $J = J^1$ of integration: $(Jy)(t) = \int_0^t y(s)ds$, the operator $D$ of differentiation: $(Dy)(t) = y'(t)$, and the operators $D^\alpha = DJ^{1-\alpha}$ of fractional differentiation:

$$(D^\alpha y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s)ds.$$

The operators $D$ and $D^\alpha$ are defined on the ranges of the operators $J$ and $J^\alpha$, respectively, and $D$ and $D^\alpha$ are left-inverse (but not right-inverse) to $J$ and $J^\alpha$, respectively, $J$ and $J^\alpha$ considered as operators from $L_p$ into $L_p$:

$$DJ = I, \; D^\alpha J^\alpha = I,$$

with $I$ denoting the identity operator: $Iy = y$.

In §2 we shall work with exponentially weighted norms in $L_p$ spaces. For $q > 0$ define $e_{-q}(t) = e^{-qt}, \; 0 \le t \le T$. Then the norm $\| \, . \, \|_{p,q}$, defined by

$$\|y\|_{p,q} = \|e_{-q} y\|_p, \text{ for } y \in L_p,$$

is a norm in $L_p$ which is equivalent to the usual $L_p$-norm. In fact, for all $y \in L_p$,

$$\|y\|_{p,q} \le \|y\|_p \le e^{qT} \|y\|_{p,q}. \tag{1.4}$$

## §2. Stability estimates

Assuming $f$ to lie in the range of $A_\alpha$ (or of $J^\alpha$ what amounts to the same) we produce an explicit positive constant $M$ such that the solution $u$ of (1.1) obeys an inequality $\|u\| \le M \|D^\alpha f\|$. Actually, we shall first derive analogous but sharper estimates in exponentially weighted norms. The constant $M$ does not depend on $f$.

¿From [SKM] §31 we take our basic

**Lemma 1** *For $f \in A_\alpha(L_p)$ the integral equation of first kind (1.1) is equivalent to the integral equation of second kind*

$$(I + B_\alpha)u = D^\alpha f, \tag{2.1}$$

*with the operator $B_\alpha : \ L_p \to L_p$ given by*

$$(B_\alpha y)(t) = \int_0^t H(t,s)y(s)ds,$$

$$H(t,s) = \frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_s^t \frac{K(t,s) - K(\tau,s)}{(t-\tau)^{1+\alpha}(\tau - s)^{1-\alpha}} d\tau,$$

*and $I + B_\alpha$ is an isomorphism of $L_p$ onto itself.*

*Comment*: Lemma 1 has as essence the factorization $A_\alpha = J^\alpha(I + B_\alpha)$.

**Lemma 2** *For $(t,s) \in V$ we have the inequality*

$$|H(t,s)| \leq \frac{m\alpha\Gamma(\lambda - \alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)}(t-s)^{\lambda - 1}. \tag{2.2}$$

**Proof**. By the Hölder condition imposed on $K$ we have

$$|H(t,s)| \leq \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_s^t \frac{(t-\tau)^\lambda}{(t-\tau)^{1+\alpha}(\tau-s)^{1-\alpha}} d\tau$$

$$= \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_s^t (t-\tau)^{\lambda-\alpha-1}(\tau - s)^{\alpha-1} d\tau$$

$$= \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)} (t-s)^{\lambda-1} \int_0^1 (1-\zeta)^{\lambda-\alpha-1}\zeta^{\alpha-1} d\zeta$$

$$= \frac{\alpha m}{\Gamma(\alpha)\Gamma(1-\alpha)} (t-s)^{\lambda-1} B(\alpha, \lambda - \alpha) = \frac{\alpha m\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)}(t-s)^{\lambda-1}.$$

□

We now ask the reader to recall our definition of the norms $\| \ . \ \|_{p,q}$ in §1.

**Lemma 3** *With $e_{-q}(t) = e^{-qt}$ we have, for $q > 0, \ \beta > 0, \ y \in L_p$ the inequality*

$$\|e_{-q}J^\beta y\|_p \leq q^{-\beta}\|e_{-q}y\|_p \tag{2.3}$$

*that can also be written in the form*

$$\|J^\beta y\|_{p,q} \leq q^{-\beta}\|y\|_{p,q}. \tag{2.4}$$

**Proof.** With $g(t) = \frac{1}{\Gamma(\beta)} e^{-qt} t^{\beta-1}, \ h(t) = e^{-qt} y(t)$, we have

$$e^{-qt}(J^\beta y)(t) = \int_0^t g(t-s)h(s)ds,$$

hence, by Young's inequality for convolutions,

$$\|e_{-q} J^\beta y\|_p \leq \|g\|_1 \|e_{-q} y\|_p,$$

and (2.3) follows from

$$\|g\|_1 \leq \int_0^\infty |g(t)|dt = q^{-\beta}.$$

□

Let us now consider the equation

$$(I + B_\alpha)w = g \tag{2.5}$$

for $g$ given in $L_p$, $w$ to be determined in $L_p$. By Lemma 1 there exists exactly one solution $w$.

**Lemma 4** *Take an arbitrary number $q > m_1^{1/\lambda}$ where $m_1 = \frac{m\alpha\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)}$ and let $w$ be the solution of (2.5). Then we have*

$$\|e_{-q} w\|_p \leq (1 - m_1 q^{-\lambda})^{-1} \|e_{-q} g\|_p, \tag{2.6}$$

*equivalently written:*

$$\|w\|_{p,q} \leq (1 - m_1 q^{-\lambda})^{-1} \|g\|_{p,q}.$$

**Proof.** Using Lemma 2 we see that for $v(t) = |w(t)|$ we have

$$v(t) \leq |g(t)| + \frac{m\alpha\Gamma(\lambda-\alpha)}{\Gamma(1-\alpha)\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} v(s)ds$$

$$= |g(t)| + m_1 (J^\lambda v)(t),$$

$$\|e_{-q} v\|_p \leq \|e_{-q} g\|_p + m_1 \|e_{-q} J^\lambda v\|_p,$$

and because of (2.3) this is

$$\leq \|e_{-q} g\|_p + m_1 q^{-\lambda} \|e_{-q} v\|_p.$$

Now $m_1 q^{-\lambda} < 1$, and we obtain (2.6). $\square$

*Remark*: The same technique of estimation yields, for all $y \in L_p$,

$$\|e_{-q} B_\alpha y\|_p \leq m_1 q^{-\lambda} \|e_{-q} y\|_p,$$

hence

$$\|B_\alpha y\|_{p,q} \leq m_1 q^{-\lambda} \|y\|_{p,q},$$

and we recognize that the operator norm of $B_\alpha : L_p \to L_p$, where $L_p$ is normed by $\| \,.\, \|_{p,q}$, is $\|B_\alpha\|_{p,q} \leq m_1 q^{-\lambda} < 1$. As a consequence, the series $\sum_{n=0}^{\infty} (-1)^n B_\alpha^n$, formally representing $(I + B_\alpha)^{-1}$, is in this norm majorized by the convergent series $\sum_{n=0}^{\infty} (m_1 q^{-\lambda})^n$. This shows (what we already know) that $I + B_\alpha$ is an isomorphism of $L_p$ onto itself.

Reformulation of Lemma 4 yields

**Theorem 1** *Let $c$ be an arbitrary positive number and put $q = (m_1 + c)^{1/\lambda}$. Then for the solution $w$ of (2.5) and the solution $u$ of (2.1) the following estimates hold:*

$$\|e_{-q} w\|_p \leq (1 + \frac{m_1}{c}) \|e_{-q} g\|_p, \tag{2.7}$$

$$\|e_{-q} u\|_p \leq (1 + \frac{m_1}{c}) \|D^\alpha f\|_p. \tag{2.8}$$

By specifying the positive number $c$ one can obtain more handsome estimates that may, however, be not quite as sharp. The choice $c = m_1$ gives $q = (2m_1)^{1/\lambda}$, hence

$$\|e_{-q} w\|_p \leq 2 \|e_{-q} g\|_p, \tag{2.9}$$

$$\|e_{-q} u\|_p \leq 2 \|e_{-q} D^\alpha f\|_p, \tag{2.10}$$

from which we, using $e^{-qT} \leq e^{-qt} \leq 1$, obtain

$$\|w\|_p \leq 2 e^{qT} \|g\|_p, \tag{2.11}$$

$$\|u\|_p \leq 2 e^{qT} \|D^\alpha f\|_p. \tag{2.12}$$

We keep these results as

**Corollary 1** *With $q = (2m_1)^{1/\lambda}$ the estimates (2.9), (2.10), (2.11), (2.12) hold for the solution $w$ of (2.5) and $u$ of (2.1).*

*Remark 1*: These estimates are stability estimates in the following sense. They exhibit the sensitivity of the solution with respect to perturbation of the right-hand side $f$ within the range of the operator $A_\alpha$. To see this, simply consider another equation $A_\alpha u_1 = f_1$ with $f_1 \in A_\alpha(L_p)$. Because $A_\alpha$ is a linear operator, $A_\alpha(u - u_1) = f - f_1$, and we conclude that

$$\|e_{-q}(u - u_1)\|_p \leq 2\|e_{-q} D^\alpha (f - f_1)\|_p,$$

$$\|u - u_1\|_p \leq 2e^{qT}\|D^\alpha(f - f_1)\|_p.$$

*Remark 2*: Use of exponentially weighted norms is not unusual in treatment of Volterra integral equations (often it is quite natural: the simplest Volterra equation of second kind, $u(t) - 1 = \int_0^t u(s)ds$ has the solution $u(t) = e^t$). See, e.g., [B1], [B2], [JW]. However, in Abel type integral equations of form (1.1) it is usually assumed that $K$ is differentiable or Lipschitz-continuous or satisfies a condition $|K(t_2, s) - K(t_1, s)| \leq |t_2 - t_1| \ln |t_2 - t_1|$ ([LRS], [B2]) which is weaker than Hölder-continuity.

## §3 Regularization

In the problem of regularization, as shortly described in §1, it is assumed that we have in hand an element $\phi \in L_p$ and a positive number $\varepsilon$ such that

$$\|A_\alpha u - \phi\|_p = \|f - \phi\|_p \leq \varepsilon,$$

and that we for the "true" solution $u$ have extra information in form of a bound $E$ on a suitable nonnegative functional of $u$ (often of some stronger norm or semi-norm of $u$). In view of Theorem 1 (or its Corollary 1) it now suffices to have a method of finding a good approximation to $D^\alpha f = D^\alpha A_\alpha u$. We assume that we dispose on a linear bounded operator $D_{\alpha,\rho} : L_p \to L_p$, depending on a (positive) parameter $\rho$, and that we have an estimate of the form

$$\|D^\alpha A_\alpha u - D_{\alpha,\rho}\phi\|_p \leq \sigma(E, \varepsilon, \rho). \tag{3.1}$$

We are lucky if by appropriate choice of $\rho = \rho(E, \varepsilon)$, putting $\sigma(E, \varepsilon, \rho(E, \varepsilon)) = \tau(E, \epsilon)$, we have $\tau(E, \varepsilon) \to 0$ as $\varepsilon \to 0$. We have successfully regularized the "approximate" integral equation $A_\alpha u \approx \phi$ if this is the case. In fact, then the operator

$$R_{E,\varepsilon} = (I + B_\alpha)^{-1} D_{\alpha,\rho(E,\varepsilon)} : \; L_p \to L_p$$

is an isomorphism of $L_p$ onto itself, and with $g = A_\alpha u - \phi = f - \phi$ we inherit from Theorem 1 and its Corollary 1 the estimates

$$\|u - R_{E,\varepsilon}\phi\|_{p,q} \leq (1 + \frac{m_1}{c})\|D^\alpha A_\alpha u - D_{\alpha,\rho(E,\varepsilon)}\phi\|_{p,q} \tag{3.2}$$

$$\text{for } q = (m_1 + c)^{1/\lambda}, \; c > 0,$$

$$\|u - R_{E,\varepsilon}\phi\|_{p,q} \leq 2^{qT}\|D^\alpha A_\alpha u - D_{\alpha,\rho(E,\varepsilon)}\phi\|_p \leq 2^{qT}\tau(E,\varepsilon) \tag{3.3}$$

$$\text{for } q = (2m_1)^{1/\lambda}.$$

*Remark*: Of course, the optimal choice of $\rho = \rho(E,\varepsilon)$ is the one that minimizes $\sigma(E,\varepsilon,\rho)$ relative to $\rho$. But not always it is possible to find the strict minimizer.

## An example of regularization

In view of (3.3) all we have to do is to use a method of regularizing the equation

$$J^\alpha w = f \tag{4.1}$$

with $w = (I + B_\alpha)u$. The extra information required on the smoothness (or regularity) of $u$ will be expressed indirectly as one for $w = D^\alpha f$. We can exploit via formula (3.3) the available literature for regularizing Abel's classical equation. See, e.g., [GW], [GV], [GY], [VG], [DRS1], [DRS2].

We present one example how to do it.

Take the method described in §1 of [GY]. Then according to the relevant Theorem 5.1 of [GY] the following statement is true (with $w$ instead of u): *Let $w$ be absolutely continuous on* $[0,T]$ *and* $w' \in L_p(0,T)$, $\alpha \in (0,1)$, $p \in [1,\infty]$,

$$\|J^\alpha w - \phi\|_p \leq \varepsilon, \; \frac{1}{T}\|w\|_p + \|w'\|_p \leq E. \tag{4.2}$$

*Then the regularization method described in §5 of [GY] yields $w_\rho$ such that*

$$\|w - w_\rho\|_p \le 8E\rho + \frac{8T^{1-\alpha}\varepsilon}{\Gamma(2-\alpha)\rho}, \tag{4.3}$$

*and the choice $\rho = \tilde{\rho} = 2T^{(1-\alpha)/2}(\Gamma(2-\alpha))^{-1/2}(\varepsilon/E)^{1/2}$ minimizes the right-hand side of this estimate and gives*

$$\|w - w_\rho\| \le 8T^{(1-\alpha)/2}(\Gamma(2-\alpha))^{-1/2}(E\varepsilon)^{1/2} = \tau(E,\varepsilon).$$

For $u_\rho = (I + B_\alpha)^{-1} w_\rho$ we then have, according to (3.3), the estimate

$$\|u - u_\rho\|_{p,q} = \|e_{-q}(u - u_\rho)\|_p \le 2^{qT}\tau(E,\varepsilon), \tag{4.4}$$

if

$$\begin{cases} \frac{1}{T}\|(I + B_\alpha)u\|_p + \|D(I + B_\alpha u)\|_p \le E, \\ \|A_\alpha u - \phi\|_p \le \varepsilon. \end{cases} \tag{4.5}$$


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