Invariance of a partial differential equation of fractional order under a certain Lie group

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Abstract

In this article a symmetry group of scaling transformations is determined for a partial differential equation of fractional order α, containing among particular cases the diffusion equation, the wave equation, and the fractional diffusion-wave equation. For its group-invariant solutions an ordinary differential equation of fractional order with the new independent variable $z = xt^{-\alpha/2}$ is derived. The derivative then is an Erdelyi-Kober derivative depending on a parameter α. Its complete solution is given in terms of generalized Wright functions.

Keywords: Fractional diffusion-wave equation, group-invariant solutions, Erdelyi-Kober derivative, generalized Wright function

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1. Introduction

In this paper we want to give group-invariant solutions to a partial differential equation of fractional order containing among particular cases the diffusion equation ($\alpha = 1$), the wave equation ($\alpha = 2$), and the so-called fractional diffusion-wave equation ($1 \le \alpha \le 2$). This equation is obtained by replacing the first or second order time-derivative in the diffusion or wave equation, respectively, by a generalized derivative of order $\alpha \ge 1$, defined in the sense of the Riemann-Liouville fractional calculus. We consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad x > 0,\ t > 0,\ D > 0,\ \alpha \ge 1, \tag{1}$$

with

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in I\!N \\ \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \int\limits_0^t (t-s)^{n-\alpha-1} u(x,s) ds, & n-1 < \alpha < n \in I\!N, \end{cases}$$

Properties and more details about the Riemann-Liouville fractional derivative can be found in texts on Fractional Calculus (see [5], [11]).
In a series of papers (see [8] and the references there) the two basic boundary-value problems for the fractional diffusion-wave equation ($1 \leq \alpha \leq 2$) were considered:
a) Cauchy problem:

$$u(x, 0+) = g(x), \ -\infty < x < +\infty; \ u(\mp\infty, t) = 0, \ t > 0,$$

b) Signaling problem:

$$u(x, 0+) = 0, \ x > 0; \ u(0+, t) = h(t), \ u(+\infty) = 0, \ t > 0.$$

By using integral transforms (Laplace, Fourier or Mellin type) the Green's functions $\mathcal{G}_c(x,t,\alpha)$ and $\mathcal{G}_s(x,t,\alpha)$ for these problems were expressed in terms of some special functions (of Wright or Mittag-Leffler type, Fox H-function) with the similarity argument $z = xt^{-\alpha/2}/\sqrt{D}$ ($x > 0, \ t > 0$). We will explain this fact in our article and determine that Equation (1) is in fact invariant under a symmetry group of scaling transformations. The method of group analysis of differential equations which we will use in our article began with the work of Sophus Lie more than a hundred years ago.
Roughly speaking a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. For partial differential equations one can determine special types of solutions, which are invariant under some subgroup of the full symmetry group of the system. These 'group-invariant' solutions are found by solving a reduced system of equations having fewer independent variables than the original system (assuming that the reduced system is easier to solve).
In recent years the ideas of the Lie group approach have been extended to difference equations and also to integro-differential equations. We refer to the handbooks [2], [3] and [4] for results and literature and especially to [1] for the case of integro-differential equations.
In [10] one can find the full symmetry groups of the diffusion and the wave equation. Here we will focus on the so-called similarity method, developed by G. D. Birkhoff in the 1930s and consider the special case of scaling transformations.
In the general case ($\alpha \notin I\!N$) one can not use the chain rule to get a reduced equation for the scale-invariant solutions of (1) as in the case of partial differential equations ($\alpha = 1, 2$). Instead we will transform Equation (1) into an ordinary differential equation of *fractional order* with the new independent variable $z = xt^{-\alpha/2}$. The derivative then is an Erdelyi-Kober derivative depending on a parameter α. For $\alpha = 1$ and $\alpha = 2$ (the diffusion and the

wave equations) this reduced equation corresponds to the ordinary differential equations well known in the literature (see [10]).
We present the general solution of the obtained differential equation of fractional order with Erdelyi-Kober derivative in terms of the Wright function.

2. The equation for the scale-invariant solutions

At first we will give the necessary definitions concerning the similarity method, though not in the most general form.
Let us consider the abstract equation for $u = u(x,t)$

$$F(u) \ = \ 0. \tag{2}$$

Definition 1 *A one-parameter family of scaling transformations, denoted by T_λ is a transformation of xtu-space of the form*

$$\bar{x} \ = \ \lambda^a x, \quad \bar{t} \ = \ \lambda^b t, \quad \bar{u} \ = \ u, \tag{3}$$

where a and b are constants and λ is a real parameter restricted to an open interval I containing $\lambda = 1$.

Definition 2 *The equation (2) is invariant under the one parameter family T_λ of scaling transformations (3) iff T_λ takes any solution u of (2) to a solution $\bar{u}$ of the same equation:*

$$\bar{u} \ = \ T_\lambda u \quad \text{and} \quad F(\bar{u}) \ = \ 0. \tag{4}$$

Definition 3 *A real-valued function $\eta(x,t,u)$ is called an invariant of the one-parameter family T_λ if it is unaffected by the transformations, in other words*

$$\eta(T_\lambda(x,t,u)) \ = \ \eta(x,t,u) \quad \text{for all} \quad \lambda \in I.$$

As is easily seen the invariant of the family of scaling transformations is $\eta(x,t,u) = xt^{-a/b}$.
If the equation (2) is a second order partial differential equation of the form

$$G(x,\ t,\ u,\ u_x,\ u_t,\ u_{xx},\ u_{tt},\ u_{xt}) \ = \ 0, \tag{5}$$

and this equation is invariant under T_λ, given by (3), then the transformation

$$u(x,t) \ = \ v(z), \quad z \ = \ xt^{-a/b} \tag{6}$$

reduces the equation (5) to a second order ordinary differential equation of the form

$$g(z,\ v,\ v',\ v'') \ = \ 0. \tag{7}$$

The new independent variable z defined in the similarity transformation (6) is then called a similarity variable. For a proof of this fact we refer in the case of general Lie group methods to [10], for the special case of similarity methods to [6].
In this article we will extend these results to the partial differential equation of fractional order (1) taking the parameter $a = 1$ in the scaling transformation. Then

$$T_\lambda(u(x,t)) \;=\; \bar{u}(\lambda x, \lambda^b t) \;=\; \bar{u}(\bar{x},\bar{t}). \tag{8}$$

In this case the abstract equation (2) takes the form

$$F(u) \;=\; \frac{\partial^\alpha u}{\partial t^\alpha} - D u_{xx} = 0.$$

Let us determine a one parameter family T_λ of scaling transformations under which Equation (1) is invariant. We have ($\bar{x} = \lambda x,\ \bar{t} = \lambda^b t,\ \bar{u} = u$):

$$u_{xx} \;=\; \frac{\partial^2}{\partial x^2}\bar{u}(\bar{x},\bar{t}) \;=\; \lambda^2 \bar{u}_{\bar{x}\bar{x}},$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} \;=\; \frac{\partial^n}{\partial t^n}\bar{u}(\bar{x},\bar{t}) \;=\; \lambda^{nb}\frac{\partial^n \bar{u}}{\partial \bar{t}^n}, \quad \text{for} \quad \alpha = n \in I\!N.$$

In the case $n-1 < \alpha < n,\ n \in I\!N$ using variable substitution we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} \;=\; \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t (t-s)^{n-\alpha-1}\bar{u}(\bar{x},\bar{s})\,ds$$

$$= \frac{\lambda^{nb}}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial \bar{t}^n}\int_0^{\bar{t}/\lambda^b} (\bar{t}\lambda^{-b} - s)^{n-\alpha-1}\cdot \bar{u}(\bar{x},\lambda^b s)\,ds$$

$$= \frac{\lambda^{b\alpha}}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial \bar{t}^n}\int_0^{\bar{t}} (\bar{t}-\tau)^{n-\alpha-1}\bar{u}(\bar{x},\tau)\,d\tau = \lambda^{b\alpha}\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha}.$$

It follows that

$$F(\bar{u}) \;=\; \lambda^{\alpha b}\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} \;-\; D\lambda^2 \bar{u}_{\bar{x}\bar{x}} \;=\; 0, \quad \text{if} \quad b = \frac{2}{\alpha}.$$

This means that the invariant for the scaling transformation T_λ of the partial differential equation of fractional order has the form

$$\eta(x,t,u) = x t^{-\alpha/2}.$$

By analogy with the partial differential equation (5) we use the transformation

$$u(x,t) \;=\; v(z), \quad z \;=\; x t^{-\alpha/2}$$

to determine the scale-invariant solutions of the partial differential equation of fractional order (1).

Let us solve for the partial derivative u_{xx} and partial derivative of fractional order $\dfrac{\partial^\alpha u}{\partial t^\alpha}$, $\alpha > 0$ in terms of derivatives of v. We find

$$u_x \;=\; v'(z)t^{-\frac{\alpha}{2}}, \quad u_{xx} \;=\; v''(z)t^{-\alpha}. \tag{9}$$

In the case $\alpha = 1$ and $\alpha = 2$ the derivative $\dfrac{\partial^\alpha u}{\partial t^\alpha}$ can be expressed in terms of a linear ordinary differential operator, applied to v, of order 1 or 2 respectively (see [10]). In the general case we have

Theorem 1 *The partial derivative of fractional order* $\dfrac{\partial^\alpha u}{\partial t^\alpha}$, $\alpha > 0$ *of the function* $u(x,t) = v(z),\ z = xt^{-\alpha/2}$ *is given by the relation*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha}(P^{1-\alpha,\alpha}_{2/\alpha} v)(z),\ z > 0 \tag{10}$$

with the Erdelyi-Kober fractional differential operator $P^{\tau,\alpha}_\beta$ *of order* $\alpha > 0$*:*

$$(P^{\tau,\alpha}_\beta g)(z) := \prod_{j=0}^{\eta-1}(\tau + j - \frac{1}{\beta}z\frac{d}{dz})(K^{\tau+\alpha,\eta-\alpha}_\beta g)(z), \tag{11}$$

$$\eta = \begin{cases} [\alpha]+1, & \alpha \notin I\!N, \\ \alpha, & \alpha \in I\!N, \end{cases}$$

$$\textit{where} \quad (K^{\tau,\alpha}_\beta g)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}g(zu^{1/\beta})\,du, & \alpha > 0, \\ g(z), & \alpha = 0 \end{cases}$$

is the Erdelyi-Kober fractional integral operator.

Proof. Let $n-1 < \alpha < n,\ n = 1, 2, 3, \ldots$. We then have ($z = xt^{-\alpha/2}$):

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}v(x\tau^{-\alpha/2})\,d\tau\right) \tag{12}$$

$$= \frac{\partial^n}{\partial t^n}\left(t^{n-\alpha}(K^{1,n-\alpha}_{2/\alpha}v)(z)\right),$$

where

$$(K^{1,n-\alpha}_{2/\alpha}v)(z) = \frac{1}{\Gamma(n-\alpha)}\int_1^\infty (u-1)^{n-\alpha-1}u^{\alpha-n-1}v(zu^{\alpha/2})\,du$$

is the Erdelyi-Kober fractional integral operator.
In view of the relation ($z = xt^{-\alpha/2}$)

$$t\frac{\partial}{\partial t}\phi(z) = tx(-\frac{\alpha}{2})t^{-\frac{\alpha}{2}-1}\phi'(z) = -\frac{\alpha}{2}z\frac{d}{dz}\phi(z) \tag{13}$$

we arrive at

$$\frac{\partial^n}{\partial t^n}\left(t^{n-\alpha}(K^{1,n-\alpha}_{2/\alpha}v)(z)\right) = \frac{\partial^{n-1}}{\partial t^{n-1}}\left(t^{n-1-\alpha}(n-\alpha-\frac{\alpha}{2}z\frac{d}{dz})(K^{1,n-\alpha}_{2/\alpha}v)(z)\right)$$

$$= \ldots = t^{-\alpha} \prod_{j=0}^{n-1} (1 - \alpha + j - \frac{\alpha}{2} z \frac{d}{dz})(K_{2/\alpha}^{1, n-\alpha} v)(z) = t^{-\alpha} (P_{2/\alpha}^{1-\alpha,\alpha} v)(z).$$

In the case $\alpha = n = 1, 2, 3, \ldots$ we use the relation (13) once again and get ($z = xt^{-n/2}$):

$$\frac{\partial^\alpha u}{\partial t^\alpha} := \frac{\partial^n}{\partial t^n} v(z) = \frac{\partial^{n-1}}{\partial t^{n-1}} (t^{-1} \cdot t \frac{\partial}{\partial t} v(z)) = \frac{\partial^{n-1}}{\partial t^{n-1}} (t^{-1} (-\frac{n}{2} z \frac{d}{dz}) v(z))$$

$$= \ldots = t^{-n} \prod_{j=0}^{n-1} (1 - n + j - \frac{n}{2} z \frac{d}{dz}) v(z) = t^{-n} (P_{2/\alpha}^{1-n,n} v)(z).$$

■

Now we substitute the expressions (9) and (10) into the partial differential equation of fractional order (1), and we find

$$t^{-\alpha} (P_{2/\alpha}^{1-\alpha,\alpha} v)(z) = D t^{-\alpha} v''(z), \; z > 0, \; D > 0.$$

As it could be expected, this equation is equivalent to one in which the variable t does not occur:

$$(P_{2/\alpha}^{1-\alpha,\alpha} v)(z) = D v''(z), \; z > 0, \; \alpha \geq 1 \tag{14}$$

with the Erdelyi-Kober fractional differential operator $P_{2/\alpha}^{1-\alpha,\alpha}$.

Remark 1 *As it follows from the definition of the Erdelyi-Kober fractional differential operator* (11) *in the case* $\alpha = n \in I\!N$ *the reduced equation* (14) *for the scale-invariant solutions is a linear ordinary differential equation of order* $\max\{2, n\}$. *In the case* $\alpha = 1$ *(the diffusion equation) this equation takes the form*

$$(P_2^{0,1} v)(z) = -\frac{1}{2} z v'(z) = D v''(z),$$

and in the case $\alpha = 2$ *(the wave equation) the form*

$$(P_1^{-1,2} v)(z) = 2 z v'(z) + z^2 v''(z) = D v''(z).$$

The complete discussion of these cases one can find in, for example, [10].

3. The scale-invariant solutions

In this part of our article we present the exact form for the scale-invariant solutions of the partial differential equation of fractional order (1). Note here that such solutions are well known for the diffusion equation ($\alpha = 1$) and for the wave equation ($\alpha = 2$) (see [10]). In the general case as to the authors' knowledge equations of type (14) were not yet considered in the literature on fractional calculus and fractional differential equations. It turns out that the solutions of Equation (14) have different structure in the cases $1 \leq \alpha < 2$, $\alpha = 2$, and $\alpha > 2$. We begin with the first case:

Theorem 2 *The scale-invariant solutions of the partial differential equation of fractional order* (1) *in the case* $1 \leq \alpha < 2$ *have the form* ($z = xt^{-\alpha/2}$):

$$v(z) \;=\; C_1 v_1(z) + C_2 v_2(z), \text{ with arbitrary real constants } C_1 \text{ and } C_2, \tag{15}$$

$$v_1(z) \;=\; W(-z/\sqrt{D}; -\frac{\alpha}{2}, 1), \quad v_2(z) = W(z/\sqrt{D}; -\frac{\alpha}{2}, 1) \tag{16}$$

with the Wright function

$$W(z; \lambda, \mu) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\lambda k + \mu)}.$$

Proof. Let us consider the case $\alpha = 1$. Then Equation (14) for the scale-invariant solutions is the linear ordinary differential equation of order 2 (see Remark 1), which has two independent solutions. Since the functions $v_1(z)$ and $v_2(z)$ are entire functions in the complex plane for $\alpha < 2$ (see [12]), we immediately have

$$-\frac{1}{2} z v_1'(z) \;=\; -\frac{1}{2} z \frac{-1}{\sqrt{D}} \sum_{k=1}^{\infty} \frac{(-z/\sqrt{D})^{k-1}}{(k-1)!\Gamma(1-\frac{k}{2})}$$

$$= \; -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(-z/\sqrt{D})^k}{(k-1)!(-\frac{k}{2})\Gamma(-\frac{k}{2})} \;=\; \sum_{k=1}^{\infty} \frac{(-z/\sqrt{D})^k}{k!\Gamma(-\frac{k}{2})} \;=\; Dv_1''(z),$$

and, analogously,

$$-\frac{1}{2} z v_2'(z) = Dv_2''(z),$$

which completes the proof in the case $\alpha = 1$.
In the case $1 < \alpha < 2$ after substituting $v'' = y$ Equation (14) takes the form

$$y(z) - \frac{1}{D}(L_\alpha y)(z) = c_1 z + c_2, \; z > 0, \; D > 0, \tag{17}$$

c_1, c_2 are arbitrary real constants,

$$(L_\alpha y)(z) = (P_{2/\alpha}^{1-\alpha,\alpha} I_{0+}^2 y)(z)$$

with the Riemann-Liouville fractional integral operator

$$(I_{0+}^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau)\, d\tau.$$

Equation (17) is a fractional integro-differential equation of the second kind of order $2 - \alpha > 0$ and it has an unique solution if the right hand side has the form $f(z) = c_1 z + c_2$ (see [7], [13]). This means that the initial equation (14) has no more than two linearly independent solutions.
Let us prove that the functions (16) satisfy Equation (14). First we have

$$v_1''(z) \;=\; \frac{1}{D} \sum_{k=2}^{\infty} \frac{(-z/\sqrt{D})^{k-2} k(k-1)}{k!\Gamma(1-\frac{\alpha}{2}k)} \;=\; \frac{1}{D} \sum_{k=0}^{\infty} \frac{(-z/\sqrt{D})^k}{k!\Gamma(1-\alpha-\frac{\alpha}{2}k)} \tag{18}$$

$$
\begin{aligned}
&= \frac{1}{D} W(-z/\sqrt{D}; -\frac{\alpha}{2}, 1-\alpha), \\
v_2''(z) &= \frac{1}{D}\sum_{k=2}^{\infty} \frac{(z/\sqrt{D})^{k-2} k(k-1)}{k!\Gamma(1-\frac{\alpha}{2}k)} = \frac{1}{D}\sum_{k=0}^{\infty} \frac{(z/\sqrt{D})^{k}}{k!\Gamma(1-\alpha-\frac{\alpha}{2}k)} \qquad (19)\\
&= \frac{1}{D} W(z/\sqrt{D}; -\frac{\alpha}{2}, 1-\alpha).
\end{aligned}
$$

Unfortunately, it is not possible to apply the operator $P_{2/\alpha}^{1-\alpha,\alpha}$ term by term to the power series (16) because of divergence of the corresponding integrals. Instead we use here methods based on the Parseval equality for the Mellin integral transform and the well known integral representation of the Wright function.

Lemma 1 *The relations*

$$
\begin{aligned}
(P_{2/\alpha}^{1-\alpha,\alpha} v_1)(z) &= \sum_{k=0}^{\infty} \frac{(-z/\sqrt{D})^{k}}{k!\Gamma(1-\alpha-\frac{\alpha}{2}k)} = W(-z/\sqrt{D}; -\frac{\alpha}{2}, 1-\alpha), \\
v_1(z) &= \sum_{k=0}^{\infty} \frac{(-z/\sqrt{D})^{k}}{k!\Gamma(1-\frac{\alpha}{2}k)} = W(-z/\sqrt{D}; -\frac{\alpha}{2}, 1), \\
(P_{2/\alpha}^{1-\alpha,\alpha} v_2)(z) &= \sum_{k=0}^{\infty} \frac{(z/\sqrt{D})^{k}}{k!\Gamma(1-\alpha-\frac{\alpha}{2}k)} = W(z/\sqrt{D}; -\frac{\alpha}{2}, 1-\alpha), \\
v_2(z) &= \sum_{k=0}^{\infty} \frac{(z/\sqrt{D})^{k}}{k!\Gamma(1-\frac{\alpha}{2}k)} = W(z/\sqrt{D}; -\frac{\alpha}{2}, 1)
\end{aligned}
$$

hold for $1 < \alpha < 2$.

Proof. In the case $1 < \alpha < 2$ the left hand side of Equation (14) has the form

$$
(P_{2/\alpha}^{1-\alpha,\alpha} v)(z) = (1-\alpha-\frac{\alpha}{2} z \frac{d}{dz})(2-\alpha-\frac{\alpha}{2} z \frac{d}{dz})(K_{2/\alpha}^{1,2-\alpha} v)(z), \qquad (20)
$$

where

$$
(K_{2/\alpha}^{1,2-\alpha} v)(z) = \frac{1}{\Gamma(2-\alpha)} \int_1^{\infty} (t-1)^{1-\alpha} t^{\alpha-3} v(z t^{\alpha/2})\, dt
$$

is the Erdelyi-Kober fractional integral operator. According to Wright ([12]),

$$
v_1(z) = \frac{e^{-Z}}{\sqrt{Z}}(A_0 + O(Z^{-1})), \quad \text{for} \quad z \to +\infty, \qquad (21)
$$

$$
\text{with} \qquad Z = (1-\delta)(\delta^{\delta} z/\sqrt{D})^{\frac{1}{1-\delta}}, \quad \delta = \frac{\alpha}{2}, \quad A_0 = \frac{1}{\sqrt{2\pi}} \frac{(1-\delta)^{\delta}}{\sqrt{\delta}},
$$

$$
v_2(z) = \frac{2}{\alpha} + O(z^{-\frac{2}{\alpha}}), \quad \text{for} \quad z \to +\infty. \qquad (22)
$$

It follows from these two relations and (20) that the operator $P_{2/\alpha}^{1-\alpha,\alpha}$ is well defined on the functions $v_1(z)$ and $v_2(z)$.
In order to get its effect on the function $v_1(z)$ we represent the operator (20) in the form ([5], [13]):

$$(P_{2/\alpha}^{1-\alpha,\alpha}v)(z) = \frac{1}{2\pi i}\int_{L_{i\infty}} \frac{\Gamma(1+\frac{\alpha}{2}s)}{\Gamma(1-\alpha+\frac{\alpha}{2}s)} v^*(s) z^{-s}\, ds, \tag{23}$$

where

$$v^*(s) = \int_0^\infty v(t) t^{s-1}\, ds$$

is the Mellin integral transform of $v(t)$ and $L_{i\infty} = \{s \in \mathbf{C} : \ \Re(s) = \gamma > 0\}$. By using results from [9] we arrive at

$$v_1^*(s) = \frac{\Gamma(s)}{\Gamma(1+\frac{\alpha}{2}s)}(1/\sqrt{D})^{-s}. \tag{24}$$

We have then

$$(P_{2/\alpha}^{1-\alpha,\alpha}v_1)(z) = \frac{1}{2\pi i}\int_{L_{i\infty}} \frac{\Gamma(1+\frac{\alpha}{2}s)}{\Gamma(1-\alpha+\frac{\alpha}{2}s)} v_1^*(s) z^{-s}\, ds$$

$$= \frac{1}{2\pi i}\int_{L_{i\infty}} \frac{\Gamma(1+\frac{\alpha}{2}s)}{\Gamma(1-\alpha+\frac{\alpha}{2}s)} \frac{\Gamma(s)}{\Gamma(1+\frac{\alpha}{2}s)}(1/\sqrt{D})^{-s} z^{-s}\, ds$$

$$= \frac{1}{2\pi i}\int_{L_{i\infty}} \frac{\Gamma(s)}{\Gamma(1-\alpha+\frac{\alpha}{2}s)}(z/\sqrt{D})^{-s}\, ds.$$

It is possible to transform in the last integral the contour $L_{i\infty}$ into the contour $L_{-\infty}$ which goes from the point $-ai-\infty$, $a>0$ to the point $bi-\infty$, $b>0$ in such a manner that $|\Im(s)| < C$ for all $s \in L_{-\infty}$, C is a constant and all the poles of the function $\Gamma(s)$, i.e. the points $s = 0, -1, -2, \ldots$ are on the left of this contour. We use then the residue theorem for the representation of this modified integral in form of the series and we get the statement of our Lemma 1 for the function $v_1(z)$.
This method couldn't be applied to the function $v_2(z)$, since the Mellin integral transform of this function doesn't exist. Instead we use the modified integral representation of the Wright function (see [12]), which has the form

$$v_2(z) = 1 + \frac{1}{2\pi i}\int_{L_-} \frac{e^\tau}{\tau}(\exp\{z\tau^{\alpha/2}/\sqrt{D}\} - 1)\, d\tau.$$

Here the contour L_- is the cut in the complex plane coinciding with the negative real semi-axis. Applying now the operator (20) and changing the order of integration, we have

$$(P_{2/\alpha}^{1-\alpha,\alpha}v_2)(z) = \frac{1}{\Gamma(1-\alpha)} + (1-\alpha-\frac{\alpha}{2}z\frac{d}{dz})(2-\alpha-\frac{\alpha}{2}z\frac{d}{dz}) \tag{25}$$

$$\frac{1}{2\pi i}\int_{L_-}\frac{e^\tau}{\tau}\frac{1}{\Gamma(2-\alpha)}\int_1^\infty (t-1)^{1-\alpha}t^{\alpha-3}(\exp\{z\tau^{\alpha/2}t^{\alpha/2}/\sqrt{D}\} - 1)\, dt\, d\tau.$$

Using the Parseval equality for the Mellin integral transform and tables of Mellin integral transforms from [9], we have:

$$\frac{1}{\Gamma(2-\alpha)}\int_1^{\infty}(t-1)^{1-\alpha}t^{\alpha-3}\exp\{z\tau^{\alpha/2}t^{\alpha/2}/\sqrt{D}\}\,dt \tag{26}$$

$$= \frac{1}{\Gamma(3-\alpha)}+\frac{1}{2\pi i}\int_{L_{i\infty}}\frac{\Gamma(1-\frac{\alpha}{2}s)\Gamma(-s)}{\Gamma(3-\alpha-\frac{\alpha}{2}s)}(-z\tau^{\alpha/2}/\sqrt{D})^s\,ds,$$

$$\text{where}\qquad \tau\in L_-,\ L_{i\infty}=\{s\in \mathbf{C}:\ 0<\Re(s)=\gamma<1\}.$$

The contour $L_{i\infty}$ in the last representation can be transformed into the contour $L_{+\infty}$, which goes from the point $-ai+\infty,\ a>0$ to the point $bi+\infty,\ b>0$ in such a manner that $|\Im(s)|<C,\ \Re(s)>0$ for all $s\in L_{+\infty}$, C is a constant and the points $s=1,2,3,\ldots$ and $s=2/\alpha,4/\alpha,6/\alpha,\ldots$ are on the right side of this contour whereas the point $s=0$ is on the left side.
We substitute the representation (26) (with the contour $L_{+\infty}$ instead of $L_{i\infty}$) into (25) and, using the identity

$$(1-\alpha-\frac{\alpha}{2}z\frac{d}{dz})(2-\alpha-\frac{\alpha}{2}z\frac{d}{dz})z^s=(1-\alpha-\frac{\alpha}{2}s)(2-\alpha-\frac{\alpha}{2}s)z^s,$$

we arrive at

$$(P^{1-\alpha,\alpha}_{2/\alpha}v_2)(z)=\frac{1}{\Gamma(1-\alpha)}+\frac{1}{2\pi i}\int_{L_-}\frac{e^{\tau}}{\tau}$$

$$\times\frac{1}{2\pi i}\int_{L_{+\infty}}\frac{\Gamma(1-\frac{\alpha}{2}s)\Gamma(-s)}{\Gamma(1-\alpha-\frac{\alpha}{2}s)}(-z\tau^{\alpha/2}/\sqrt{D})^s\,ds\,d\tau.$$

Changing the order of integration in the last formula and using the well known representation of the Γ-function (see [9])

$$\frac{1}{\Gamma(\beta)}=\frac{1}{2\pi i}\int_{L_-}e^{\tau}\tau^{-\beta}\,d\tau,\quad \text{if}\quad \Re(\beta)<1,$$

we finally get

$$(P^{1-\alpha,\alpha}_{2/\alpha}v_2)(z)=\frac{1}{\Gamma(1-\alpha)}+\frac{1}{2\pi i}\int_{L_{+\infty}}\frac{\Gamma(1-\frac{\alpha}{2}s)\Gamma(-s)}{\Gamma(1-\alpha-\frac{\alpha}{2}s)}(-z/\sqrt{D})^s$$

$$\times\frac{1}{2\pi i}\int_{L_-}e^{\tau}\tau^{-(1-\frac{\alpha}{2}s)}\,d\tau\,ds=\frac{1}{\Gamma(1-\alpha)}+\frac{1}{2\pi i}\int_{L_{+\infty}}\frac{\Gamma(-s)}{\Gamma(1-\alpha-\frac{\alpha}{2}s)}(-z/\sqrt{D})^s$$

$$=\frac{1}{\Gamma(1-\alpha)}+\sum_{k=1}^{\infty}\frac{(z/\sqrt{D})^k}{k!\Gamma(1-\alpha-\frac{\alpha}{2}k)}=W(z/\sqrt{D};-\frac{\alpha}{2},1-\alpha).$$

This last relation completes our proof of Lemma 1. The statement of Theorem 2 follows now by comparing relations (18) and (19) and the results proven in Lemma 1. ■

Remark 2 *In the case $\alpha = 1$ (i.e. for the diffusion equation) the scale-invariant solutions* (16) *have the form:*

$$v_1(z) = \sum_{k=0}^{\infty} \frac{(-z/\sqrt{D})^k}{k!\Gamma(1-\frac{1}{2}k)} = 1 + \sum_{n=0}^{\infty} \frac{(-z/\sqrt{D})^{2n+1}}{(2n+1)!\Gamma(\frac{1}{2}-n)}$$

$$= 1 - \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (z/(2\sqrt{D}))^{2n+1}}{(n)!(n+\frac{1}{2})} = 1 - \pi\, erf(z/(2\sqrt{D})),$$

and, analogously,

$$v_2(z) = 1 + \pi\, erf(z/(2\sqrt{D})),$$

where erf is the error function, which is in agreement with the known results (see [10]).

Remark 3 *The scale-invariant solutions* (16) *can also be used in the case $\alpha = 2$ (wave equation). In this case we should write them in the form:*

$$u_1(z) = (v_1(z) + v_2(z))/2 = \sum_{k=0}^{\infty} \frac{(z^2/D)^k}{\Gamma(1+2k)\Gamma(1-\alpha k)},$$

$$u_2(z) = D \cdot \Gamma(1-\frac{\alpha}{2})(v_2(z) - v_1(z))/2 = \Gamma(1-\frac{\alpha}{2}) z \sum_{k=0}^{\infty} \frac{(z^2/D)^k}{\Gamma(2+2k)\Gamma(1-\frac{\alpha}{2}-\alpha k)}.$$

Using the relation ($\alpha = 2$)

$$\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(1-\frac{\alpha}{2}-\alpha k)} = (-\frac{\alpha}{2})(-\frac{\alpha}{2}-1) \times \ldots \times (-\frac{\alpha}{2}-(2k-1)) = (2k)!,\ k = 0, 1, 2, \ldots$$

we arrive for $\alpha = 2$ at

$$u_1(z) = 1,$$

$$u_2(z) = z \sum_{k=0}^{\infty} \frac{(z^2/D)^k (2k)!}{(1+2k)!} = z \sum_{k=0}^{\infty} \frac{(z^2/D)^k}{1+2k} = -\frac{\sqrt{D}}{2} \ln \frac{\sqrt{D}-z}{\sqrt{D}+z},\ z < \sqrt{D}.$$

Remark 4 *In a series of papers (see* [8] *and references there) some boundary-value problems for Equation* (1) *were considered. Using the asymptotic formulae* (21) *and* (22) *we can get the boundary values of signaling type for our scale-invariant solutions of Equation* (1)*:*

$$\begin{cases} u_1(x, 0+) = v_1(+\infty) = 0, & x > 0, \\ u_1(0+, t) = v_1(0) = 1, & t > 0, \\ u_1(+\infty, t) = v_1(+\infty) = 0, & t > 0, \end{cases}$$

$$\begin{cases} u_2(x, 0+) = v_2(+\infty) = \frac{2}{\alpha}, & x > 0, \\ u_2(0+, t) = v_2(0) = 1, & t > 0, \\ u_2(+\infty, t) = v_2(+\infty) = \frac{2}{\alpha}, & t > 0. \end{cases}$$

Now we consider the case $2 < \alpha$:

Theorem 3 *The scale-invariant solutions of the partial differential equation of fractional order* (1) *in the case* $2 < \alpha,\ \alpha \notin I\!N$ *have the form* ($z = xt^{-\alpha/2}$):

$$v(z) = \sum_{j=0}^{[\alpha]} C_j v_j(z) \text{ with arbitrary real constants } C_j,\ 0 \le j \le [\alpha], \tag{27}$$

$$v_j(z) = z^{-2+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k}{\Gamma(\alpha - j + \alpha k)\Gamma(-1 + \frac{2}{\alpha}(1+j) - 2k)}. \tag{28}$$

Proof. After substituting $(P_{2/\alpha}^{1-\alpha,\alpha} v)(z) = y(z)$ Equation (14) has the form

$$y(z) - D(L_\alpha y)(z) = \sum_{j=0}^{[\alpha]} c_j z^{-4+\frac{2}{\alpha}(1+j)},$$

where $c_j,\ 0 \le j \le [\alpha]$ are arbitrary real constants and

$$(L_\alpha y)(z) = \frac{d^2}{dz^2}(K_{2/\alpha}^{1-\alpha,\alpha} y)(z).$$

This last equation is an integro-differential equation of second kind and of order $\alpha - 2 > 0$ which has a unique solution if the right hand side has the form $f(z) = \sum_{j=0}^{[\alpha]} c_j z^{-4+\frac{2}{\alpha}(1+j)}$ (see [7], [13]). This means that Equation (14) has no more than $[\alpha]+1$ linearly independent solutions. Let us prove that the functions $v_j(z),\ 0 \le j \le [\alpha]$ are its solutions. Firste note that the series in the definition of these functions determine entire functions. Indeed, using the well known formula

$$\frac{1}{\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}\Gamma(z)$$

we can define the majorizing series

$$S(x) = \sum_{k=0}^{\infty} \frac{1}{\pi} \left| \frac{\Gamma(2 + 2k - \frac{2}{\alpha}(1+j))}{\Gamma(2 - j + \alpha k)} \right| x^k$$

with the radius $R = +\infty$ of convergence:

$$R = \lim_{k\to\infty} \left| \frac{\Gamma(2 + 2k - \frac{2}{\alpha}(1+j))}{\Gamma(2 - j + \alpha k)} \frac{\Gamma(2 - j + \alpha + \alpha k)}{\Gamma(4 + 2k - \frac{2}{\alpha}(1+j))} \right|$$

$$= \lim_{k\to\infty} (2k)^{-2}(\alpha k)^{\alpha} = +\infty,\ \alpha > 2.$$

This means that we can differentiate these series term by term and we have

$$v_j''(z) = z^{-4+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k(-2 - 2k + \frac{2}{\alpha}(1+j))(-3 - 2k + \frac{2}{\alpha}(1+j))}{\Gamma(\alpha - j + \alpha k)\Gamma(-1 + \frac{2}{\alpha}(1+j) - 2k)} \tag{29}$$

$$= z^{-4+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k}{\Gamma(\alpha - j + \alpha k)\Gamma(-3 + \frac{2}{\alpha}(1+j) - 2k)}.$$

We can also apply the operator $P_{2/\alpha}^{1-\alpha,\alpha}$ term by term to the series defining the functions $v_j(z),\ 0 \le j \le [\alpha]$. Specifying the well known formula (see [5]) for our case we get

$$P_{2/\alpha}^{1-\alpha,\alpha} x^p = x^p \frac{\Gamma(1 - \frac{\alpha}{2}p)}{\Gamma(1 - \alpha - \frac{\alpha}{2}p)},\ 1 - \frac{\alpha}{2}p > 0.$$

Using the last formula we arrive at

$$(P_{2/\alpha}^{1-\alpha,\alpha} v_j)(z) = z^{-2+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k \Gamma(\alpha - j + \alpha k)}{\Gamma(\alpha - j + \alpha k)\Gamma(-1 + \frac{2}{\alpha}(1+j) - 2k)\Gamma(-j + \alpha k)}$$

$$= z^{-2+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k}{\Gamma(-1 + \frac{2}{\alpha}(1+j) - 2k)\Gamma(-j + \alpha k)}$$

$$= Dz^{-4+\frac{2}{\alpha}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k}{\Gamma(-3 + \frac{2}{\alpha}(1+j) - 2k)\Gamma(\alpha - j + \alpha k)}.$$

Comparing this relation with (29) we get the statement of our Theorem 3. ■
Finally we consider the case of natural number $2 < \alpha = n \in I\!N$. Using exactly the same method as in the proof of Theorem 3 we have

Theorem 4 *The scale-invariant solutions of the partial differential equation* ($2 < n \in I\!N$)

$$\frac{\partial^n u}{\partial t^n} = D u_{xx},\ t > 0,\ x > 0,\ D > 0$$

have the form ($z = x/t^{n/2}$):

$$v(z) = \sum_{j=0}^{n-1} C_j v_j(z) \text{ with arbitrary real constants } C_j,\ 0 \le j \le n-1, \tag{30}$$

$$v_j(z) = z^{-2+\frac{2}{n}(1+j)} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k}{\Gamma(n - j + nk)} \frac{\Gamma(-1 + \frac{2}{n}(1+j))}{\Gamma(-1 + \frac{2}{n}(1+j) - 2k)},\ 0 \le j \le n-2,$$

$$v_{n-1}(z) = 1,$$

where for $j = \frac{n}{2} - 1 = m - 1$ *in the case* $n = 2m,\ 1 < m \in I\!N$

$$\frac{\Gamma(-1 + \frac{2}{n}(1+j))}{\Gamma(-1 + \frac{2}{n}(1+j) - 2k)} = (-2 + \frac{2}{n}(1+j))(-3 + \frac{2}{n}(1+j)) \times \ldots$$

$$\times (-1 - 2k + \frac{2}{n}(1+j)) = (2k)!.$$

Remark 5 *In the case $\alpha = n = 2$ we can use the results of Theorem* 4 *for $z > \sqrt{D}$. We have*

$$v_1(z) = 1,$$

$$v_0(z) = z^{-1} \sum_{k=0}^{\infty} \frac{(Dz^{-2})^k (2k)!}{\Gamma(2+2k)} = \frac{1}{2\sqrt{D}} \sum_{k=0}^{\infty} \frac{(\sqrt{D}z^{-1})^{2k+1}}{2k+1} = -\frac{1}{2\sqrt{D}} \ln \frac{z-\sqrt{D}}{z+\sqrt{D}}.$$

Combining these relations with Remark 3 *we get the well known (see* [10]*) scale-invariant solutions of the wave equation:*

$$u(x,t) = v(x/t) = C_1 + C_2 \ln \left| \frac{x - \sqrt{D}t}{x + \sqrt{D}t} \right|.$$

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