Signalling Problem and Dirichlet-Neumann map for time-fractional diffusion-wave equations

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Abstract

The time-fractional, spatially one-dimensional, diffusion-wave equation is considered. For the Dirichlet and the Neumann condition prescribed on the boundary of the spatial positive half-line and zero-initial condition at the origin of time the solutions are derived via the method of Laplace transforms, and it is shown that the Dirichlet-Neumann map is given by a time-fractional differential operator whose order is (in analogy to the cases of the classical diffusion and wave equation) half the order of the time-fractional derivative.

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§1. Introduction

Consider the following initial-boundary value problem: *To determine a function* $u = u(x,t)$ *for* $x > 0,\ t > 0$ *so that the diffusion equation (or heat equation)*

$$u_t(x,t) = u_{xx}(x,t) \tag{1}$$

is fulfilled together with the initial and boundary conditions

$$u(x,0) = 0 \quad for \quad x > 0, \qquad u(0,t) = \phi(t) \quad for \quad t \geq 0. \tag{2}$$

It is well known that this problem has a unique solution if the function ϕ is smooth and $u(x,t)$ is required to tend to zero as $x \to \infty$. Putting

$$\psi(t) := u_x(0,t) \tag{3}$$

then we also know that the *Dirichlet* data $\phi(t)$ and the *Neumann* data $\psi(t)$ are connected by the formula

$$\phi(t) = -\frac{1}{\sqrt{\pi}} \int_0^t (t-\tau)^{-\frac{1}{2}}\, \psi(\tau)\, d\tau\,, \quad t > 0\,. \tag{4}$$

See, e.g., [4] or [12] or books in which the heat conduction equation $u_t = u_{xx}$ is treated in detail. If ϕ is given and ψ is unknown then (4) is an Abel integration of first kind for the function ψ, and by Abel's solution formula (see [1]) or books on integral equations) we have

$$\psi(t) = -\frac{1}{\sqrt{\pi}} \frac{d}{d\,t} \int_0^t (t-\tau)^{-\frac{1}{2}}\, \phi(\tau)\, d\tau\,, \quad t > 0. \tag{5}$$

In this relation we have an example of a *Dirichlet-Neumann map* (namely the mapping of the Dirichlet data ϕ on the Neumann data ψ) which in customary function spaces is an ill-posed operation. If, for example, $u(x,t)$ is the temperature at point x in instant t of a half-space $x > 0$ with a wall at $x = 0$, then the influx $-\psi(t)$ of heat through the wall can be found from measurements of the inner (at $x = 0+$) wall temperature $\phi(t)$ by solving the Abel integral equation (4). The map (5) and some other interesting Dirichlet-Neumann maps are considered in [7].

The aim of this note is to generalize formulas (4) and (5) to the fractional diffusion-wave equation. To this purpose we replace the first-order time-derivative in (1) by a time-derivative of fractional order α, where $0 < \alpha \leq 2$. With $\alpha = 1$ we again have the heat equation $u_t = u_{xx}$, with $\alpha = 2$ (a case to be discussed separately) we have the wave equation $u_{tt} = u_{xx}$, and our intention is to discuss the whole scale of equations. On Cauchy problems (namely initial-value problems) for such equations there exists an extensive literature, let us only quote [6], [14], [15], [22], [23], [24], and Mittag-Leffler functions play a decisive role in constructing fundamental solutions. Interesting aspects of similarity properties are discussed in the papers [2], [5], [13]. A related topic that recently has found special attention is the space-fractional diffusion equation for which we refer to [10] and [11].

§2. Basic notions of fractional calculus

To formulate the initial-boundary value problem that we want to treat we need the basic notions of fractional integration and differentiation. For sufficiently well behaved functions (the reader may consult [8], [9], [12], [19], [20], [21]) we define operators J^α and D^α of fractional integration and differentiation for $\alpha > 0$ (in the sequel operating on the time variable t when acting on functions of x and t). We remark that from the references just quoted [8], [9], [12] use the same notation as we here. The defining formulas are the following ones, for a generic function w and for $\alpha > 0,\ t > 0$:

$$(J^\alpha w)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w(\tau)\, d\tau, \tag{6}$$

$$(D^\alpha w)(t) = (D^m J^{m-\alpha} w)(t) \tag{7}$$

where $D = \frac{d}{dt}$ is the common differentiation operator and the natural number m is chosen so that $m-1 < \alpha \le m$. For completeness we put

$$D^0 = J^0 = I = \text{identity operator}, \tag{8}$$

$$(J^\alpha w)(0) = \lim_{t \to 0+} (J^\alpha w)(t).$$

The operators $J^\alpha,\ \alpha \ge 0$, form a semigroup, we have

$$J^\alpha J^\beta = J^{\alpha+\beta} \quad \text{for} \quad \alpha \ge 0\,, \quad \beta \ge 0\,,$$

and the operator D^α is left-inverse to the operator J^α, but not right-inverse. We have

$$\begin{cases} D^\alpha J^\alpha w = w \text{ generally,} \\ \text{but } J^\alpha D^\alpha w = w \ \text{ if } \ w(0) = w'(0) = \ldots = w^{(m-1)}(0)\,, \\ \text{where } m-1 < \alpha \le m \in \mathbf{N}. \end{cases} \tag{9}$$

With these notations the relations (4) and (5), connecting the Dirichlet data $\phi(t)$ to the Neumann data $\psi(t)$, can be written in the form

$$\phi = -J^{\frac{1}{2}} \psi, \quad \psi = -D^{\frac{1}{2}} \phi, \tag{10}$$

so $-\psi$ is obtained from ϕ by differentiation of order $1/2$.

In our treatment of the general problem we will use the technique of Laplace transforms. For a generic function $w(t),\ t \ge 0$, the Laplace transform is defined as

$$\tilde{w}(s) = (\mathcal{L} w)(s) = \int_0^\infty w(t) e^{-st}\, dt, \tag{11}$$

and if the function w is locally integrable and does not grow too fast as $t \to \infty$ then there is an abscissa σ of convergence so that $\tilde{w}(s)$, the Laplace transform of $w(t)$, exists for $\Re(s) > \sigma$. For the general theory of the Laplace transform and for the technique of using it we refer to [3]. Like we do for the operators J^α and D^α we apply the Laplace transform also as acting on the time variable t. The juxtaposition of a function $w(t)$ with its Laplace transform $\tilde{w}(s)$ we denote by writing

$$w(t) \div \tilde{w}(s). \tag{12}$$

¿From [3] or from the definition of the operator J^α and general working rules for the Laplace transform we take the important formula (for $\alpha \geq 0$)

$$\mathcal{L}(J^\alpha w)(s) = s^{-\alpha} \tilde{w}(s), \tag{13}$$

in other notation

$$(J^\alpha w)(t) \div s^{-\alpha} \tilde{w}(s). \tag{14}$$

§3. The general problem

For a fixed value α with $0 < \alpha < 2$ we consider the *time-fractional diffusion-wave equation*

$$(D^\alpha u)(x,t) = u_{xx}(x,t) \tag{15}$$

in the quarter-plane $x > 0,\ t > 0$, with initial condition (for $x > 0$)

$$\begin{cases} u(x,0) = 0 & \text{if } 0 < \alpha \leq 1, \\ u(x,0) = \frac{\partial}{\partial t} u(x,0) = 0 & \text{if } 1 < \alpha \leq 2, \end{cases} \tag{16}$$

and one of the two boundary conditions: *either the Dirichlet condition* (D) *or the Neumann condition* (N).

$$u(0,t) = \phi(t), \quad t \geq 0, \tag{D}$$
$$u_x(0,t) = \psi(t), \quad t \geq 0. \tag{N}$$

Furthermore we require a spatial boundary condition at infinity, namely

$$u(x,t) \to 0 = u(\infty,t) \quad \text{as} \quad x \to \infty. \tag{17}$$

For simplicity we assume ϕ and ψ to be smooth functions.
Remarks: The fractional differential equation (15) describes a "slow" diffusion process if $0 < \alpha < 1$, whereas in the case $1 < \alpha < 2$ it describes a process intermediate between diffusion and wave propagation. In both these cases we have integro-differential equations (in view of the definitions (7) of D^α), i.e. processes with memory. Only the particular cases $\alpha = 1$ (common diffusion) and $\alpha = 2$ (common wave propagation) are memoryless. In a picturesque language the variant with the boundary condition (D) is called the "signalling problem" (see [15] and [16]), $\phi(t)$ being the signal propagating from the boundary $x = 0$ into the interior $x > 0$.

§4. Solution of the Dirichlet problem

Applying fractional integration J^α to the fractional differential equation (15) and observing the initial conditions (16) by which according to (9) J^α and D^α commute we get

$$u(x,t) = J^\alpha u_{xx}(x,t), \tag{18}$$

hence by Laplace transformation (we use (14)) the s-parameterized system of ordinary differential equations in x, of second order,

$$\tilde{u}(x,s) = s^{-\alpha}\tilde{u}_{xx}(x,s), \quad x > 0, \quad s > 0, \tag{19}$$

with boundary conditions (see (D) and (17))

$$\tilde{u}(0,s) = \tilde{\phi}(s), \quad \tilde{u}(\infty,s) = 0. \tag{20}$$

Its solution is

$$\tilde{u}(x,s) = \tilde{\phi}(s)\exp(-xs^{\alpha/2}). \tag{21}$$

Denoting now the inverse Laplace transform of $\exp(-xs^{\alpha/2})$ by $G_D(x,t)$, namely

$$G_D(x,t) \div \exp(-xs^{\alpha/2}) = \tilde{G}_D(x,s) \tag{22}$$

and using the convolution theorem for Laplace transforms we obtain

$$u(x,t) = \int_0^t G_D(x,t-\tau)\phi(\tau)\,d\tau. \tag{23}$$

In [10], [15], [16], [17] [18] various relations of this Green function G_D to special functions and to extremal stable probability densities are worked out. The interested reader is referred to these papers.

§5. Solution of the Neumann problem and the Dirichlet-Neumann map

Applying J^α to (15) and again observing (16) and (9) we get

$$u(x,t) = J^\alpha u_{xx}(x,t), \tag{24}$$

hence once more the s-parameterized system of second order (in x) ordinary differential equations

$$\tilde{u}(x,s) = s^{-\alpha}\tilde{u}_{xx}(x,s),\ \ x > 0,\ \ s > 0, \tag{25}$$

but now with boundary conditions (from (N) and (17))

$$\tilde{u}_x(0,s) = \tilde{\psi}(s),\ \ \tilde{u}(\infty,s) = 0. \tag{26}$$

Its solution is given by

$$\tilde{u}(x,s) = -\tilde{\psi}(s)s^{-\alpha/2}\exp(-xs^{\alpha/2}). \tag{27}$$

With the Green function $G_N(x,t)$ as inverse Laplace transform of

$$-s^{-\alpha/2}\exp(-xs^{\alpha/2}),$$

namely

$$G_N(x,t) \div -s^{-\alpha/2}\exp(-xs^{\alpha/2}) = \tilde{G}_N(x,s), \tag{28}$$

we now get

$$u(x,t) = \int_0^t G_N(x,t-\tau)\psi(\tau)\,d\tau. \tag{29}$$

Assume now that $u(x,t)$ is a solution to (15) obeying both conditions (D) and (N). Comparison of (21) and (27) then teaches us the relation

$$\tilde{\phi}(s) = -s^{-\alpha/2}\tilde{\psi}(s), \tag{30}$$

consequently by (14)

$$\phi(t) = -(J^{\alpha/2}\psi)(t), \tag{31}$$

$$-(D^{\alpha/2}\phi)(t) = \psi(t). \tag{32}$$

Result: *If* $u(x,t)$ *is solution to the equation* (15) *and obeys the initial condition* (16) *and both boundary conditions* (D) *and* (N) *then the Dirichlet-Neumann map* $\phi \mapsto \psi$ *is realized by the fractional differentiation operator* $-D^{\alpha/2}$, *namely*

$$\psi = -D^{\alpha/2}\phi. \tag{33}$$

Remark 1: In our analysis in which we have assumed $0 < \alpha < 2$ we have tacitly used the fact that $\exp(-xs^{\alpha/2})$ (for $x > 0$) is the Laplace transform of a genuine function. The proof for this fact can be taken from [15] or [16]. The analysis breaks down (or degenerates) in the special case $\alpha = 2$, the inverse Laplace transform of e^{-xs} being the generalized function $\delta(t-x)$. In fact, then

$$\delta(t-x) \div \exp(-xs),\ u(x,t) = \phi(t-x),\ u_x(x,t) = -\phi_t(t-x) = -D^1\phi(t-x),$$

and inserting $x = 0$ we obtain

$$\psi(t) = -(D^1\phi)(t),$$

so *the result* (33) *is still true in the special case* $\alpha = 2$. However, in general we no longer have fulfilment of (17).

Remark 2: Shifting the boundary $x = 0$ to an arbitrary interior point $x > 0$ and there carrying out the analogous investigation (replacing $\phi(t)$ by $u(x,t)$, $\psi(t)$ by $u_x(x,t)$) we find that *for all* $x > 0,\ t > 0$ there holds the relation

$$u_x(x,t) = -D^{\alpha/2}u(x,t). \tag{34}$$

This result can be motivated by the formal factorization

$$D_t^{\alpha} - D_x^2 = (D_t^{\alpha/2} - D_x)(D_t^{\alpha/2} + D_x),$$

where $D_x = \frac{\partial}{\partial x}$. In this product of operators we must keep the second factor, so we have $(D_t^{\alpha/2} + D_x)u = 0$, which in the special case $\alpha = 2$ corresponds to an outgoing wave

$$u(x,t) = \phi(t-x).$$

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