CHARACTERIZATIONS OF SUBSPACES OF IMPORTANT TYPES OF CONVERGENCE SPACES IN THE REALM OF CONVENIENT TOPOLOGY

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Dedicated to Professor George E. Strecker on his 60th birthday

Abstract. Since in Convenient Topology we are mainly concerned with semiuniform convergence spaces, the question arises how the subspaces of important types of convergence spaces such as topological spaces, pretopological spaces (=closure spaces in the sense of Čech [5]), limit spaces (in the sense of Kowalsky [10] and Fischer [6]) or Kent convergence spaces can be characterized when they are considered as semiuniform convergence spaces (provided all convergence spaces fulfill a certain symmetry condition). This paper presents the solution. Furthermore, the relationships to other important subconstructs of the construct **SUConv** of semiuniform convergence spaces are investigated.

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0. Introduction

In the realm of Convenient Topology we are mainly concerned with semiuniform convergence spaces and their invariants, i.e. Convenient Topology consists essentially in the study of (full and isomorphism–closed) subconstructs of the construct **SUConv** of semiuniform convergence spaces (cf. [16]). Thus, the study of uniform spaces as well as the study of topological semiuniform convergence spaces (=symmetric topological spaces) belongs to Convenient Topology. Though there is a difference of a "topological" nature between removing a point from the usual topological space $\mathbb{R}_t$ of real numbers and removing a closed interval of length one, the obtained topological spaces are homeomorphic. But if we do the same with the usual uniform space $\mathbb{R}_u$ of real numbers, we obtain non–isomoprhic uniform spaces. The reason why uniform spaces behave "well" and topological spaces behave "badly" with respect to the formation of subspaces becomes clear in the framework of semiuniform convergence spaces, namely a subspace of a uniform spaces is uniform, but a subspace of a topological semiuniform convergence space need not be topological. Thus, the question arises how the 'subspaces of (symmetric) topological spaces' (=subtopological spaces) can be characterized. We solve this question by means of a completion for a certain class of filter spaces (filter spaces have been introduced by Katětov [9]).

The analogous question in the realm of nearness spaces introduced by Herrlich [8] has been solved by Bentley [2]. Surprisingly, our characterization of subtopological spaces is nothing else than an alternative description of Bentley's subtopological spaces.

By means of a completion for arbitrary filter spaces we solve additionally the problem of characterizing the subspaces of symmetric Kent convergence spaces, symmetric limit spaces and symmetric pretopological spaces and investigate the relationships between all these constructs. Furthermore, it turns out that subtopological spaces have a more comprehensive description than the subspaces of the other above mentioned types of convergence spaces.
The terminology of this article corresponds to [1] and [12].

Conventions. 1) A filter on a set X is not allowed to contain the empty set.
2) Subconstructs are always assumed to be full and isomorphism–closed.

1. Preliminaries

1.1. A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that the following are satisfied:

$UC_1)$ $\dot{x} \times \dot{x} \in \mathcal{J}_X$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in A\}$,
$UC_2)$ $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$,
$UC_3)$ $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$, where $F^{-1} = \{(y, x) : (x, y) \in F\}$.

A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiunifrom convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{J}_X) \subset \mathcal{J}_Y$. A seminuniform convergence space $(X, \mathcal{J}_X)$ is called a *uniform limit space* provided that the following are satisfied:

$UC_4)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$,
$UC_5)$ $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$ (whenever $\mathcal{F} \circ \mathcal{G}$ exists, i.e. $F \circ G = \{(x, y) : \exists z \in X \text{ with } (x, z) \in G \text{ and } (z, y) \in F\} \neq \phi$ for every $F \in \mathcal{F}, G \in \mathcal{G}$), where $\mathcal{F} \circ \mathcal{G}$ is the filter generated by the filter base $\{F \circ G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$.

The construct of all semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**, whereas **ULim** denotes its subconstruct of all uniform limit spaces. The construct **Unif** of uniform spaces (and uniformly continuous maps) in the sense of A. Weil [17] is bireflectively embedded into **ULim** (cf. e.g. [12;4.2.8.1.]) and **ULim** is a bireflective subconstruct of **SUConv** (cf. [14;2.2]). All these constructs are topological and initial structures are easy to describe, e.g. if X is a set, $(X_i, \mathcal{J}_{X_i})_{i \in I}$ a family of semiuniform convergence spaces and $(f_i : X \to X_i)_{i \in I}$ is a family of maps, then $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : (f_i \times f_i)(\mathcal{F}) \in \mathcal{J}_{X_i} \text{ for each } i \in I\}$ is the initial **SUConv**–structure on X with respect to the given data.

1.2. A *generalized convergence space* is a pair (X, q), where X is a set and $q \subset F(X) \times X$ (provided $F(X)$ denotes the set of all filters on X) such that the following are satisfied:
$C_1)$ $(\dot{x}, x) \in q$ for each $x \in X$,
$C_2)$ $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{F} \subset \mathcal{G}$.
If (X, q) is a generalized convergence space, then we say that $\mathcal{F} \in F(X)$ *converges* to $x \in X$ provided that $(\mathcal{F}, x) \in q$. A generalized convergence space (X, q) is called

α) a *Kent convergence space* provided that the following is satisfied:

C_3) $(\mathcal{F}\cap\dot{x},x)\in q$ whenever $(\mathcal{F},x)\in q$,

β) a *limit space* provided that the following is satisfied:

C_4) $(\mathcal{F},x)\in q$ and $(\mathcal{G},x)\in q$ imply $(\mathcal{F}\cap\mathcal{G},x)\in q$,

γ) a *pretopological space* provided that the following is satisfied:

C_5) $(\mathcal{U}_q(x),x)\in q$ where $\mathcal{U}_q(x)=\bigcap\{\mathcal{F}\in F(X):(\mathcal{F},x)\in q\}$ denotes the *neighborhood filter* of $x\in X$,

δ) a *topological space* provided that it is a pretopological space and for each $x\in X$ the following is satisfied:

C_6) For each $U\in\mathcal{U}_q(x)$ there is some $V\in\mathcal{V}_q(x)$ such that $U\in\mathcal{U}_q(y)$ for all $y\in V$.

ε) *symmetric* provided that the following is satisfied:

S) $(\mathcal{F},x)\in q$ and $y\in\bigcap\limits_{F\in\mathcal{F}}F$ imply $(\mathcal{F},y)\in q$.

A map $f:(X,q)\to(X',q')$ between generalized convergence spaces is called *continuous* provided that $(f(\mathcal{F}),f(x))\in q'$ whenever $(\mathcal{F},x)\in q$. Every semiuniform convergence space $(X,\mathcal{J}_X)$ has an *underlying (symmetric) Kent convergence space* $(X,q_{\gamma_{\mathcal{J}_X}})$ iff $\mathcal{F}\cap\dot{x}\in\gamma_{\mathcal{J}_X}$ where $\gamma_{\mathcal{J}_X}=\{\mathcal{F}\in F(X):\mathcal{F}\times\mathcal{F}\in\mathcal{J}_X\}$. A semiuniform convergence $(X,\mathcal{J}_X)$ is called a *convergence space* provided that $\mathcal{J}_X=\{\mathcal{F}\in F(X\times X)$: there are some $x\in X$ and some $\mathcal{G}\in F(X)$ with $(\mathcal{G},x)\in q_{\gamma_{\mathcal{J}_X}}$ and $\mathcal{F}\supset\mathcal{G}\times\mathcal{G}\}$, i.e. iff it is 'generated' by its convergent filters. The construct **Conv** of convergence spaces (and uniformly continuous maps) is (concretely) isomorphic to the construct $\mathbf{KConv_s}$ of symmetric Kent convergence spaces (and continuous maps) and it is a bicoreflective subconstruct of **SU-Conv** (the bicoreflector assings to each semiuniform convergence spaces its underlying Kent convergence space!). Let $\mathbf{KConv_s}$, $\mathbf{Lim_s}$, $\mathbf{PrTop_s}$ and $\mathbf{Top_s}$ denote the constructs of symmetric Kent convergence spaces, symmetric limit spaces, symmetric pretopological spaces and symmetric topological spaces ($=R_0$–topological spaces) respectively, then each of the constructs in the following list is a bireflective subconstruct of the preceding ones:

$$\mathbf{KConv_s}\supset\mathbf{Lim_s}\supset\mathbf{PrTop_s}\supset\mathbf{Top_s}.$$

For each generalized convergence space (X,q) the *closure* $\overline{A}$ of a subset A of X is defined by $\overline{A}=\{x\in X$: there is some $\mathcal{G}\in F(X)$ with $(\mathcal{G},x)\in q$ and $A\in\mathcal{G}\}$; furthermore, $O\subset X$ is called *q–open* provided that $X\backslash O=\overline{X\backslash O}$, and $\mathcal{X}_q=\{O\subset X:O$ is q–open$\}$ is a topology on X.

All constructs in this section are topological and initial structures can be easily described, e.g. if X is a set, $(X_i,q_i)_{i\in I}$ a family of symmetric Kent convergence spaces and $(f_i:X\to X_i)_{i\in I}$ a family of maps, then $q=\{(\mathcal{F},x)\in F(X)\times X:(f_i(\mathcal{F}),f_i(x))\in q_i$ for each $i\in I\}$ is the initial $\mathbf{KConv_s}$–structure on X with respect to the given data.

1.3. A *filter space* is a pair (X,γ) where X is a set and $\gamma\subset F(X)$ such that the following are satisfied:

F_1) $\dot{x}\in\gamma$ for each $x\in X$,

F_2) $\mathcal{G}\in\gamma$ whenever $\mathcal{F}\in\gamma$ and $\mathcal{F}\subset\mathcal{G}$.

If (X,γ) is a filter space, then the elements of γ are called *Cauchy filters*. A filter space (X,γ) is called a *Cauchy space* provided that the following is satisfied:

F_3) If $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters such that every member of $\mathcal{F}$ meets every member of $\mathcal{G}$, then $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter.
A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that the filter $f(\mathcal{F})$ generated by $\{f[F] : F \in \mathcal{F}\}$ belongs to γ' for each $\mathcal{F} \in \gamma$.
For every semiuniform convergence space $(X, \mathcal{J}_X), (X, \gamma_{\mathcal{J}_X})$ is called the *underlying filter space.*
The construct **Fil** of filter spaces (and Cauchy continuous maps) is topological and initial structures can be easily described, namely if X is a set, $(X_i, \gamma_i)_{i \in I}$ a family of filter spaces and $(f_i : X \to X_i)_{i \in I}$ a family of maps, then $\gamma = \{\mathcal{F} \in F(X) : f_i(\mathcal{F}) \in \gamma_i$ for each $i \in I\}$ is the initial **Fil**–structure on X with respect to the given data. The subconstruct **Chy** of all Cauchy spaces is bireflective in **Fil** (cf. [13;3.1.]).
Every filter space (X, γ) has an *underlying symmetric Kent convergence space* (X, q_γ), where $(\mathcal{F}, x) \in q_\gamma$ iff $\mathcal{F} \cap \dot{x} \in \gamma$.
A filter space (X, γ) is called *complete* provided that each $\mathcal{F} \in \gamma$ converges in (X, q_γ). Obviously, the construct **CFil** of complete filter spaces (and Cauchy continuous maps) is (concretely) isomorphic to $\mathbf{KConv_s}$ and bicoreflective in **Fil** (the bicoreflector assigns to each filter space its underlying symmetric Kent convergence space!). A semiuniform convergence space $(X, \mathcal{J}_X)$ is called **Fil**–*determined* provided that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there some $\mathcal{G} \in \gamma_{\mathcal{J}_X}$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$, i.e. iff it is 'generated' by its Cauchy filters. The construct **Fil-D-SUConv** of all **Fil**–determined semiuniform convergence spaces (and uniformly continuous maps) is (concretely) isomorphic to **Fil** and it is bireflective and bicoreflective in **SUConv** (cf. [14]); the bicoreflector assigns to each semiuniform convergence space its underlying filter space.

1.4. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *topological* provided it is a convergence space and the corresponding symmetric Kent convergence space is topological. It is easily checked that the construct **T-SUConv** of topological semiuniform convergence spaces (and uniformly continuous maps) is (concretely) isomorphic to $\mathbf{Top_s}$ and that it is closed under formation of products and closed (!) subspaces in **SUConv**. Thus, subspaces of topological semiuniform convergence spaces need not be topological, e.g. $\mathbb{R}\backslash\{0\}$ is an open subspace of the usual topological space $\mathbb{R}_t$ of real numbers, but it is not topological since it is not complete.

2. A filter space completion and its applications

2.1 Theorem. *Let* (X, γ) *be a filter space. Put* $\hat{X} = X \cup \{\mathcal{F} \in \gamma$*:* $\mathcal{F}$ *is non–convergent*$\}$ *and* $\hat{\gamma} = \{\mathcal{H} \in F(\hat{X}) : \mathcal{H} \supset \mathcal{G}$ *for some* $\mathcal{G} \in \hat{\mathcal{B}}\}$ *where* $\hat{\mathcal{B}} = \{i(\mathcal{F}) \cap \dot{x} : (\mathcal{F}, x) \in q_\gamma\} \cup \{i(\mathcal{F}) \cap \dot{\mathcal{F}} : \mathcal{F} \in \gamma$ *is non–convergent*$\}$ *and* $i : X \to \hat{X}$ *denotes the inclusion map. Then* $(\hat{X}, \hat{\gamma})$ *is a complete filter space containing* (X, γ) *as a dense subspace.*

Proof. 0) Obviously, $(\hat{X}, \hat{\gamma})$ is a filter space.

1) (X, γ) is a subspace of $(\hat{X}, \hat{\gamma})$, i.e. $\gamma = \{\mathcal{F} \in F(X) : i(\mathcal{F}) \in \hat{\gamma}\}$: If $\mathcal{F} \in \gamma$, then $i(\mathcal{F}) \in \hat{\gamma}$. Conversely, let $\mathcal{F} \in F(X)$ such that $i(\mathcal{F}) \in \hat{\gamma}$. Then there is some

$(\mathcal{G}, x) \in q_\gamma$ with $i(\mathcal{F}) \supset i(\mathcal{G}) \cap \dot{x}$ or there is some non–convergent $\mathcal{G} \in \gamma$ with $i(\mathcal{F}) \supset i(\mathcal{G}) \cap \dot{\mathcal{G}}$. In the first case $\mathcal{G}' = \mathcal{G} \cap \dot{x} \in \gamma$ and $i(\mathcal{G}') = i(\mathcal{G}) \cap \dot{i(x)} = i(\mathcal{G}) \cap \dot{x} \subset i(\mathcal{F})$; thus, $\mathcal{F} = i^{-1}(i(\mathcal{F})) \supset i^{-1}(i(\mathcal{G}')) = \mathcal{G}'$ and consequently, $\mathcal{F} \in \gamma$. In the second case $\mathcal{F} = i^{-1}(i(\mathcal{F})) \supset i^{-1}(i(\mathcal{G}) \cap \dot{\mathcal{G}}) = \mathcal{G}$ which implies $\mathcal{F} \in \gamma$.

2) X is dense in $(\hat{X}, \hat{\gamma})$, i.e. $\hat{X} \subset \overline{X} = \{y \in \hat{X}$: there is some $\mathcal{G} \in F(\hat{X})$ with $(\mathcal{G}, y) \in q_{\hat{\gamma}}$ and $X \in \mathcal{G}\}$: If $y \in \hat{X}$, then $y \in X$ or $y = \mathcal{F} \in \gamma$ is non–convergent. In the first case $y \in \overline{X}$, since $X \in i(\dot{y}) \xrightarrow{q_{\hat{\gamma}}} i(y) = y$. In the second case $y \in \overline{X}$, since $X \in i(\mathcal{F}) \xrightarrow{q_{\hat{\gamma}}} \mathcal{F} = y$.

3) $(\hat{X}, \hat{\gamma})$ is complete: If $\mathcal{G} \in \hat{\gamma}$, then $\mathcal{G} \supset i(\mathcal{F}) \cap \dot{x}$ with $(\mathcal{F}, x) \in q_\gamma$ or $\mathcal{G} \supset i(\mathcal{F}) \cap \dot{\mathcal{F}}$ where $\mathcal{F} \in \gamma$ is non–convergent. In the first case it follows from $i(\mathcal{F}) \cap \dot{x} \xrightarrow{q_{\hat{\gamma}}} x$ that $\mathcal{G} \xrightarrow{q_{\hat{\gamma}}} x$. In the second case it follows that $\mathcal{G} \cap \dot{\mathcal{F}} \in \hat{\gamma}$, i.e. $\mathcal{G} \xrightarrow{q_{\hat{\gamma}}} \mathcal{F}$.

2.2 Proposition. *Let (X, γ) be a filter space and $(\hat{X}, \hat{\gamma})$ its above completion. If $(\hat{X}, q_{\hat{\gamma}})$ is the underlying Kent convergence space of $(\hat{X}, \hat{\gamma})$, then the following are valid:*

1. $x \in X : (\hat{\mathcal{F}}, x) \in q_{\hat{\gamma}} \Longleftrightarrow \hat{\mathcal{F}} \cap \dot{x} \supset i(\mathcal{F}) \cap \dot{x}$ *for some* $\mathcal{F} \in F(X)$ *with* $\mathcal{F} \cap \dot{x} \in \gamma$
2. $\mathcal{F} \in \hat{X} \backslash X : (\hat{\mathcal{F}}, \mathcal{F}) \in q_{\hat{\gamma}} \Longleftrightarrow \hat{\mathcal{F}} \cap \dot{\mathcal{F}} \supset i(\mathcal{F}) \cap \dot{\mathcal{F}}$.

Proof. 1. $'' \Rightarrow ''$. The case $\mathcal{F} \cap \dot{x} \supset i(\mathcal{F}) \cap \dot{\mathcal{F}}$ for some non–convergent $\mathcal{F} \in \gamma$ cannot occur since otherwise for each $F \in \mathcal{F}$ there were some $\hat{F} \in \hat{\mathcal{F}}$ such that $\hat{F} \cup \{x\} \subset F \cup \{\mathcal{F}\}$, and consequently, $x \in F$, i.e. $\mathcal{F} \subset \dot{x}$ which would imply $\mathcal{F} \cap \dot{x} = \mathcal{F} \in \gamma$, i.e. $\mathcal{F} \xrightarrow{q_\gamma} x$ – a contradiction.

2. $'' \Rightarrow ''$. The case $\hat{\mathcal{F}} \cap \dot{\mathcal{F}} \supset i(\mathcal{G}) \cap \dot{x}$ for some $(\mathcal{G}, x) \in q_\gamma$ cannot occur, since $X \in \mathcal{G}$ and thus $X \cup \{x\} = X \in i(\mathcal{G}) \cap \dot{x}$ but otherwise $X \notin \hat{\mathcal{F}} \cap \dot{\mathcal{F}}$, i.e. $X \not\supset \hat{F} \cup \{\mathcal{F}\}$ for each $\hat{F} \in \hat{\mathcal{F}}$, because $\mathcal{F} \notin X$. Consequently, it remains the case that there is some non–convergent $\mathcal{G} \in \gamma$ such that $\hat{\mathcal{F}} \cap \dot{\mathcal{F}} \supset i(\mathcal{G}) \cap \dot{\mathcal{G}}$. Let $G \in \mathcal{G} \subset i(\mathcal{G})$. Then $G \cup \{\mathcal{G}\} = \hat{F} \cup \{\mathcal{F}\}$ for some $\hat{F} \in \hat{\mathcal{F}}$. Thus, $\mathcal{F} = \mathcal{G}$, since $\mathcal{F} \notin G \subset X$. Hence, $\hat{\mathcal{F}} \cap \dot{\mathcal{F}} \supset i(\mathcal{F}) \cap \dot{\mathcal{F}}$.
The inverse implications in both cases are trivial.

2.3 Definitions. 1) A filter space (X, γ) is called
a) a *sublimit space* provided that (X, q_γ) is a limit space,
b) *subpretopological* provided that (X, q_γ) is a pretopological space,
c) *weakly subtopological* provided that (X, q_γ) is a topological space.

2) **SubLim**, **SubPrTop** and $\mathbf{SubTop_w}$ denote the constructs of sublimit spaces, subpretopological and weakly subtopological filter spaces (and Cauchy continuous maps) respectively.

2.4 Proposition. *Each of the constructs in the following list is a bireflective subconstruct of the preceding ones:*

$$\mathbf{Fil} \supset \mathbf{SubLim} \supset \mathbf{SubPrTop} \supset \mathbf{SubTop_w}$$

Proof. Let $\mathcal{A}$ be the construct **SubLim**, **SubPrTop** or $\mathbf{SubTop_w}$ respectively and $\mathcal{B}$ the construct $\mathbf{Lim_s}$, $\mathbf{PrTop_s}$ or $\mathbf{Top_s}$ respectively. It suffices to prove that $\mathcal{A}$ is bireflective in **Fil**. Let $(f_i : (X, \gamma) \to (X_i, \gamma_i))_{i \in I}$ be an initial source in **Fil** such that $(X_i, \gamma_i) \in |\mathcal{A}|$

for each $i \in I$. Then $(f_i : (X, q_\gamma) \to (X, q_{\gamma_i}))_{i \in I}$ is an initial source in $\mathbf{KConv_s}$. Since $\mathcal{B}$ is bireflective in $\mathbf{KConv_s}$ and by assumption all (X_i, q_{γ_i}) belongs to $|\mathcal{B}|$ it follows that $(X, q_\gamma) \in |\mathcal{B}|$, i.e. $(X, \gamma) \in |\mathcal{A}|$.

2.5 Definitions. 1) A filter space (X, γ) is called
a) a *limit filter space* provided that it is a complete sublimit space,
b) *pretopological* provided that it is complete and subpretopological,
c) *topological* provided that it is complete and weakly subtopological.
2) **LimFil**, **PrTop-Fil** and **Top-Fil** denote the constructs of limit filter spaces, pretopological and topological filter spaces (and Cauchy continuous maps) respectively.

2.6 Remark. The following isomorphisms are concrete:

1. **Lim-Fil** $\cong$ $\mathbf{Lim_s}$
2. **PrTop-Fil** $\cong$ $\mathbf{PrTop_s}$
3. **Top-Fil** $\cong$ $\mathbf{Top_s}$

(Note: (**Fil** $\cong$ $\mathbf{KConv_s}$ [cf. 1.3]).

2.7 Theorem. *Let (X, γ) be a filter space.*
1) *The following are equivalent:*
(a) (X, γ) is a sublimit space,
(b) $(\hat{X}, \hat{\gamma})$ is a sublimit space,
(c) $(\hat{X}, \hat{\gamma})$ is a limit space,
(d) (X, γ) is a dense subspace (in **Fil**) *of some limit filter space,*
(e) (X, γ) is a subspace (in **Fil**) *of some limit filter space,*
(f) $(X, \gamma)(= (X, \mathcal{J}_\gamma))$ is a dense subspace (in **SUConv**) *of some (symmetric) limit space (Y, q) $(= (Y, \mathcal{J}_{\gamma_q}))$.*
(g) $(X, \gamma)(= (X, \mathcal{J}_\gamma))$ is a subspace (in **SUConv**) *of some (symmetric) limit space $(Y, q)(= (Y, \mathcal{J}_{\gamma_q}))$.*
2) *The following are equivalent:*
(a) (X, γ) is subpretopological,
(b) $(\hat{X}, \hat{\gamma})$ is subpretopological,
(c) $(\hat{X}, \hat{\gamma})$ is pretopological,
(d) (X, γ) is a dense subspace (in **Fil**) *of some pretopological filter space.*
(e) (X, γ) is a subspace (in **Fil**) *of some pretopological filter space.*
(f) $(X, \gamma)(= (X, \mathcal{J}_\gamma))$ is a dense subspace (in **SUConv**) *of some (symmetric) pretopological space $(Y, q)(= (Y, \mathcal{J}_{\gamma_q}))$,*
(g) $(X, \gamma)(= (X, \mathcal{J}_\gamma))$ is a subspace (in **SUConv**) *of some (symmetric) pretopological space $(Y, q)(= (Y, \mathcal{J}_{\gamma_q}))$.*

Proof. Since 1) and 2) can be proved similarly, we restrict ourselves to the proof of 2):
(a) $\Rightarrow$ (b). Let $y \in \hat{X}$. Then $y \in X$ or $y \in \hat{X} \backslash X$. In each case we have to prove that

$(\mathcal{U}_{q_{\hat\gamma}}(y), y) \in q_{\hat\gamma}$, where $\mathcal{U}_{q_{\hat\gamma}}(y) = \bigcap\{\hat{\mathcal{F}} \in F(\hat X) : (\hat{\mathcal{F}}, y) \in q_{\hat\gamma}\}$. In the first case, using 2.2.1., we obtain

$$\Big(\bigcap_{\hat{\mathcal{F}} \xrightarrow{q_{\hat\gamma}} y} \hat{\mathcal{F}}\Big) \cap \dot{y} = \bigcap_{\hat{\mathcal{F}} \xrightarrow{q_{\hat\gamma}} y} (\hat{\mathcal{F}} \cap \dot{y}) \supset \bigcap_{\mathcal{F} \xrightarrow{q_{\gamma}} y} i(\mathcal{F}) \cap \dot{y} = i\Big(\bigcap_{\mathcal{F} \xrightarrow{q_{\gamma}} y} \mathcal{F}\Big) \cap \dot{y}\ ;$$

by assumption, $\bigcap_{\mathcal{F} \xrightarrow{q_{\gamma}} y} \mathcal{F} \to y$ and also by 2.2.1., $\bigcap_{\hat{\mathcal{F}} \xrightarrow{q_{\hat\gamma}} y} \hat{\mathcal{F}} \to y$. In the second case, put $y = \mathcal{F}$ with $\mathcal{F} \in \gamma$ non-convergent. Then, using 2.2.2, $(\bigcap_{\hat{\mathcal{F}} \to \mathcal{F}} \hat{\mathcal{F}}) \cap \dot{\mathcal{F}} = \bigcap_{\hat{\mathcal{F}} \to \mathcal{F}} \hat{\mathcal{F}} \cap \dot{\mathcal{F}} = i(\mathcal{F}) \cap \dot{\mathcal{F}} \in \hat\gamma$, which implies $\bigcap_{\hat{\mathcal{F}} \to \mathcal{F}} \hat{\mathcal{F}} \to \mathcal{F}$.
(b) $\Rightarrow$ (c). Since $(\hat X, \hat\gamma)$ is complete, this implication is obvious.
(c) $\Rightarrow$ (d). Obvious.
(d) $\Rightarrow$ (e). Obvious.
(e) $\Rightarrow$ (a). This implication follows from the fact that every pretopological filter space is subpretopological and that by 2.4. **SubPrTop** is closed under formation of subspaces in **Fil**.
Furthermore, (d) $\Leftrightarrow$ (f) (resp. (e) $\Leftrightarrow$ (g)), since subspaces in **Fil** are formed as in **SUConv** (note: **Fil** is bireflectively embedded in **SUConv**).

2.8 Remarks. 1) It follows also from 2.1. that *the construct of all subspaces (in* **Fil** *[resp.* **SUConv**]*) of symmetric Kent convergence spaces is* **Fil** *itself* (cf. also [15;3.6]).

2) **SubLim** has also been studied by Bentley, Herrlich and Lowen–Colebunders [3] in the realm of merotopic spaces; it has been denoted there by **C**. In particular, they proved that it is not cartesian closed. Furthermore, it contains **Chy** as a bireflective subconstruct.

3. Subtopological spaces

3.1 Definition. A filter space (X, γ) is called *subtopological* provided that each $\mathcal{F} \in \gamma$ contains some $\mathcal{G} \in \gamma$ with a q_γ–open base $\mathcal{B}$ (i.e. for each $B \in \mathcal{B}, B \in \mathcal{X}_{q_\gamma}$).

3.2 Proposition. *Let (X, γ) be a filter space. Then the following are equivalent:*
(1) *(X, γ) is subtopological,*
(2) *Each $\mathcal{F} \in \gamma$ contains some $\mathcal{G} \in \gamma$ such that for each $G \in \mathcal{G}, int_{\mathcal{X}_{q_\gamma}} \in \mathcal{G}$,*
(3) *For each Cauchy filter $\mathcal{F}$ in (X, γ) the topological neighborhood filter of $\mathcal{F}$, denoted by $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F})$, is a Cauchy filter in (X, γ), where $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F}) = \{U \subset X : U \supset O_F \supset F$ for some $F \in \mathcal{F}$ and some $O_F \in \mathcal{X}_{q_\gamma}\}$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious.

(1) $\Rightarrow$ (3). Let $\mathcal{F} \in \gamma$. Then there is some $\mathcal{G} \in \gamma$ with $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}$ has a q_γ–open base. Thus, $\mathcal{G} \subset \mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F})$ and consequently, $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F}) \in \gamma$.

(3) $\Rightarrow$ (1). For each $\mathcal{F} \in F(X)$, $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F}) \subset \mathcal{F}$ has the q_γ–open base $\mathcal{B}$ =

$\{int_{\mathcal{X}_{q_\gamma}} U : U \in \mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F})\}$. By assumption, $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F}) \in \gamma$ provided that $\mathcal{F} \in \gamma$. Thus, (X, γ) is subtopological.

3.3 Proposition. *Every subtopological space (X, γ) is weakly subtopological.*

Proof. In order to prove that (X, q_γ) is topological it suffices to prove

$$(*) \qquad (\mathcal{F}, x) \in q_\gamma \Longleftrightarrow \mathcal{F} \supset \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x) .$$

Let $(\mathcal{F}, x) \in q_\gamma$. If $U \in \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x)$, then there is some $O \in \mathcal{X}_{q_\gamma}$ with $x \in O \subset U$. Consequently, $O \in \mathcal{F}$ and thus $U \in \mathcal{F}$. Hence $\mathcal{F} \supset \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x)$. Conversely, let $\mathcal{F} \supset \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x)$. In order to prove that $(\mathcal{F}, x) \in q_\gamma$ it suffices to prove $(\mathcal{U}_{\mathcal{X}_{q_\gamma}}(x), x) \in q_\gamma$. Since $\dot{x} \in \gamma$, $\mathcal{U}_{\mathcal{X}_{q_\gamma}}(\dot{x}) = \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x) = \mathcal{U}_{\mathcal{X}_{q_\gamma}}(x) \cap \dot{x} \in \gamma$, i.e. $(\mathcal{U}_{\mathcal{X}_{q_\gamma}}(x), x) \in q_\gamma$.

3.4 Remark. *A weakly subtopological filter space need not be subtopological* as the following example shows: Let X be the set $\mathbb{R}\backslash\{0\}$ and $\mathcal{X}$ the topology induced by the usual topology on $\mathbb{R}$. Put $\gamma = \gamma_{q_\mathcal{X}} \cup \{\mathcal{F} \in F(X) : \mathcal{F} \supset \mathcal{G}\}$ where $\mathcal{G}$ denotes the elementary filter of the sequence $(1/n)$ in X. Then (X, γ) is a filter space whose underlying (symmetric) Kent convergence space is the topological space $(X, q_\mathcal{X})$. Thus, (X, γ) is weakly subtopological, but not subtopological, since $\mathcal{G} \in \gamma$ does not have a $q_\mathcal{X}$–open base.

3.5 Proposition. *The construct* **SubTop** *of all subtopological filter spaces (and Cauchy continuous maps) is a bireflective subconstruct of* **Fil**.

Proof. 1) **SubTop** is closed under formation of subspaces in **Fil**: Let $(X', \gamma') \in |\mathbf{SubTop}|$ and (X, γ) as subspace in **Fil**, i.e. $X \subset X'$ and $\gamma = \{\mathcal{F} \in F(X) : i(\mathcal{F}) \in \gamma'\}$, where $i : X \to X'$ denotes the inclusion map. By assumption, for each $\mathcal{F} \in \gamma$, there is some $\mathcal{G} \in \gamma'$ such that $\mathcal{G} \subset i(\mathcal{F})$ and $\mathcal{G}$ has a $q_{\gamma'}$–open base. Thus, $\mathcal{F} \supset i^{-1}(\mathcal{G})$, where $i^{-1}(\mathcal{G})$ is a Cauchy filter in (X, γ) with a q_γ–open base $(i : (X, q_\gamma) \to (X', q'_\gamma)$ as well as $i : (X, q_{\mathcal{X}_{q_\gamma}}) \to (X', q_{\mathcal{X}_{q_{\gamma'}}})$ are continuous, i.e. $i : (X, \mathcal{X}_{q_\gamma}) \to (X', \mathcal{X}_{q_{\gamma'}})$ is continuous!)

2) **SubTop** is closed under formation of products in **Fil**. Let $((X_i, \gamma_i))_{i\in I}$ be a family of **SubTop**–objects and $((\prod_{i\in I} X_i, \gamma), (p_i)_{i\in I})$ their product in **Fil**, i.e. $\gamma = \{\mathcal{F} \in F(\Pi X_i) : p_i(\mathcal{F}) \in \gamma_i$ for each $i \in I\}$, where $p_i : \Pi X_i \to X_i$ denotes the i–th projection. If $\mathcal{F} \in \gamma$, then by assumption there is some $\mathcal{G}_i \in \gamma_i$ for each $i \in I$ such that $\mathcal{G}_i \subset p_i(\mathcal{F})$ and $\mathcal{G}_i$ contains $int_{\mathcal{X}_{q_{\gamma_i}}} G_i$ for each $G_i \in \mathcal{F}_i$. Hence, the product filter $\prod_{i\in I} \mathcal{F}_i \subset \mathcal{F}$ contains the product filter $\prod_{i\in I} \mathcal{G}_i \in \gamma$ (note: $p_i[\Pi\mathcal{G}_i] = \mathcal{G}_i \in \gamma_i$ for each $i \in I$) which has a q_γ–open base $\mathcal{B} = \{\prod_{i\in I} int_{\mathcal{X}_{q_{\gamma_i}}} G_i : G_i \in \mathcal{G}_i$ for each $i \in I$ and $G_i \neq X_i$ for at most finitely many $i \in I\}$ (note: $\prod_{i\in I} int_{\mathcal{X}_{q_{\gamma_i}}} G_i \in \mathcal{X}$ provided that $(\prod_{i\in I} X_i, \mathcal{X})$ denotes the product of $((X_i, \mathcal{X}_{q_{\gamma_i}}))_{i\in I}$ in $\mathbf{Top_S}$.)

3) All indiscrete **Fil**–objects, i.e. all filter spaces (X, γ) such that $\gamma = F(X)$, belong to $|\mathbf{SubTop}|$, since for each $\mathcal{F} \in F(X), \mathcal{U}_{\mathcal{X}_{q_\gamma}}(\mathcal{F}) \in F(X)$.

It follows from 1), 2) and 3) that **SubTop** is bireflective in **Fil** (cf. [12;2.2.11.2]).

3.6 Theorem. *Let (X, γ) be a subtopological filter space. Put $X^* = X \cup \{\mathcal{F} \in \gamma : \mathcal{F}$*

has a q_γ–open base and is non–convergent$\}$ and $\gamma^* = \{\mathcal{H} \in F(X^*) : \mathcal{H} \supset \mathcal{G}$ for some $\mathcal{G} \in \mathcal{B}^*\}$ where $\mathcal{B}^* = \{i(\mathcal{F}) \cap \dot{x} : (\mathcal{F}, x) \in q_\gamma\} \cup \{i(\mathcal{F}) \cap \dot{\mathcal{F}} : \mathcal{F} \in \gamma$ has a q_γ–open base and is non–convergent$\}$ and $i : X \to X^*$ denotes the inclusion map. Then (X^*, γ^*) is a complete filter space containing (X, γ) as a dense subspace.

The *proof* of 3.5. is omitted, since it is similar to 2.1.

3.7 Proposition. *Let (X, γ) be a subtopological filter space and (X^*, γ^*) its above completion (cf. 3.6.). If (X^*, q_{γ^*}) is the underlying Kent convergence space of (X^*, γ^*), then the following is valid:*

1. $x \in X : (\mathcal{F}^*, x) \in q_{\gamma^*} \iff \mathcal{F}^* \cap \dot{x} \supset i(\mathcal{F}) \cap \dot{x}$ for some $\mathcal{F} \in F(X)$ with $\mathcal{F} \cap \dot{x} \in \gamma$

2. $\mathcal{F} \in X^* \backslash X : (\mathcal{F}^*, \mathcal{F}) \in q_{\gamma^*} \iff \mathcal{F}^* \cap \dot{\mathcal{F}} \supset i(\mathcal{F}) \cap \dot{\mathcal{F}}$.

Proof. 3.7. is proved analogously to 2.2.

3.8 Corollary. *Let (X, γ) be a subtopological filter space and (X^*, γ^*) its above completion. The neighborhood filters of the points of (X^*, q_{γ^*}) are obtained as follows:*

1. $x \in X : \mathcal{U}_{q_{\gamma^*}}(x) = i(\mathcal{U}_{q_\gamma}(x))$,

2. $\mathcal{F} \in X^* \backslash X : \mathcal{U}_{q_{\gamma^*}}(\mathcal{F}) = i(\mathcal{F}) \cap \dot{\mathcal{F}}$.

Proof. 1. It suffices to prove $(i(\mathcal{U}_{q_\gamma}(x)) \subset \mathcal{U}_{q_{\gamma^*}}(x)$, since the inverse inclusion follows from the continuity of i.

$$i(\mathcal{U}_{q_\gamma}(x)) = i(\bigcap_{\mathcal{F} \cap \dot{x} \in \gamma} \mathcal{F}) = \bigcap_{\mathcal{F} \cap \dot{x} \in \gamma} i(\mathcal{F}) = \bigcap_{\mathcal{F} \cap \dot{x} \in \gamma} i(\mathcal{F} \cap \dot{x}) = \bigcap_{\mathcal{F} \cap \dot{x} \in \gamma} i(\mathcal{F}) \cap \dot{x} \subset \bigcap_{\mathcal{F}^* \cap \dot{x} \in \gamma^*} \mathcal{F}^* \cap \dot{x} = \bigcap_{\mathcal{F}^* \xrightarrow{q_{\gamma^*}} x} \mathcal{F}^* = \mathcal{U}_{q_{\gamma^*}}(x) \text{ (cf. 3.7.1.)}$$

$$2.\ \mathcal{U}_{q_{\gamma^*}}(\mathcal{F}) = \bigcap_{\mathcal{F}^* \xrightarrow{q_{\gamma^*}} \mathcal{F}} \mathcal{F}^* = \bigcap_{\mathcal{F}^* \xrightarrow{q_{\gamma^*}} \mathcal{F}} \mathcal{F}^* \cap \dot{\mathcal{F}} = i(\mathcal{F}) \cap \dot{\mathcal{F}} \text{ (cf. 3.7.2.)}$$

3.9 Proposition. *Every topological filter space is subtopological.*

Proof. Let (X, γ) be a topological filter space. Then (X, γ) is weakly subtopological,i.e. (X, q_γ) is topological, and complete, i.e. from $\mathcal{F} \in \gamma$ it follows the existence of some $x \in X$ with $\mathcal{F} \xrightarrow{q_\gamma} x$ or equivalently $\mathcal{F} \supset \mathcal{U}_{q_\gamma}(x)$, where $\mathcal{U}_{q_\gamma}(x)$ has a q_γ–open base. Thus, (X, γ) is also subtopological.

3.10 Theorem. *Let (X, γ) be a filter space. Then the following are equivalent:*

(1) (X, γ) is subtopological,

(2) (X^*, γ^*) is weakly subtopological,

(3) (X^*, γ^*) is topological,

(4) (X, γ) is a dense subspace(in **Fil**) *of some topological filter space,*

(5) (X, γ) is a subspace(in **Fil**) *of some topological filter space.*

Proof.
(1) $\Rightarrow$ (2). In order to prove that (X^*, q_{γ^*}) is topological, it suffices to show that C_6) is satisfied, since by 3.8., C_5) is valid.
1. $x \in X$: If $U_x^* \in \mathcal{U}_{q_{\gamma^*}}(x)$, then, by 3.8.1., there is some $U_x \in \mathcal{U}_{q_\gamma}(x)$ with $U_x \subset U_x^*$. Since (X, q_γ) is topological, there is some $V_x \in \mathcal{U}_{q_\gamma}(x) \subset \mathcal{U}_{q_{\gamma^*}}(x)$ such that for each $z \in V_x$, $U_x \in \mathcal{U}_{q_\gamma}(z) \subset i(\mathcal{U}_{q_\gamma}(z)) = \mathcal{U}_{q_{\gamma^*}}(z)$.
2. $\mathcal{F} \in X^* \backslash X$: Let $U_{\mathcal{F}}^* \in \mathcal{U}_{q_{\gamma^*}}(\mathcal{F}) = i(\mathcal{F}) \cap \dot{\mathcal{F}}$ (cf. 3.8.2.), i.e. $U_{\mathcal{F}}^* \supset F \cup \{\mathcal{F}\}$ for some $F \in \mathcal{F}$. Put $V_{\mathcal{F}}^* = (int_{\mathcal{X}_{q_\gamma}} F) \cup \{\mathcal{F}\}$. Then $V_{\mathcal{F}}^* \subset U_{\mathcal{F}}^*$ and since by assumption, $int_{\mathcal{X}_{q_\gamma}} F \in \mathcal{F}, V_{\mathcal{F}}^* \in \mathcal{U}_{q_{\gamma^*}}(\mathcal{F})$. Let $z \in V_{\mathcal{F}}^*$. Then a) $z = \mathcal{F}$ or b) $z \in int_{\mathcal{X}_{q_\gamma}} F$. In case a), $U_{\mathcal{F}}^* \in \mathcal{U}_{q_{\gamma^*}}(\mathcal{F})$ by assmuption. In case b), $U_{\mathcal{F}}^* \in \mathcal{U}_{q_{\gamma^*}}(z)$ since $int_{\mathcal{X}_{q_\gamma}} F \in \mathcal{U}_{q_\gamma}(z) \subset i(\mathcal{U}_{q_\gamma}(z)) = \mathcal{U}_{q_{\gamma^*}}(z)$ and $U_{\mathcal{F}}^* \supset int_{\mathcal{X}_{q_\gamma}} F$.
(2) $\Rightarrow$ (3). Since (X^*, γ^*) is complete, this implication is obvious.
(3) $\Rightarrow$ (4). Obvious.
(4) $\Rightarrow$ (5). Obvious.
(5) $\Rightarrow$ (1). Let (Y, η) be a topological filter space containing (X, γ) as a subspace. Since every topological filter space is subtopological (cf. 3.9.) and **SubTop** is bireflective in **Fil** (and thus closed under formation of subspaces), (X, γ) is also subtopological.

3.11 Definition. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called *subtopological* provided that it is **Fil**–determined and its corresponding filter space $(X, \gamma_{\mathcal{J}_X})$ is subtopological.

3.12 Corollary. *Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space. Then the following are equivalent:*
(1) $(X, \mathcal{J}_X)$ is subtopological,
(2) $(X, \mathcal{J}_X)$ is a subspace (in **SUConv***) of some topological semiuniform convergence space.*
(3) $(X, \mathcal{J}_X)$ is a dense subspace (in **SUConv***) of some topological semiuniform convergence space.*

Proof. Since **Fil-D-SUConv** $\cong$ **Fil** is bireflective in **SUConv** (and thus subspaces in **Fil** are formed as in **SUConv**) and the construct **T-SUConv** of all topological semiuniform convergence spaces (and uniformly continuous maps) is (concretely) isomorphic to $\mathbf{Top_s}$ as well as to **Top-Fil**, the above corollary is an immediate consequence of 3.10.

3.13 Remarks. 1) Obviously, the construct $\mathbf{Sub_{SUConv}T\text{-}SUConv}$ of all subspaces (in **SUConv**) of topological semiuniform convergence spaces (and uniformly continuous maps) is (concretely) isomorphic to **SubTop**.

2) Bentley [2] defines 'subtopological spaces' to be subspaces (in the construct **Near** of nearness spaces) of topological nearness spaces (= symmetric topological spaces) and shows that they are those filter spaces (= filtermerotopic spaces) which are nearness spaces. Since there is no difference in forming subspaces of symmetric topological spaces in **Near**, **Mer** (= construct of merotopic spaces in the sense of Katětov [9]) or **Fil** respectively it follows from 3.10. that a *filter space (= filtermerotopic space)*

is subtopological iff it is a nearness space.

3.14 Definition. A Cauchy space (X,γ) is called *uniformizable* provided that there is a uniformity $\mathcal{W}$ on X such that γ is the set of all Cauchy filters in $(X,\mathcal{W})$ (the uniformity $\mathcal{W}$ is called a *compatible uniformity*).

3.15 Remark. The construct **UChy** of all uniformizable Cauchy spaces (and Cauchy continuous maps) is a bireflective subconstruct of **Chy** and a bicoreflective subconstruct of **Unif** (cf. [11;1.3.]). Obviously, *the uniformizable Cauchy spaces are exactly the subspaces in* **Fil** *of all completely uniformizable topological spaces.*

3.16 Proposition. **UChy** *is a bireflective subconstruct of* **SubTop**.

Proof. Since **Chy** is bireflective in **Fil**, 3.16. follows immediately from 3.15.

3.17 Remark. Frič and Kent [7] studied completely regular Cauchy spaces as a generalization of uniformizable Cauchy spaces. Their definition coincides with the definition of completely regular filter spaces used by Bentley and Lowen–Colebunders [4], where a filter space is called *completely regular* provided that it is a subspace (in **Fil**) of some completely regular topological space (considered as filter space). A filter space (X,γ) is completely regular iff for each $\mathcal{F}\in\gamma$ the subfilter $\mathcal{G}=\{G\subset X : F$ is completely within G for some $F\in\mathcal{F}\}$ belongs to γ; here F is completely within G provided that there is a Cauchy continuous map $f:(X,\gamma)\to([0,1],\gamma_t)$ such that $f[F]\subset\{0\}$ and $f[X\backslash G]\subset\{1\}$, where γ_t denotes the set of all convergent filters on the unit interval $[0,1]$ endowed with the usual topology (cf. [4]). The construct **CRegFil** of completely regular filter spaces (and Cauchy continuous maps) in bireflective in **Fil** (cf. [4;3.2.]). The restriction of the bicoreflector $\mathcal{C}:\mathbf{Fil}\to\mathbf{KConv_s}$ leads to the result that the construct **CRegTop** of completely regular topological spaces (and continuous maps) is bicoreflectively embedded into **CRegFil**. **CRegFil** is related to other important constructs mentioned in this paper by means of the following diagram, where r (resp. c) stands for embedding as a bireflective (resp. bicoreflective) subconstruct:

$$\begin{tikzcd}
\mathbf{SUConv} & & \mathbf{Fil} \arrow[ll, "c", "r"'] & \mathbf{KConv_s} \arrow[l, "c"'] \\
 & & \mathbf{SubLim} \arrow[u, "r"] & \mathbf{Lim_s} \arrow[l, "c"'] \arrow[u, "r"] \\
 & & \mathbf{SubPrTop} \arrow[u, "r"] & \mathbf{PrTop_s} \arrow[l, "c"'] \arrow[u, "r"] \\
 & & \mathbf{SubTop} \arrow[u, "r"] & \mathbf{Top_s} \arrow[l, "c"'] \arrow[u, "r"] \\
\mathbf{ULim} \arrow[uuuu, "r"] & \mathbf{Chy} \arrow[l, "c"'] \arrow[uuur, "r"] & \mathbf{CRegFil} \arrow[l, "r"'] \arrow[u, "r"] & \mathbf{CRegTop} \arrow[l, "c"'] \arrow[u, "r"] \\
\mathbf{Unif} \arrow[u, "r"] & & \mathbf{UChy} \arrow[ll, "c"'] \arrow[u, "r"] &
\end{tikzcd}$$

4. Problem Find a nice characterization of subspaces in **Fil** of symmetric pseudotopological spaces (considered as filter spaces) where a generalized convergence space (X, q) is called pseudotopological provided that $(\mathcal{F}, x) \in q$ whenever $(\mathcal{U}, x) \in q$ for each ultrafilter $\mathcal{U} \supset \mathcal{F}$.

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