Partial pseudo-differential equations of fractional order: well-posedness of the Cauchy and multi-point value problems

S.R. Umarov*, Yu.F. Luchko** and R. Gorenflo**

* Mechanics and Mathematics Department
Tashkent State University
Tashkent, Uzbekistan

** Department of Mathematics and Computer Science
Free University of Berlin
Berlin, Germany

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Abstract

This paper is devoted to the Cauchy and multi-point value problems for partial pseudo-differential equations of fractional order. The used pseudo-differential operators are associated with the symbols which may have singularities. The solvability theorems for these problems in the space $\Psi_{G,p}(\mathbb{R}^n),\ 1 < p < \infty$, of functions in L_p whose Fourier transforms are compactly supported in a domain $G \subset \mathbb{R}^n$ and in its dual space $\Psi'_{-G,q}(\mathbb{R}^n),\ q = p/(p-1)$, are proved. The representations of the solutions in terms of pseudo-differential operators are constructed and some of their properties are given. The obtained results are then used to get solvability theorems in the Sobolev spaces $H^s(\mathbb{R}^n),\ s \in \mathbb{R}$.

1 Introduction

In the recent few decades differential equations of fractional order and integral equations of Abel type have been recognized as useful tools for description of memory and hereditary properties of various processes. In many cases they lead to more adequate models than those based on integer order derivatives. It should be noted that both ordinary and partial differential equations have been already successfully used by several researchers in modeling of real processes. Since in our paper we deal with partial differential equations, we will mention some applications of these equations; for numerous examples of models in terms of ordinary differential equations we refer to the papers by Gorenflo and Mainardi [GM97], Kempfle [Kem98], Mainardi [Mai97] and to the recent book by Podlubny [Pod99].

It seems that the first paper devoted to the construction of a particular model by aid of partial differential equations of fractional order was by Gerasimov [Ger48], who considered the movement of a viscous fluid between two moving surfaces. Some decades later Oldham and Spanier [OS74] used such equations (in their case of the order one-half) to describe some processes in electro-analytical chemistry (generalized Fick's law). Later on, the main attention was given to models based on the fractional diffusion-wave equation obtained by replacing the first- or second-order time derivative in the diffusion or wave equation, respectively, by a fractional derivative. This equation has been used for modeling of various processes including the behavior of viscoelastic materials (see Glöcke and Nonnenmacher [GN91], Hrusa and Nohel [HN85], Mainardi [Mai97], Renardy, Hrusa and Nohel [RHN87] and references there), diffusion in special types of porous media, which exhibit a fractal geometry (Nigmatullin [Nig86]) and for modeling of of the grey noise (Schneider [Sch90]). We note that nonlinear partial differential equations of fractional order have also appeared in applications. We refer, in particular, to the paper by Diethelm and Freed [DF99] where such equations were proposed for description of behavior of some viscoplastic materials.

Another direction for applications of the fractional diffusion-wave equation is connected with the fact that the fundamental solution of the Cauchy and Signaling problems for this equation can be interpreted in terms of the Lévy stable probability distributions (Gorenflo and Mainardi [GM98b], Mainardi, Paradisi and Gorenflo [MPG99], Schneider and Wyss [SW89]). As the next step in this direction various new random walk models approximating space-fractional diffusion process were constructed by Gorenflo and Mainardi (see, for example, [GM98a], [GM99] and [GM]). These random walk models can be used to describe some processes in finance (see Scalas, Gorenflo and Mainardi [SGM]). This direction is under especially extensive development nowadays.

As to the mathematical theory of partial differential equations of fractional order we mention here only some results most close to the subject of our paper. There are a lot of papers devoted to the study of the time-fractional diffusion-wave equation. We refer, in particular, to the papers by Fujita [Fuj90], Mainardi [Mai97] and Schneider and Wyss [SW89]. Fujita [Fuj90] proved the existence of a unique global solution of the Cauchy problem for the time-fractional diffusion-wave equation in the space $C([0,\infty); S(I\!R))$ where $S(I\!R)$ is the Schwartz space of rapidly decreasing functions. Let us note that existence and uniqueness results for initial value problems for general ordinary differential equations of fractional order can be found in Samko, Kilbas and Marichev [SKM93]. These authors give also some general results for boundary related value problems. Interesting results for solution of particular boundary value problems are presented in Djrbashian [Djr93], where the completeness of a system of eigenfunctions given in terms of Mittag-Leffler type functions is proved. Returning to the time-fractional diffusion-wave equation we refer to the paper by Mainardi [Mai97], who found an explicit form of the Green functions for the Cauchy and Signaling problems in terms of the Wright functions in the one-dimensional case, and to the paper by Schneider and Wyss [SW89], who considered the multi-dimensional case. Schneider [Sch86] (see also Gorenflo and Mainardi [GM98b], Mainardi, Paradisi and Gorenflo [MPG99]) showed that this Green function can be interpreted as a probability density. We note also the papers by Buckwar and Luchko [BL98], Gorenflo, Luchko and Mainardi [GLM99], Luchko and Gorenflo [LG98], who gave an explicit form of the scale-invariant solutions of some partial differential equations of fractional order.

More recently, some results have been obtained for the abstract Cauchy problem for the linear fractional differential equation $D_*^\alpha x(t) = Ax(t)$, where $A : \mathcal{D}(A) \to X$, $\mathcal{D}(A) \subset X$ is a linear unbounded closed operator in a Banach space X, and $D_*^\alpha x(t)$ is the Caputo fractional derivative. In the papers by Bazhlekova [Baz98] and Kochubei [Koc89] the necessary and sufficient conditions for solvability of the abstract Cauchy problem in the case $0 < \alpha < 1$ were given, by extending the conditions of the Hille-Yosida theorem from $\alpha = 1$ to $\alpha \in (0, 1]$. The case $0 < \alpha \leq 2$ was considered by El-Sayed [ElS95]. In the paper by Kostin [Kos93] the solvability conditions for the abstract Cauchy problem with the Riemann-Liouville fractional derivative instead of the Caputo derivative have been obtained in the case $\alpha > 0$. Gorenflo, Luchko and Zabrejko [GLZ99] described the set of initial data for which the solution of the abstract Cauchy problem can be represented by means of the Mittag-Leffler operator function.

As to boundary value problems for partial differential equations of fractional order in bounded and unbounded domains, there are a lot of applications including computer tomography, electrodynamics, electro-statics and elasticity theory. For mathematical investigations of such problems we refer, for example, to the works by Päivärinta and Rempel [PR92] and Natterer [Nat86], where the equation $\Delta^{\pm 1/2}u = f$ in two dimensions was studied in domains having piecewise smooth boundaries without sharp peaks, and the papers by Turmetov and Umarov [TU94] and Nazarova [Naz97], who considered some boundary value problems for elliptic equations of fractional order in the upper semi-space. We note that in this context another problem formulation is also natural. Namely, partial differential equations of integer order with boundary conditions given through operators of fractional calculus can be considered (see, for example, Nakhushev [Nah75] and Umarov [Uma93], [Uma94]). Finally, operators of fractional calculus have been used by several authors including Alimov [Ali72] and Samko, Kilbas and Marichev [SKM93] as a useful tool for investigation of various boundary value problems for partial differential equations of integer order and in spectral theory.

The space-fractional diffusion equation in one space dimension obtained from the classical diffusion equation by replacing the second-order space derivative by the pseudo-differential operator with the symbol $-|\xi|^\alpha,\ 0 < \alpha \leq 2$ has been also considered. Saichev and Zaslavsky [SZ97] have investigated it both from the analytical point of view and from that of applications in modern physics. The explicit formula for the Green function for the relevant Cauchy problem was given by Mainardi, Paradisi and Gorenflo [MPG99]. In a series of papers (see, for example, [GM98a], [GM99] and [GM]) Gorenflo and Mainardi developed a theory of random walk models approximating space-fractional diffusion processes. They showed that for properly scaled transition to vanishing space and time steps these models converge in distribution to the corresponding time-parameterized stable probability distribution which, in its turn, can be interpreted as the fundamental solution to the space-fractional diffusion equation. We should mention also the recent paper by Gorenflo, Iskenderov and Luchko [GIL] where the explicit form of the Green function for the Cauchy problem for that diffusion equation fractional both in time and in space is presented. In the paper by Scalas, Gorenflo and Mainardi [SGM] it was shown that this equation can be interpreted in the theory of continuous-time finance. In it the time fractional derivative is taken in the Caputo sense and the space fractional derivative is understood as a pseudo-differential operator with the symbol $-|\xi|^\alpha,\ 0 < \alpha \leq 2$.

In the present paper we consider first in detail the Cauchy problem for pseudo-differential equations of fractional order α, $m - 1 < \alpha < m \in I\!N$,

$$D_*^\alpha u(t,x) = A(D)u(t,x),\ t > 0,\ x \in I\!R^n, \tag{1}$$

$$\frac{\partial^k u(0,x)}{\partial t^k} = \varphi_k(x),\ x \in I\!R^n,\ k = 0, ..., m-1, \tag{2}$$

where the $\varphi_k,\ k = 0, ..., m-1$, are given functions in certain spaces described later, $D = (D_1, ..., D_n)$, $D_j = -i\frac{\partial}{\partial x_j},\ j = 1, \ldots, n$, $A(D)$ is a pseudo-differential operator with the symbol $A(\xi)$ which is a continuous function defined in an open domain $G \subset I\!\!R^n$, and D_*^α is the operator of fractional differentiation of order α ($\alpha > 0$) in the Caputo sense (see, for example, [Cap67], [GLM99], [GM97], [Luc99], [LG99], [Pod99])

$$D_*^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \ t > 0. \tag{3}$$

We also give some basic results for solution of the more general multi-point value problem for pseudo-differential equations of fractional order

$$D_*^{\alpha_m} u(t,x) + \sum_{k=1}^{m-1} A_k(D) D_*^{\alpha_k} u(t,x) + A_0(D) u(t,x) = 0, \ t \in (0,T), \ x \in I\!\!R^n, \tag{4}$$

$$\sum_{j=0}^{m-1} \Gamma_{kj}(D) \frac{\partial^k u(t_{kj},x)}{\partial t^k} = \varphi_k(x), \ k = 0,...,m-1, \ x \in I\!\!R^n, \tag{5}$$

where $D_*^{\alpha_k}$ is the Caputo fractional derivative operator as in (3) of order α_k, $k-1 < \alpha_k \leq k$, $k = 1,...,m$, the $A_k(D)$, $k = 0,...,m-1$, and $\Gamma_{kj}(D)$, $k,j = 0,...,m-1$, are pseudo-differential operators whose symbols $A_k(\xi)$, $k = 0,...,m-1$, $\Gamma_{kj}(\xi)$, $k,j = 0,...,m-1$, are continuous functions in an open domain $G \subset I\!\!R^n$, $t_{kj} \in [0,T]$, $0 < T < \infty$, and the φ_k, $k = 0,...,m-1$, are given functions.

We note that the Cauchy problem (1), (2) and some boundary value problems are particular cases of the multi-point value problem (4), (5).

In the case $\alpha_k \in I\!\!N$, $k = 1,...,m$, the Cauchy problem for pseudo-differential equations with analytic symbols or with symbols having singularities was studied, for example, by Dubinskij [Dub81], Umarov [Uma86] and Tran Duc Van [Van89]. Several authors including Antipko and Borok [AB92], Borok [Bor71], Umarov and Nazarova [UN97], Ptashnik [Pta84], Umarov and Saydamatov [US97] and Umarov [Uma97], [Uma98] (see also references therein) considered multi-point value problems with integer α_k, $k = 1,...,m$.

For studying the described Cauchy and more general multi-point value problems we use the corresponding properties of pseudo-differential operators with symbols which may have singularities. These operators were introduced by Dubinskij [Dub81], [Dub82] in the case of analytic symbols defined in some domain G and used by several authors including [Uma86], [Van89], [VHG90]. Later on, the systematic presentation of this approach was given in the books [Dub91], [VH94]. The construction of pseudo-differential operators used by Dubinskij and other authors essentially uses the analyticity of symbols, because they define a pseudo-differential operator with the help of local elements of differential operators of infinite order whose symbols are defined by power series.

In our constructions we choose another approach which does not require gluing of local elements of pseudo-differential operators. This approach allows us to consider non-analytic symbols which may have non-integrable or other type singularities on the boundary of G or in its exterior.

The choice of such wide class of symbols leads to certain restrictions in something else. Indeed, as the space of initial data we should take a sufficiently narrow class of entire functions of finite exponential type. Endowing this set of functions with an appropriate convergence turns it into a locally convex topological space. This space will be denoted by $\Psi_{G,p}(\mathbb{R}^n)$. We prove that for initial data from $\Psi_{G,p}(\mathbb{R}^n)$ the Cauchy problem (1), (2) can always be solved, and its solution is unique.

It should be remarked that this construction has a dual character. Namely, since the space of initial data is a locally convex space and satisfies all requirements of basic spaces of test functions we are able to construct a space of distributions as dual to $\Psi_{G,p}(\mathbb{R}^n)$. This space of distributions is sufficiently large, but now the solvability should be considered in a weaker sense, namely, in the sense of distributions.

We also note that under certain conditions on G (see [Uma97]) the space $\Psi_{G,p}(\mathbb{R}^n)$ is densely embedded into the Sobolev spaces $H_p^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. This fact allows to transfer the obtained results to Sobolev spaces by means of the closure of solution operators. In this framework it becomes important to study the problem of closure of pseudo-differential operators defined on $\Psi_{G,p}(\mathbb{R}^n)$ to the classical Sobolev spaces. In the case of analytic symbols this problem was studied in [Uma97].

To illustrate the general Cauchy problem (1), (2) we now consider three examples.

The first example is the time-fractional differential equation

$$D_*^\alpha u(t,x) = \Delta u(t,x), \ t > 0, \ x \in \mathbb{R}^n, \ \alpha > 0,$$

where Δ is the Laplace operator with the symbol $A(\xi) = -|\xi|^2$. As usual, we denote by $|\cdot|$ the Euclidean norm in the vector space $\mathbb{R}^n$. The Cauchy problem for this equation represents a fractional model of diffusion in the case $0 < \alpha < 1$ and processes intermediate between diffusion and wave propagation in the case $1 < \alpha < 2$. We prove later the relaxation property of the solution in the case $0 < \alpha < 1$ and the oscillation-relaxation property in the case $1 < \alpha < 2$.

The second example is the space-fractional equation

$$\frac{\partial u(t,x)}{\partial t} = D_0^\beta u(t,x), \ t > 0, \ x \in \mathbb{R}^n, \ \beta > 0,$$

where the symbol of D_0^β is $-|\xi|^\beta$ which is a continuous function. We note that the pseudo-differential operator D_0^β is connected with the inversion of the fractional Riesz potential (see, for example, [SKM93], [Rub96]) and can be rewritten in the form

$$D_0^\beta f(x) = \frac{-1}{c_{n,l}(\beta)} \int_{R^n} \frac{(\Delta_h^l f)(x)}{|h|^{n+\beta}} dh,$$

Here

$$(\Delta_h^l f)(x) = (\tau_{-\frac{h}{2}} - \tau_{\frac{h}{2}})^l f(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x + (\frac{l}{2} - k)h)$$

is the centered finite difference of an even order $l > \beta$ with the step $h \in I\!R^n$, $(\tau_h f)(x) = f(x-h)$ is the shift operator, and $c_{n,l}(\beta)$ is a constant defined by

$$c_{n,l}(\beta) = 2^{l-\beta} i^l \int_{I\!R^n} \sin^l y_1 |y|^{-n-\beta} dy,$$

y_1 being the first coordinate of the vector $y = (y_1, \ldots, y_n) \in I\!R^n$. In the one-dimensional case and under the condition $0 < \beta \leq 2$ this equation describes symmetric Lévy-Feller diffusion processes [Fel52]. Symmetric and non-symmetric random walk models approximating Lévy-Feller diffusion processes were presented in detail by Gorenflo and Mainardi [GM98a], [GM98b], [GM99].

The third example is the time- and space-fractional differential equation

$$D_*^\alpha u(t,x) = D_0^\beta u(t,x), \ t > 0, \ x \in I\!R^n, \ \alpha, \beta > 0.$$

This equation may be considered as a generalization of the first two equations. With $\beta = 2$ we obtain the first, with $\alpha = 1$ the second equation. In Section 4 we obtain a smoothness theorem for the solution of the Cauchy problem for this equation in the Sobolev spaces $H^s(I\!R^n)$, $s \in I\!R$ for all values $\alpha \in (0,2]$. This theorem is in agreement with the classical results for the Cauchy problem for the wave equation. Unfortunately, $\alpha = 1$ is an exceptional value, in this case our theorem is weaker than the classical result.

Our paper is organized as follows. In Section 2 we give the description of the space $\Psi_{G,p}(I\!R^n)$ introduced in [Uma97], define pseudo-differential operators with singular symbols and study their properties. In Section 3 we prove solvability theorems for the Cauchy problem (1), (2) in the space $\Psi_{G,p}(I\!R^n)$ and in its dual space $(\Psi_{G,p}(I\!R^n))^* = \Psi'_{-G,q}(I\!R^n)$, $q = p/(p-1)$. Moreover, we study there some additional properties of solutions coming from the use of the Caputo fractional derivative. In Section 4 we study the closability properties of pseudo-differential operators defined on $\Psi_{G,p}(I\!R^n)$ in the Sobolev spaces $H^s(I\!R^n)$, $s \in I\!R$. Using the obtained results we also prove some theorems on solvability in these spaces and consider in detail the model equations given above as three examples. Finally in Section 5 we consider the general multi-point value problem (4), (5) and outline the related results.

2 Pseudo-differential operators and spaces of functions and functionals

Let be given an open domain in $G \subset I\!R^n$ and a system of open sets $\{g_k\}_{k=0}^{\infty}$ forming locally finite covering of G, i.e., $G = \bigcup_{k=0}^{\infty} g_k$, $g_k \subset\subset G$. We assume that any compact set $K \subset G$ has a nonempty intersection with finitely many sets g_k and denote by $\{\phi_k\}_{k=0}^{\infty}$ a smooth partition of unity of G.

For a function $f \in S$, the space of rapidly decreasing test functions, we define its Fourier transform $\hat{f}$ as

$$\hat{f}(\xi) \;=\; Ff(\xi) = \int_{I\!R^n} f(x)e^{-ix\xi}dx, \tag{6}$$

and as usual we extend the operator F to the space S' of tempered distributions (for definition of S' see [GS68]).

Let a function f be in $L_p(I\!R^n)$, $1 \leq p \leq \infty$, and let its Fourier transform have a compact support in G. The set of all such functions endowed with the convergence defined below is denoted by $\Psi_{G,p}(I\!R^n)$.

Definition 1. *A sequence of functions $f_m \in \Psi_{G,p}(I\!R^n)$ is said to converge to an element $f_0 \in \Psi_{G,p}(I\!R^n)$ iff:*

(a) there exists a compact set $K \subset G$ such that supp $\hat{f}_m \subset K$ for all $m \in I\!N$;

(b) $\|f_m - f_0\|_p = (\int_{R^n} |f_m - f_0|^p dx)^{\frac{1}{p}} \to 0$ for $m \to \infty$.

In the case $p = 2$ we write simply $\Psi_G(I\!R^n)$ instead of $\Psi_{G,2}(I\!R^n)$.

According to the Paley-Wiener-Schwartz theorem, the elements of $\Psi_{G,p}(I\!R^n)$ are entire functions of exponential type which, restricted to $I\!R^n$, are in the space $L_p(I\!R^n)$. In the case $p = 2$ this space was introduced and studied by Dubinskij [Dub82] (see also [Dub91] and [VH94]), who denoted it by $H^{\infty}(G)$.

We can represent $\Psi_{G,p}(I\!R^n)$ as an inductive limit of Banach spaces constructed with the help of compact sets covering G. Let $G_N = \cup_{k=1}^{N} g_k$ and $\kappa_N(\xi) = \sum_{k=1}^{N} \phi_k(\xi)$. It is clear that $G_N \subset G_{N+1}$, $N \in I\!N$ and $G_N \to G$ for $N \to \infty$.

Denote by Ψ_N the set of functions $f \in L_p(I\!R^n)$ satisfying the conditions (i)–(iii):

(i) supp $Ff \subset G_N$;
(ii) supp $Ff \cap \text{supp } \phi_j = \emptyset$ for $j > N$;
(iii) $p_N(f) = \|F^{-1}\kappa_N Ff\|_p < \infty$.

Lemma 1. *For $N \in I\!N$ the relations*
(a) $\Psi_N \hookrightarrow \Psi_{N+1}$,
(b) $\Psi_N \hookrightarrow L_p$
are valid, where $\hookrightarrow$ denote the operation of continuous embedding.

P r o o f. Let $f \in \Psi_N$, i.e., $\text{supp}\, Ff \subset G_N$. According to the definition of G_N we have $\text{supp}\, Ff \subset G_{N+1}$. Moreover

$$p_{N+1}(f) = \|F^{-1}\kappa_{N+1}Ff\|_p \leq \|F^{-1}\kappa_N Ff\|_p + \|F^{-1}\phi_{N+1}Ff\|_p = p_N(f).$$

The last equality follows from the fact that supp $\phi_{N+1} \cap \text{supp } Ff = \emptyset$ which gives us $\|F^{-1}\phi_{N+1}Ff\|_p = 0$. Thus the first assertion is true. It is evident that $\Psi_N \subset L_p$. It remains to prove that this embedding is continuous. Indeed, since for $f \in \Psi_N$ we have supp $Ff \cap \text{supp } \phi_k = \emptyset$ for every $k > N$, it follows that

$$\|f\|_p = \|F^{-1}Ff\|_p = \|F^{-1}\sum_{k=0}^{\infty}\phi_k Ff\|_p = \|F^{-1}\kappa_N Ff\|_p = p_N(f) < \infty. \quad (7)$$

■

Corollary 1. $\Psi_{G,p}(I\!R^n) = \text{ind}\lim_{N\to\infty}\Psi_N$.

P r o o f. We have to show that the topology of the inductive limit agrees with the convergence introduced above. Let $f_m \to f_0$ in $\Psi_{G,p}(I\!R^n)$. According to the definition of convergence in $\Psi_{G,p}(I\!R^n)$ there is a compact set $K \subset G$ such that $\text{supp}\,\hat{f}_m \subset K$ for all $m \in I\!N$. Since K has non-empty intersection only with a finite number of sets from the system $\{g_k\}_{k=0}^{\infty}$, then $K \subset G_N$ for all N starting from some index N_0. Taking into account this inclusion and the fact that every κ_N is a multiplier in $L_p(I\!R^n)$ we have the relation

$$p_N(f_m - f_0) = \|F^{-1}\kappa_N F(f_m - f_0)\|_p \leq C_p\|f_m - f_0\|_p \to 0$$

for $m \to \infty$ for all $N \geq N_0$.

Vice versa, let $p_N(f_m - f_0) \to 0$ for $m \to \infty$ for every N greater than some number N_0. Then supp $\hat{f}_m \subset G_{N_0}$. Hence, there exists a compact set $K \subset G_{N_0} \subset G$ such that supp $\hat{f}_m \subset K$. Moreover, assertion (b) of Lemma 1 implies that for every fixed $N \geq N_0$ we have

$$\|f_m - f_0\|_p \leq p_N(f_m - f_0) \to 0$$

as $m \to \infty$. ■

Let $q = p/(p-1),\ 1 \leq p < \infty$. Denote by $\Psi'_{-G,q}(I\!R^n)$ the space of all linear bounded functionals defined on the space $\Psi_{G,p}(I\!R^n)$ endowed with the weak topology. Namely, we say that a sequence of functions $g_N \in \Psi'_{-G,q}(I\!R^n)$ converges to an element $g_0 \in \Psi'_{-G,q}(I\!R^n)$ in the weak sense if for all $f \in \Psi_{G,p}(I\!R^n)$ the sequence of numbers $< g_N, f >$ converges to $< g_0, f >$ as $N \to \infty$. We recall that the notation $< g, f >$ means the value of $g \in \Psi'_{-G,q}(I\!R^n)$ on an element $f \in \Psi_{G,p}(I\!R^n)$.

It follows from Corollary 1 that $\Psi'_{-G,q}(I\!R^n)$ is the projective limit of Banach spaces being conjugate to Ψ_N.

One of the important questions for our further aims is whether the space $\Psi_{G,p}(I\!R^n)$ can be densely embedded in some classical spaces intermediate between $\Psi_{G,p}(I\!R^n)$ and $\Psi'_{-G,q}(I\!R^n)$. For Besov and Lizorkin-Triebel spaces this problem was studied in [Uma97]. For the reader's convenience we reproduce the proof in the case of the Sobolev spaces H^s_p, $1 \leq p < \infty$, $s \in I\!R$, consisting of all tempered distributions $f \in S'$ for which

$$\|f\|_{p,s} = (\int_{R^n} |F^{-1}(1+|\xi|^2)^{\frac{s}{2}} Ff|^p dx)^{\frac{1}{p}} = \|F^{-1}(1+|\xi|^2)^{\frac{s}{2}} Ff\|_p < \infty.$$

In the case $p = 2$ we write simply H^s instead of H^s_2.

Lemma 2 (continuity and density lemma). *Let $1 < p < \infty$, $s \in I\!R$ and $G \subset I\!R^n$. Then the embeddings*

$$\Psi_{G,p} \hookrightarrow H^s_p \hookrightarrow \Psi'_{-G,q} \tag{8}$$

are continuous in the corresponding topologies. The first embedding is dense if and only if the complement of G in $I\!R^n$ has zero measure.

The proof of the density statement of this important lemma will be given in Section 6. The continuity of the first embedding follows easily from (7) and embeddings theorems for the Sobolev spaces H^s_p. By using this result and applying the standard duality techniques we get the continuity of the second embedding.

Now we consider some properties of a pseudo-differential operator $A(D)$ with a symbol $A(\xi)$ defined and continuous in a domain $G \subset I\!R^n$. Outside of G or on its boundary $A(\xi)$ may have singularities of arbitrary type. For a function $\varphi \in \Psi_{G,p}(I\!R^n)$ the operator $A(D)$ is defined by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^n} \int_G A(\xi) F\varphi(\xi) e^{ix\xi} d\xi, \tag{9}$$

where $F\varphi$ is the Fourier transform given by (6). Generally speaking, $A(D)$ has no sense even for functions in the space $C_0^\infty(I\!R^n)$. In fact, let ξ_0 be a non-integrable singular point of $A(\xi)$ and denote by $O(\xi_0)$ some neighborhood of

ξ_0. Let us take a function $\varphi \in C_0^\infty(I\!R^n)$ with $F\varphi(\xi) > 0$ for $\xi \in O(\xi_0)$ and $F\varphi(\xi_0) = 1$. Then it is easy to verify that $A(D)\varphi(x) = \infty$. On the other hand, for $\varphi \in \Psi_{G,p}(I\!R^n)$ the integral in (9) is convergent due to the compactness of supp $F\varphi \subset G$.

We define the operator $A(-D)$ acting in the space $\Psi'_{-G,q}(I\!R^n)$ by the formula

$$< A(-D)f, \varphi > = < f, A(D)\varphi >, \ f \in \Psi'_{-G,q}(I\!R^n), \ \varphi \in \Psi_{G,p}(I\!R^n). \quad (10)$$

Lemma 3. *The spaces $\Psi_{G,p}(I\!R^n)$ and $\Psi'_{-G,q}(I\!R^n)$ are invariant with respect to the action of an arbitrary pseudo-differential operator $A(D)$ whose symbol is continuous in G. Moreover, if $A(\xi)k_N(\xi)$ is a multiplier in L_p for every $N \in I\!N$, then this operator acts continuously.*

P r o o f. Let $\varphi \in \Psi_{G,p}(I\!R^n)$, i.e., supp $F\varphi \subset K$ for an appropriate compact set $K \subset G$ and $\varphi \in L_p(I\!R^n)$. Evidently, supp $F(A(D)\varphi) =$ supp $(A(\xi)F\varphi) \subset K$ and $A(D)\varphi \in L_p(I\!R^n)$. Hence, $A(D)\Psi_{G,p}(I\!R^n) \subset \Psi_{G,p}(R^n)$.

Further, let $\varphi_m \to \varphi$ in $\Psi_{G,p}(I\!R^n)$ and $A(\xi)k_N(\xi)$ be a multiplier in L_p for every $N \in I\!N$. Since $\Psi_{G,p}(I\!R^n)$ can be represented as the inductive limit of Ψ_N as $N \to \infty$ (see Corollary 1), there exists an index $N > 0$ such that $p_N(\varphi_m - \varphi) \to 0$ as $m \to \infty$. Then we have the estimate

$$p_N(A(D)\varphi_m - A(D)\varphi) = p_N(A(D)(\varphi_m - \varphi))$$

$$= \|F^{-1}A(\xi)k_N(\xi)F(\varphi_m - \varphi)\|_p \le C_p\|\varphi_m - \varphi\|_p \to 0, \ m \to \infty.$$

■

Remark 1. *In the case $p = 2$ an arbitrary pseudo-differential operator whose symbol is continuous in G acts continuously without the additional condition that $A(\xi)k_N(\xi)$ is a multiplier in L_2 for every $N \in I\!N$.*

Lemma 4. *Let the complement of $G \subset I\!R^n$ be of zero measure in $I\!R^n$ and the symbol $A(\xi)$ satisfy the estimate*

$$|A(\xi)| \le C(1 + |\xi|)^\ell, \ C > O, \ \xi \in I\!R^n, \ \ell \in I\!R.$$

Then for the operator $A(D) : \ \Psi_G(I\!R^n) \to \Psi_G(I\!R^n)$ corresponding to $A(\xi)$ there exists a continuous closure

$$\overline{A}: \ H^s(I\!R^n) \to H^{s-\ell}(I\!R^n).$$

P r o o f. It follows from Lemma 2 that $\overline{\Psi_G(I\!\!R^n)} = H^s(I\!\!R^n)$ for arbitrary $s \in I\!\!R$. Let $\varphi \in \Psi_G(I\!\!R^n)$. Taking into account the condition for the symbol $A(\xi)$ we have the estimate

$$\|A(D)\varphi(x)\|^2_{2,s-\ell} = \int_{R^n} |A(\xi)\hat{\varphi}(\xi)|^2(1+|\xi|)^{2(s-\ell)}d\xi$$

$$\leq C\int_{I\!\!R^n} |\hat{\varphi}|^2(1+|\xi|)^{2s}d\xi = C\|\varphi\|^2_{2,s}.$$

These two facts yield the existence of the closure of the operator $A(D)$ which we denote by $\overline{A}$. It follows from the last estimate that $\overline{A}: \ H^s(I\!\!R^n) \to H^{s-\ell}(I\!\!R^n)$ is continuous. ■

Remark 2. *The analogous assertion is valid for $p \neq 2$ too. The interested reader is referred to [Uma97] for details.*

3 Well-posedness of the Cauchy problem in $\Psi_{G,p}(I\!\!R^n)$ and $\Psi'_{-G,q}(I\!\!R^n)$

The main purpose of this section is to prove solvability theorems for the Cauchy problem (1), (2) and to give some properties of its solutions. It should be noted that, for the dual theory, we always assume that the right-hand side of the equation (1) is replaced with $A(-D)u(t,x)$ defined by the formula (10). Otherwise there arises confusion connected with the disagreement of actions of pseudo-differential operators and their domains.

At the first stage, using some formal manipulations, we get a representation for the solution of (1), (2). One important point is that the solution operators are also pseudo-differential. Then we study the properties of their symbols and use them to prove the solvability theorems.

Applying formally the Fourier transform to the equations (1), (2) we get

$$D^\alpha_* \hat{u}(t,\xi) = A(\xi)\hat{u}(t,\xi), \ t > 0, \ \xi \in G, \tag{11}$$

$$\frac{\partial^k \hat{u}(0,\xi)}{\partial t^k} = \hat{\varphi}_k(\xi), \ \xi \in G, \ k = 0, ..., m-1. \tag{12}$$

The problem (11), (12) is an initial value problem for an ordinary differential equation of fractional order α depending on the parameter ξ. Gorenflo and Mainardi in their recent work [GM97] (see also [Luc99] or [LG99]) have studied such problems. Their method is applicable in our case as well. It gives us the representation

$$\hat{u}(t,\xi) = \sum_{k=1}^{m} J^{k-1}E_\alpha(t^\alpha A(\xi))\hat{\varphi}_{k-1}(\xi)$$

for the solution of (11), (12). In this formula J^k (for $k \in I\!N_0$) is the k-th power of the common integration operator with lower limit 0,

$$J^k f(t) = \underbrace{\int_0^t \cdots \int_0^\tau}_{k-\text{times}} f(\tau_1) d\tau_1 \ = \ \int_0^t \frac{(t-\tau)^{k-1}}{\Gamma(k)} f(\tau)\, d\tau \quad \text{for} \quad k \geq 1,$$

$$J^0 \ = \ I \ = \ \text{identity operator},$$

and $E_\alpha(z)$ is the Mittag-Leffler function (see [EMOT55], [Djr66])

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

Introducing the notation

$$B_k(\alpha; t, \xi) = J^{k-1} E_\alpha(A(\xi)t^\alpha), \ k = 1, ..., m \tag{13}$$

and applying the inverse Fourier transform we obtain the solution of the Cauchy problem (1), (2) in the form

$$u(t,x) = \sum_{k=1}^{m} B_k(\alpha; t, D)\varphi_{k-1}(x). \tag{14}$$

Here the pseudo-differential operator $B_k(\alpha; t, D)$, $k = 1, ..., m$, has the symbol $B_k(\alpha; t, \xi)$ given by (13). We call it the k-th solution operator of the Cauchy problem (1), (2).

Definition 2. *Let $m - 1 < \alpha < m \in I\!N$. A function*

$$u(t,x) \in C^{(m)}(t > 0; \Psi_{G,p}(I\!R^n)) \cap C^{(m-1)}(t \geq 0; \Psi_{G,p}(I\!R^n))$$

is called a strong solution of the problem (1), (2) if it satisfies the equation (1) and the initial conditions (2) pointwise.

Definition 3. *Let $m - 1 < \alpha < m \in I\!N$. A function*

$$u(t,x) \in C^{(m)}(t > 0; \Psi'_{-G,q}(I\!R^n)) \cap C^{(m-1)}(t \geq 0; \Psi'_{-G,q}(I\!R^n))$$

is called a weak solution of the problem (1), (2) (with $A(-D)$ defined by (10) instead of $A(D)$) if it satisfies the equation (1) and the conditions (2) in the following sense: for arbitrary $v \in \Psi_{G,p}(I\!R^n)$ the equalities

$$< D_*^\alpha u(t,x), v(x) > = < u(t,x), A(D)v(x) >,$$

$$lim_{t \to +0} < u^{(k)}(t,x), v(x) > = < \varphi_k(x), v(x) >, \ k = 0, ..., m-1,$$

hold.

To prove solvability theorems we need some auxiliary lemmas. Before formulating these we introduce some notations. Let $C^\ell(t \geq 0; C(G))$ be the space of functions $f(t,\xi)$ continuous with respect to ξ at any fixed $t \in [0,\infty)$ and having continuous derivatives up to the order ℓ with respect to t up to the bound $t = 0$ for fixed ξ. The corresponding meaning is given to $C^\ell(t > 0; C(G))$. Analogously, we write $C^\infty(t > 0; C(G))$ for the space of functions which are infinitely often differentiable with respect to t.

Lemma 5. *For every $k = 1, ..., m$ the following assertions are valid:*
(i) $B_k(\alpha; t, \xi) \in C^{m+k-2}(t \geq 0; C(G))$;
(ii) $B_k(\alpha; t, \xi) \in C^\infty(t > 0; C(G))$.

P r o o f. The symbol $B_1(\alpha; t, \xi) = E_\alpha(A(\xi)t^\alpha)$ is a superposition of the Mittag-Leffler function $E_\alpha(z)$ which is an entire function and of the function $\psi(\xi, t) = A(\xi)t^\alpha$. The continuity of $A(\xi)$ implies that for every fixed $t > 0$ the function $B_1(\alpha; t, \xi)$ is also continuous in the domain G. Moreover, the function $\phi(t) = t^\alpha$ is infinitely often differentiable for $t > 0$. Hence, $B_1(\alpha; t, \xi) \in C^\infty(t > 0; C(G))$. For $m - 1 < \alpha < m \in I\!N$ it is easy to check that $\phi(t) = t^\alpha$ has all derivatives up to order $m-1$, and these are continuous up to $t = +0$. This implies

$$B_1(\alpha; t, \xi) \in C^{m-1}(t \geq 0; C(G)).$$

Similarly we have (J^{k-1} acting with respect to the variable t)

$$B_k(\alpha; t, \xi) = J^{k-1} E_\alpha(A(\xi)t^\alpha) = J^{k-1} B_1(\alpha; t, \xi).$$

This function is $m + k - 2$ times differentiable with respect to t for every fixed ξ due to the fact that the integration operator J^{k-1} increases the order of differentiability of $B_1(\alpha; t, \xi)$ by $k - 1$. Hence, $B_k(\alpha; t, \xi) \in C^{m+k-2}(t \geq 0; C(G))$. ■

Lemma 6. *For $m - 1 < \alpha < m \in I\!N$ the following relations are true:*
(i) $\frac{\partial^{k-1} B_k(\alpha;t,\xi)}{\partial t^{k-1}} \to 1$ *for* $t \to 0,\ k = 1, ..., m$;
(ii) $\frac{\partial^\ell B_k(\alpha;t,\xi)}{\partial t^\ell} \to 0$ *for* $t \to 0,\ \ell = 0, ..., m-1,\ k = 1, ..., m,\ l \neq k-1$.

P r o o f. According to the definition of the solution operators we have

$$\frac{\partial^{k-1} B_k(\alpha; t, \xi)}{\partial t^{k-1}} = E_\alpha(A(\xi)t^\alpha),\ k = 1, ..., m.$$

This relation and the formula $E_\alpha(0) = 1$ give us (i). Now let $k-1 < \ell \leq m-1$. It is not difficult to verify that the derivative $\frac{\partial^\ell B_k(\alpha;t,\xi)}{\partial t^\ell}$ can be represented as a linear combination of expressions of the type

$$t^{\alpha-j}E_{\alpha}^{(j)}(A(\xi)t^{\alpha}),\ j \le \ell.$$

Since $m-1<\alpha<m\in I\!N$, then all of them tend to zero if $t\to+0$. In the case $0\le\ell<k-1$ we have

$$\frac{\partial^{\ell}B_k(\alpha;t,\xi)}{\partial t^{\ell}} = J^{k-\ell-1}E_{\alpha}(A(\xi)t^{\alpha})$$

$$= \frac{1}{(k-\ell-1)!}\int_0^t (t-\tau)^{k-\ell-2}E_{\alpha}(A(\xi)t^{\alpha})d\tau \to 0 \ \text{ for } \ t\to+0.$$

■

Theorem 1. *Let $m-1<\alpha<m\in I\!N$ and $\varphi_j\in\Psi_{G,p}(I\!R^n),\ j=0,...,m-1$. Then the Cauchy problem (1), (2) has a unique strong solution. This solution is given by the representation (14).*

P r o o f. Let $\varphi_j\in\Psi_{G,p}(I\!R^n),\ j=0,...,m-1$. According to the construction every term in the right-hand side of (14) satisfies (at least formally) the equation (1) and, due to Lemma 6, conditions (2). It follows from Lemma 5 that for the symbol of the k-th solution operator we have the inclusion

$$B_k(\alpha;t,\xi)\in C^{m+k-2}(t\ge 0;C(G))\cap C^{\infty}(t>0;C(G)).$$

Lemma 3 yields

$$B_k(\alpha;t,D)\varphi_{k-1}(x)\in\Psi_{G,p}(I\!R^n) \ \text{ for every fixed } \ t>0,$$

$$B_k(\alpha;t,D)\varphi_{k-1}(x)\in C^{m+k-2}(t\ge 0)\cap C^{m}(t>0) \ \text{ for every fixed } \ x\in I\!R^n.$$

Hence,

$$u(t,x)\in C^{(m)}(t>0;\Psi_{G,p}(I\!R^n))\cap C^{(m-1)}(t\ge 0;\Psi_{G,p}(I\!R^n)),$$

and $u(t,x)$ is a strong solution of the Cauchy problem (1), (2). Its uniqueness follows from the representation formula (14) and the $\Psi_{G,p}(I\!R^n)$-continuity of pseudo-differential operators with continuous symbols (see Lemma 3). ■

Theorem 2. *Let $m-1<\alpha<m\in I\!N$ and $\varphi_j\in\Psi'_{-G,q}(I\!R^n),\ j=0,...,m-1$. Then the Cauchy problem (1), (2) has a unique weak solution. This solution is given by*

$$u(t,x)=\sum_{k=1}^{m}B_k(\alpha;t,-D)\varphi_{k-1}(x), \tag{15}$$

where $B_k(\alpha;t,-D),\ k=1,...,m$, is the k-th solution operator with the symbol $B_k(\alpha;t,\xi)$.

P r o o f. Let $\varphi_j \in \Psi'_{-G,q}(I\!R^n),\ j = 0,...,m-1$. Lemma 3 implies that every term in the right-hand side of (15), namely, $u_k(t,x) = B_k(\alpha;t,-D)\varphi_{k-1}(x),\ k = 1,\dots,m$, is a functional from the space $\Psi'_{-G,q}(I\!R^n)$. Further, to prove the theorem we have to show that $u_k(t,x),\ k = 1,...,m$, satisfies the equation (1) and the initial conditions (2) in the weak sense. In fact, let $v \in \Psi_{G,p}(I\!R^n)$ be an arbitrary function. We have

$$< D_*^\alpha u_k(t,x) - A(-D)u_k(t,x), v(x) >$$

$$= < [D_*^\alpha B_k(\alpha;t,-D) - A(-D)B_k(\alpha;t,-D)]\varphi_{k-1}(x), v(x) >$$
$$= < \varphi_{k-1}(x), [D_*^\alpha B_k(\alpha;t,D) - A(D)B_k(\alpha;t,D)]v(x) > .$$

Due to the definition of $B_k(\alpha;t,D)$ we have $D_*^\alpha B_k(\alpha;t,D) \equiv A(D)B_k(\alpha;t,D)$. Hence, $u_k(x,t)$ satisfies the equation (1) in a weak sense. Moreover, by using Lemma 6 it is not difficult to verify that

$$\frac{\partial^{k-1}}{\partial t^{k-1}} B_k(\alpha;0,D) = \delta_{k,\ell} \cdot I,\ k = 1,...,m,\ \ell = 1,...,m,$$

where I is the identity operator and $\delta_{k,\ell}$ is Kronecker's symbol. This gives us the fulfillment of (2) in the weak sense. ■

Let $BC(t \geq 0, \Psi_G(I\!R^n))$ be the set of functions $f(t,x)$ such that
(a) $f(t,x)$ as a function of x is in the space $\Psi_G(I\!R^n)$ for every fixed $t > 0$;
(b) $f(t,x) \in C(t > 0; \Psi_G(I\!R^n))$;
(c) $\|f(t,x)\|_2 \leq C$ for all $t \in [0,T]$, where T is an arbitrary (but fixed) positive number and C is a constant not depending on f (C may depend on T).

Similarly, let $BC(t \geq 0, \Psi'_{-G}(I\!R^n))$ be the set of functions $f(t,x)$ such that
(a) $f(t,x)$ is in $\Psi'_{-G}(I\!R^n)$ for every fixed $t > 0$;
(b) $f(t,x) \in C(t > 0; \Psi'_{-G}(I\!R^n))$;
(c) for a fixed $T < \infty$ the estimate $| < f(t,x), v(x) > | \leq C$ holds for all $t \in [0,T]$ and for all $v \in \Psi_G(I\!R^n)$ with a constant C not depending on f (C may depend on v and T).

Lemma 7. *For every $k = 1,...m$ and every fixed $\xi \in K \subset\subset G$ there exists a positive constant $C_{\alpha,K}$ such that the inequality*

$$|D_*^\alpha B_k(\alpha;t,\xi)| \leq C_{\alpha,K} < \infty,\ 0 \leq t \leq T,\ m-1 < \alpha < m \in I\!N$$

holds.

P r o o f. Recalling and taking into account $D_*^\alpha f = J^{m-\alpha}D^m f$ we have

$$D_*^\alpha B_k(\alpha;t,\xi) = J^{m-\alpha}D^m B_k(\alpha;t,\xi) = J^{m-\alpha}D^m J^{k-1} E_\alpha(A(\xi)t^\alpha)$$

$$= J^{m-\alpha} D^{m-k+1} E_\alpha(A(\xi)t^\alpha) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} D^{m-k+1} E_\alpha(A(\xi)t^\alpha) d\tau.$$

It is easy to see from the last expression that the most irregular case is $k = 1$, i.e.,

$$D_*^\alpha B_1(\alpha; t, \xi) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} D^m E_\alpha(A(\xi)t^\alpha) d\tau.$$

By using the series expansion of the Mittag-Leffler function we get the asymptotics (for $A(\xi) \neq 0$)

$$D^m E_\alpha(A(\xi)t^\alpha) = O(t^{\alpha-m}),\ t \to 0.$$

Hence the last integral is asymptotically equivalent to the integral

$$I_\alpha \ = \ c_\alpha \int_0^t \frac{(t-\tau)^{m-1-\alpha}}{t^{m-\alpha}} d\tau.$$

Using the substitution $\tau = ts$ we reduce the last integral to the Euler beta-function, namely,

$$I_\alpha \ = \ c_\alpha \int_0^1 s^{(\alpha+1-m)-1} (1-s)^{m-\alpha-1} ds \ = \ c_\alpha B(\alpha+1-m, m-\alpha),$$

which is a finite number due to the inequalities $m-1 < \alpha < m$. ∎

Theorem 3. *Let the conditions of Theorem 1 be fulfilled. Then the strong solution of the Cauchy problem (1), (2) given by (14) has the following properties:*

(a) $u(t,x) \in C^\infty(t > 0; \Psi_G(\mathbb{R}^n))$;

(b) $D_*^\alpha u(t,x) \in BC(t \geq 0, \Psi_G(\mathbb{R}^n))$.

P r o o f. (a) is an implication of the corresponding property of symbols of the solution operators. Let us prove (b). Lemma 7 implies

$$\|D_*^\alpha u(t,x)\|_2 \leq \sum_{k=1}^m \int_{R^n} |D_*^\alpha B_k(\alpha; t, D)\varphi_{k-1}(x)|^2 dx$$

$$= \ C \sum_{k=1}^m \int_{R^n} |D_*^\alpha B_k(\alpha; t, \xi)|^2 |\varphi_{k-1}(\xi)|^2 d\xi \leq C_t \sum_{k=1}^m \|\varphi_{k-1}\|^2 < \infty$$

for all $t \in [0, T]$, where T is an arbitrary (but fixed) number. ∎

Theorem 4. *Let the conditions of Theorem 2 be fulfilled. Then the weak solution of the problem (1), (2) given by (15) has the following properties:*

(i) $u(t,x) \in C^\infty(t > 0; \Psi'_{-G}(\mathbb{R}^n))$;

(ii) $D_*^\alpha u(t,x) \in BC(t \geq 0, \Psi'_{-G}(\mathbb{R}^n))$.

P r o o f. (i) follows from the corresponding property of the symbols of the solution operators. We check (ii). Let v be an arbitrary element of $\Psi_G(\mathbb{R}^n)$. It can be easily verified that for $g \in \Psi'_{-G}(\mathbb{R}^n)$ the equality

$$< g(x), v(x) > = \ (2\pi)^{-n} < \hat{g}(\xi), \hat{v}(\xi) >$$

is valid. According to the definition of the space $\Psi_G(\mathbb{R}^n)$ $\operatorname{supp}\hat{v}$ is a compact subset of G. Hence, $\hat{g}$ is locally square-integrable in G. Moreover, the inequality

$$| < g, v > | \leq C \|\hat{g}\|_{L_2(\operatorname{supp}\hat{v})} \|v\|_2$$

is valid. Taking this fact into account we see that

$$| < D_*^\alpha u(t,x), v(x) > | \ \leq \ \sum_{k=1}^{m} | < D_*^\alpha B_k(\alpha; t, \xi) \hat{\varphi}_k(\xi), \hat{v}(\xi) > |$$

$$\leq \sum_{k=1}^{m} | < D_*^\alpha B_k(\alpha; t, \xi) | \| \hat{\varphi}_k(\xi) \|_{L_2(\operatorname{supp}\hat{v})} \|v\|_2 \ \leq \ C_v \ < \ \infty.$$

■

4 Well-posedness of the Cauchy problem in the Sobolev spaces H^s

We first state and prove the theorem basic for this section.

Theorem 5. *Let $\varphi_k \in H^s(\mathbb{R}^n),\ k = 0, ..., m-1,\ s \in \mathbb{R}$, and the estimate*

$$|E_\alpha(A(\xi)t^\alpha| \ \leq \ C(1 + |\xi|)^\ell, \ C > 0, \ \xi \in \mathbb{R}^n$$

be valid for a fixed $\ell \in \mathbb{R}$. Then there exists a unique solution of the Cauchy problem (1), (2) in the space

$$C^m(t > 0; H^{s-\ell}(\mathbb{R}^n)) \cap C^{m-1}(t \geq 0; H^{s-\ell}(\mathbb{R}^n)).$$

This solution is given by the formula

$$u(t,x) = \sum_{k=1}^{m} \overline{B}_k(\alpha; t, D) \varphi_{k-1}(x), \tag{16}$$

where $\overline{B}_k(\alpha; t, D)$ means the closure of the k-solution operator $B_k(\alpha; t, D)$ with the symbol (13) in the corresponding Sobolev space.

P r o o f. Let $\varphi_k \in H^s(\mathbb{R}^n),\ k = 0, ..., m-1$. We can choose any domain G whose complement $\mathbb{R}^n \setminus G$ has zero measure. In particular, let us take $G =$

$I\!R^n$. Lemma 2 gives us the relation $\overline{\Psi_G(I\!R^n)} = H^s(I\!R^n)$. Hence, for every φ_k we can choose an approximating sequence of functions $\varphi_{k,N} \in \Psi_G(I\!R^n),\ N = 0, 1, 2, \ldots$. For fixed N, according to Theorem 1, there exists a unique solution of the Cauchy problem (1), (2) (in which the $\varphi_k,\ k = 0, 1 \ldots, m-1$, are replaced by $\varphi_{k,N},\ k = 0, 1 \ldots, m-1$, respectively) given by the formula

$$u_N(t,x) = \sum_{k=1}^{m} B_k(\alpha; t, D)\varphi_{k-1,N}(x). \tag{17}$$

We recall that $B_1(\alpha; t, \xi) = E_\alpha(A(\xi)t^\alpha)$. It is not difficult to check that if $E_\alpha(A(\xi)t^\alpha)$ satisfies the estimate of the theorem then the k-th integral ($k \in I\!N$) with respect to t satisfies an estimate of the same type, namely

$$|B_k(\alpha; t, \xi)| \ \leq\ C(1 + |\xi|)^\ell,\ C > 0,\ \xi \in I\!R^n,\ \ell \in I\!R.$$

Indeed, for $k = 2$ we have

$$|B_2(\alpha; t, \xi)| = |\int_0^t E_\alpha(A(\xi)\tau^\alpha)\, d\tau| \ \leq C_t |E_\alpha(A(\xi)t^\alpha)| \ \leq C_t^*(1+|\xi|)^\ell,\ \xi \in I\!R^n.$$

For other values of k we proceed analogously. Thus we have the same estimate as in the theorem for the symbols of all operators $B_k(\alpha; t, D),\ k = 1, ..., m$. Taking into account this fact and Lemma 4 we get a closure of (17) in the H^s-Sobolev norm by taking $N \to \infty$. ■

Now we consider the case $A(D) = \Delta$ with the Laplace operator Δ. Then the Cauchy problem (1), (2) gives us a model for the fractional heat conduction process ($0 < \alpha < 1$) and the process intermediate between diffusion and wave propagation ($1 < \alpha < 2$). So we proceed in these two cases separately.

Let $0 < \alpha < 1$. We consider the Cauchy problem for the time-fractional equation

$$D_*^\alpha u(t,x) = \Delta u(t,x),\ t > 0,\ x \in I\!R^n, \tag{18}$$

$$u(0,x) = \varphi(x),\ x \in I\!R^n. \tag{19}$$

Then we have only one solution operator, namely, $B_1(\alpha; t, D)$ whose symbol is given as $B_1(\alpha; t, \xi) = E_\alpha(-|\xi|^2 t^\alpha)$. From the well known asymptotics at infinity of the Mittag-Leffler function with negative argument (see, for example, [EMOT55], [Djr66], [Djr93]) we obtain

$$B_1(\alpha; t, \xi) = O(1 + |\xi|)^{-2},\ |\xi| \to \infty. \tag{20}$$

Similarly, if $1 < \alpha < 2$, we deal with the Cauchy problem for the time-fractional equation

$$D_*^\alpha u(t,x) = \Delta u(t,x),\ t > 0,\ x \in I\!R^n, \tag{21}$$

$$u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ x\in I\!R^n. \tag{22}$$

In this case we have two solution operators, $B_1(\alpha;t,D)$ and $B_2(\alpha;t,D)$ with the symbols $B_1(\alpha;t,\xi)=E_\alpha(-|\xi|^2t^\alpha)$ and $B_2(\alpha;t,\xi)=JE_\alpha(-|\xi|^2t^\alpha)$, respectively. The symbol $B_2(\alpha;t,\xi)$ has the same asymptotics (20) for $|\xi|\to\infty$ as $B_1(\alpha;t,\xi)$.

Applying Theorem 5 and the asymptotic formula (20) we get the following result:

Theorem 6. *Let $0<\alpha<1$ and $\varphi\in H^s(I\!R^n)$. Then the Cauchy problem (18), (19) has a unique solution in the space*

$$C^\infty(t>0;H^{s+2}(I\!R^n))\cap C(t\geq 0;H^s(I\!R^n)).$$

This solution is given by the formula

$$u(t,x)=\overline{B}_1(\alpha;t,D)\varphi(x). \tag{23}$$

Using the representation formula (23) for the solution of the Cauchy problem (18), (19) we can study its behavior for large t.

Theorem 7. *Let $0<\alpha<1$, $T>0$ fixed, and $\varphi\in H^s(I\!R^n)$. Then there exists a positive constant C_α such that for the solution of the Cauchy problem (18), (19) given by (23) the estimate*

$$\|u(t,x)\|_{2,s+2}\leq C_\alpha T^{-\alpha}\|\varphi\|_{2,s}\ \ for\ \ t\geq T \tag{24}$$

holds.

P r o o f. Following [GM97] we have for the symbol $B_1(\alpha;t,\xi)$ the representation $(0<\alpha<1)$

$$B_1(\alpha;t,\xi)=\int_0^\infty e^{-rt|\xi|^{\frac{2}{\alpha}}}K_{\alpha,0}(r)dr, \tag{25}$$

where

$$K_{\alpha,0}(r)=\frac{1}{\pi}\frac{r^{\alpha-1}\sin(\alpha\pi)}{r^{2\alpha}+2r^\alpha\cos(\alpha\pi)+1}.$$

For $t\geq T$ the representation (25) can be rewritten as

$$B_1(\alpha;t,\xi)=\int_0^\infty e^{-rT|\xi|^{\frac{2}{\alpha}}}e^{-r(t-T)|\xi|^{\frac{2}{\alpha}}}K_{\alpha,0}(r)dr,$$

which gives us the estimate

$$|B_1(\alpha;t,\xi)|\leq\int_0^\infty e^{-rT|\xi|^{\frac{2}{\alpha}}}K_{\alpha,0}(r)dr,\ t\geq T.$$

By using the Watson lemma we obtain for large values of $T|\xi|^{\frac{2}{\alpha}}$ the asymptotic relation

$$|B_1(\alpha;t,\xi)| \ = \ O\left(\frac{1}{T^\alpha|\xi|^2}\right), \ t\geq T, \ |\xi|\to\infty,$$

which implies

$$\|u(t,x)\|^2_{2,s+2} = \int_{R^n}|B_1(\alpha;t,\xi)|^2|\hat{\varphi}|^2(1+|\xi|^2)^{s+2}d\xi \leq \ \frac{C^2_\alpha}{T^{2\alpha}}\|\varphi\|^2_{2,s}, \ t\geq T.$$

■

Corollary 2. *Taking $T\to+\infty$ in the estimate (24) we get, for all $\alpha\in(0,1)$, the relaxation property of the solution of the Cauchy problem (18), (19):*

$$\lim_{t\to\infty}\|u(t,x)\|_{2,s+2} = 0.$$

For $s>\frac{n}{2}-2$ it follows from the embedding theorems for the Sobolev spaces H^s that this limit is uniform with respect to $x\in I\!R^n$.

Let us consider now the Cauchy problem (21), (22).

Theorem 8. *Let $1<\alpha<2$, $\varphi\in H^s(I\!R^n)$, and $\psi\in H^s(I\!R^n)$. Then the Cauchy problem (21), (22) has a unique solution in the space*

$$C^\infty(t>0;H^{s+2}(I\!R^n))\cap C^1(t\geq 0;H^s(I\!R^n)).$$

This solution is given by the formula

$$u(t,x) = \overline{B}_1(\alpha;t,D)\varphi(x)+\overline{B}_2(\alpha;t,D)\psi(x). \tag{26}$$

Theorem 9. *Let $1<\alpha<2$, $T>0$ fixed, and $\varphi,\psi\in H^s(I\!R^n)$. Then there exists a positive constant C_α such that for the solution (26) of the Cauchy problem (21), (22) the estimate*

$$\|u(t,x)\|_{2,s+2}\leq\frac{C_\alpha}{T^{\alpha-1}}(\frac{\|\varphi\|_{2,s}}{T}+\|\psi\|_{2,s}) \ \ \textit{for} \ \ t\geq T \tag{27}$$

holds.

P r o o f. Once again we take from [GM97] the formulas for the symbols $B_j(\alpha;t,\xi)$ thus arriving at the representations $(1<\alpha<2)$

$$B_1(\alpha;t,\xi) = \int_0^\infty e^{-rt|\xi|^{\frac{2}{\alpha}}}K_{\alpha,0}(r)dr+\frac{2}{\alpha}e^{t|\xi|^{\frac{2}{\alpha}}\cos\frac{\pi}{\alpha}}\cos(t|\xi|^{\frac{2}{\alpha}}\sin\frac{\pi}{\alpha}),$$

$$B_2(\alpha;t,\xi) = \int_0^\infty e^{-rt|\xi|^{\frac{2}{\alpha}}} K_{\alpha,1}(r)dr + \frac{2}{\alpha} e^{t|\xi|^{\frac{2}{\alpha}}\cos\frac{\pi}{\alpha}} \cos(t|\xi|^{\frac{2}{\alpha}} \sin(\frac{\pi}{\alpha} - \frac{\pi}{2})),$$

where

$$K_{\alpha,1}(r) = -r^{-1} K_{\alpha,0}(r).$$

For $1 < \alpha < 2$ the main asymptotic terms come from the first terms of the right-hand sides of these representations. Thus, analogously to the previous case, we use the Watson lemma to obtain the asymptotics ($t \geq T$)

$$|B_1(\alpha;t,\xi)| \;=\; O\left(\frac{1}{T^\alpha |\xi|^2}\right), \;|B_2(\alpha;t,\xi)| \;=\; O\left(\frac{1}{T^{1-\alpha}|\xi|^2}\right), \;|\xi| \to \infty.$$

Taking these estimates into account we have for $t \geq T$

$$\|B_1(\alpha;t,D)\varphi\|_{2,s+2} \leq \frac{C_{1,\alpha}}{T^\alpha} \|\varphi\|_{2,s}, \quad \|B_2(\alpha;t,D)\psi\|_{2,s+2} \leq \frac{C_{2,\alpha}}{T^{1-\alpha}} \|\psi\|_{2,s},$$

which implies the estimate (27). ■

Corollary 3. *Taking in the estimate (27) $T \to \infty$ we arrive, for all $\alpha \in (1,2)$, at the oscillation-relaxation property of the solution of the Cauchy problem (21), (22):*

$$\lim_{t\to\infty} \|u(t,x)\|_{2,s+2} = 0.$$

For $s > \frac{n}{2} - 2$ it follows from the embedding theorems for the Sobolev spaces H^s that this limit is uniform with respect to $x \in I\!R^n$.

Remark 3. *The asymptotic expressions used for $1 < \alpha < 2$ are not valid for $\alpha = 2$. The symbols of the solution operators in this case are given as*

$$B_1(t,\xi) = \cos(t|\xi|), \quad B_2(t,\xi) = \frac{\sin(t|\xi|)}{|\xi|}.$$

They have pure oscillatory behavior which is inherited by the solution as well.

The theorems analogous to the given above are also true for the Cauchy problem for the time- and space-fractional equation

$$D_*^\alpha u(t,x) = D_0^\beta u(t,x), \; t > 0, \; x \in I\!R^n, \; \alpha, \beta > 0.$$

In this case the solution operators have the symbols

(i) $B_1(\alpha;t,\xi) = E_\alpha(-t^\alpha|\xi|^\beta)$, if $0 < \alpha < 1$;

(ii) $B_1(\alpha;t,\xi) = E_\alpha(-t^\alpha|\xi|^\beta)$, $B_2(\alpha;t,\xi) = JE_\alpha(-t^\alpha|\xi|^\beta)$, if $1 < \alpha < 2$.

From Theorem 5 and using the same technique as in the proofs of Theorems 6 and 8 we get, that the solution is in the space

$$C^\infty(t > 0; H^{s+\beta}(I\!R^n)) \cap C(t \geq 0; H^s(I\!R^n)),$$

if the initial data are in the space $H^s(I\!R^n)$.

Apart from the case of non-integer α the situation concerning smoothness of solutions for $\alpha = 1$ and $\alpha = 2$ is similar to the classical results for the Cauchy problems for the heat conduction and the wave propagation equations, respectively. We formulate the corresponding results without proofs which can be easily obtained analogously to the classical cases.

First we consider the Cauchy problem for the space-fractional equation with the time derivative of the second order:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = D_0^\beta u(t,x),\ t > 0,\ x \in I\!R^n,\ \beta > 0, \tag{28}$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),\ x \in I\!R^n. \tag{29}$$

It is easy to see that the solution operators have the symbols

$$B_1(t,\xi) = \cos(t|\xi|^\beta), \quad B_2(t,\xi) = \frac{\sin(t|\xi|^\beta)}{|\xi|^\beta},$$

satisfying the estimates

$$|B_1(t,\xi)| \le 1, \quad |B_2(t,\xi)| \le \frac{1}{|\xi|^\beta}.$$

Theorem 10. *Let $\beta > 0$, $\varphi \in H^s(I\!R^n)$, $\psi \in H^{s-\beta}(I\!R^n)$. Then the problem (28), (29) has a unique solution in the space*

$$C(t \ge 0; H^s(I\!R^n)) \cap C^1(t \ge 0; H^{s-\beta}(I\!R^n)).$$

This solution is given by the formula

$$u(t,x) = \overline{B}_1(\alpha; t, D)\varphi(x) + \overline{B}_2(\alpha; t, D)\psi(x).$$

The solution operator of the Cauchy problem for the space-fractional equation with the time derivative of the first order

$$\frac{\partial u(t,x)}{\partial t} = D_0^\beta u(t,x),\ t > 0,\ x \in I\!R^n,\ \beta > 0, \tag{30}$$

$$u(0,x) = \varphi(x),\ x \in I\!R^n, \tag{31}$$

has the symbol

$$B_1(t,\xi) = e^{-t|\xi|^\beta},$$

and for every $l > 0$ there exists a positive constant C_l such that the estimate

$$|B_1(t,\xi)| \leq \; C_l(1+|\xi|)^{-\ell}, \; \xi \in I\!R^n$$

holds. Correspondingly, we arrive at the result:

Theorem 11. *Let $\beta > 0$ and $\varphi \in H^s(I\!R^n)$. Then the problem (30), (31) has a unique solution in the space*

$$C^\infty(t > 0; \cap_{s \in I\!R} H^s(I\!R^n)) \cap C(t \geq 0; H^s(I\!R^n)).$$

This solution is given by the formula

$$u(t,x) = \overline{B}_1(\alpha; t, D)\varphi(x).$$

Remark 4. *The same technique used above for the Sobolev spaces $H^s(I\!R^n)$ is still applicable for more general Sobolev spaces $H^s_p(I\!R^n)$, Besov spaces $B^s_{pq}(I\!R^n)$ and Triebel-Lizorkin spaces $F^s_{pq}(I\!R^n)$.*

5 The multi-point value problems for partial differential equations of fractional order

In this section we deal with the general multi-point value problem (5) for partial differential equation of fractional order (4).

Denote by $\Delta(s,\xi)$ the characteristic function of the equation (4), namely,

$$\Delta(s,\xi) = s^{\alpha_m} + \sum_{k=1}^{m-1} A_k(\xi)s^{\alpha_k} + A_0(\xi).$$

Let

$$c_\beta(t,\xi) = L^{-1}[\frac{s^\beta}{\Delta(s,\xi)}](t),$$

where L^{-1} means the inverse Laplace transform.

Let us first consider the Cauchy problem for the equation (4) with the initial data

$$u^{(k)}(0,x) = \psi_k(x), \; k = 0, ..., m-1. \tag{32}$$

Using the method developed in Section 3 we obtain the system $\{B_k(t,D)\}_{k=0}^{m-1}$ of solution pseudo-differential operators with the symbols

$$B_k(t,\xi) = c_{\alpha_m - k - 1}(t,\xi) + \sum_{j=k+1}^{m-1} A_j(\xi)c_{\alpha_j - k - 1}(t,\xi). \tag{33}$$

Then we get the formal representation of the solution of the Cauchy problem (4), (32) as

$$u(t,x) = \sum_{k=0}^{m-1} B_k(t,D)\psi_k(x).$$

Now we construct the system of solution operators of the general problem (4), (5). Applying the Fourier transform with respect to the variable x we get

$$D_*^{\alpha_m}\hat{u}(t,\xi) + \sum_{k=1}^{m-1} A_k(\xi) D_*^{\alpha_k}\hat{u}(t,\xi) + A_0(\xi)\hat{u}(t,\xi) = 0,\ t\in(0,T),\ \xi\in G, \tag{34}$$

$$\sum_{j=0}^{m-1} \Gamma_{kj}(\xi)\frac{\partial^k \hat{u}(t_{kj},\xi)}{\partial t^k} = \hat{\varphi}_k(\xi),\ \xi\in G. \tag{35}$$

We look for a solution of the multi-point value problem (34), (35) in the form

$$\hat{u}(t,\xi) = \sum_{k=0}^{m-1} f_k(\xi) B_k(t,\xi) \tag{36}$$

with $B_k(t,\xi),\ k=0,\dots,m-1$, given by (33). It is clear that $\hat{u}(t,\xi)$ satisfies (34). Substituting it into (35) we get a system of linear algebraic equations

$$M(\xi)F(\xi) = \hat{\Phi}(\xi). \tag{37}$$

Here $M(\xi)$ is a square matrix of the order m with the entries

$$m_{kl} = \sum_{j=0}^{m-1} \Gamma_{kj}(\xi) B_l^{(j)}(t_{kj},\xi),\ k,l=0,...,m-1,$$

$F(\xi) = (f_0(\xi),...,f_{m-1}(\xi))$ is an unknown vector, and $\hat{\Phi}(\xi) = (\hat{\varphi}_0(\xi),\dots,\hat{\varphi}_{m-1}(\xi))$ is the initial data vector. Denote by M_0 the set of all points $\xi\in G$ such that $\mathrm{Det}M(\xi)=0$. If $\xi\notin M_0$, then the equation (37) has a unique solution

$$F(\xi) = M^{-1}(\xi)\hat{\Phi}(\xi). \tag{38}$$

We note that M_0 is the singular set for the symbols of the solution operators. Inserting the representation (38) of the vector $F(\xi) = (f_0(\xi),...,f_{m-1}(\xi))$ into (36) and applying the inverse Fourier transform we get the solution of the general multi-point value problem (4), (5) as

$$u(t,x) = \sum_{k=0}^{m-1} U_k(t,D)\varphi_k(x), \tag{39}$$

where the $U_k(t, D),\ k = 0, ..., m-1$ are solution pseudo-differential operators with the symbols

$$U_k(t, \xi) = (M^*)^{-1}(\xi) B(t, \xi),$$

$(M^*)^{-1}(\xi)$ is the matrix inverse to the Hermitian conjugate of $M(\xi)$, and $B(t, \xi)$ is the transpose of the vector row $(B_0(t, \xi), ..., B_{m-1}(t, \xi))$ with the components given by (33).

The vector-function $U(t, \xi)$ with the components $U_0(t, \xi), ..., U_{m-1}(t, \xi)$ depends on operators $A_k(D),\ k = 0, ..., m-1$, and $\Gamma_{kj}(D),\ k = 0, ..., m-1,\ j = 0, ..., m-1$, given in (4), (5), respectively. Its behavior may be of different nature: it may have singularities in G if $M_0 \cap G \neq \emptyset$, it may increase or decrease at infinity. The well-posedness of the multi-point value problem (4), (5) in the classical Sobolev, Besov, etc. spaces essentially depends on the behavior of the vector function $U(t, \xi)$. In particular, we have the following result:

Theorem 12. *Let $\varphi_k \in \Psi_{G',p}(I\!R^n),\ k = 0, ..., m-1,\ G' = G \setminus M_0$. Then the multi-point value problem (4), (5) has a unique solution in the space $C^{(m)}((T_1, T_2); \Psi_{G',p}(I\!R^n))$. This solution is given by the formula (39).*

The analogous theorem is valid for the dual problem too. Under the dual problem we mean a problem obtained by replacing the operators $A_k(D),\ k = 0, ..., m-1$, and $\Gamma_{kj}(D),\ k = 0, ..., m-1,\ j = 0, ..., m-1$, by their dual operators (see formula (10)).

Theorem 13. *Let $\varphi_k \in \Psi'_{-G',q}(I\!R^n),\ k = 0, ..., m-1,\ G' = G \setminus M_0$. Then the dual multi-point value problem has a unique solution in the space $C^{(m)}((T_1, T_2); \Psi'_{-G',q}(I\!R^n))$. This solution is given by the formula*

$$u(t, x) = \sum_{k=0}^{m-1} U_k(t, -D)\varphi_k(x).$$

For the Sobolev spaces H^s we get the following result:

Theorem 14. *Let for all $t \in (T_1, T_2)$ and $k = 0, ..., m-1$ the estimates*

$$\left|\frac{\partial^k u_j(t, \xi)}{\partial t^k}\right| \leq C(1 + |\xi|)^{s_j - \ell_k},\ C > 0,\ \xi \in I\!R^n,\ j = 0, ..., m-1$$

with certain numbers $s_k \in I\!R,\ \ell_k \in I\!R,\ k = 0, \ldots, m-1$, be valid. Then for any $\varphi_j \in H^{s_j},\ j = 0, ..., m-1$, there exists a unique solution $u(t, x)$ of the multi-point value problem (4), (5) such that $\frac{\partial^k u(t,x)}{\partial t^k} \in C^{m-1-k}((T_1, T_2); H^{\ell_k})$ and

$$\sup_{t \in (T_1, T_2)} \sum_{k=0}^{m-1} \left\|\frac{\partial^k u}{\partial t^k}\right\|_{2,\ell_k} \leq C \sum_{j=0}^{m-1} \|\varphi_j\|_{2,s_j}.$$

6 Proof of the density lemma

In this section we give a proof of the density statement of Lemma 2 in Section 2. We recall that G is an open domain in $I\!R^n$, $\{\phi_j(\xi)\}_{j=0}^{\infty}$ is a smooth partition of unity corresponding to the locally finite covering $\{g_j\}_{j=0}^{\infty}$ of G, $G_N = \cup_{k=0}^{N} g_k$ and $\kappa_N(\xi) = \phi_0(\xi) + ... + \phi_N(\xi)$. Denote by $\gamma_N(\xi)$ the function $1-\kappa_N(\xi) = \sum_{k=N+1}^{\infty} \phi_k(\xi)$ and by $m_n(\cdot)$ the n-dimensional Lebesgue measure.

Density lemma. *Let $1 < p < \infty$, $s \in I\!R$ and $G \in I\!R^n$. Then the embedding*

$$\Psi_{G,p}(I\!R^n) \hookrightarrow H_p^s(I\!R^n) \tag{40}$$

is dense if and only if $m_n(I\!R^n \setminus G) = 0$.

P r o o f. *Sufficiency.* Let the complement of G in $I\!R^n$ have the n-dimensional measure zero, i.e., $m_n(I\!R^n \setminus G) = 0$. We note that the case $s = 0$ can be taken without loss of generality, because the Sobolev spaces H_p^s form a scale of densely embedded spaces. Moreover, since S is dense in L_p it suffices to prove the lemma for $f \in S$.

We take the functions $\psi_0,\ \psi \in S$ satisfying the conditions:
(i) $0 \le \psi_0,\ \psi \le 1$;
(ii) supp $\psi_0 \subset \{|\xi| \le 2\}$, supp $\psi \subset \{1 \le |\xi| \le 4\}$;
(iii) $\sum_{j=0}^{\infty} \psi_j(\xi) = 1,\ \psi_j(\xi) = \psi(2^{1-j}\xi),\ j = 0, 1, 2 \ldots.$

Then it is known (see [Tri77]), that the sequence of functions

$$f_M(x) = \sum_{j=0}^{M} F^{-1}\psi_j F f(x),\ M = 0, 1, ...$$

tends to f in the norm of L_p if $M \to \infty$. Obviously,

$$\text{supp } F f_M \subset \{|\xi| \le 2^{M+1}\} = B_M.$$

So we can consider the functions $f \in S$ with supp $Ff \subset B_M$.

As an approximating sequence we take the sequence of functions

$$f_N(x) = F^{-1}\kappa_N(\xi) F f(x),\ N = 0, 1, \ldots.$$

Obviously, $f_N \in \Psi_{G,p}(R^n)$ for all $N \ge 0$. We have

$$f(x) - f_N(x) = F^{-1}(Ff - \kappa_N(\xi)Ff) = F^{-1}(\gamma_N(\xi))Ff(\xi)).$$

Putting $w_N =$ supp $(\gamma_N \cdot \Omega_{B_M})$, where $\Omega_{B_M}(\xi)$ is the characteristic function of the ball B_M, we see that the set w_N is a compact subset of $(I\!R^n \setminus G_N) \cap B_M$.

Evidently, $m_n(w_N) \to 0$ for $N \to \infty$ since $m_n((I\!R^n \setminus G_N) \cap B_M) \to 0$ for $N \to \infty$.

To prove the lemma we have to show that the sequence of functions $F^{-1}\gamma_N Ff \to 0$ in the norm of L_p for all $p \in (1, \infty)$ if $N \to \infty$. For $p \in [2, \infty)$ we consider the expression

$$\|F^{-1}\gamma_N Ff\|_p = \sup_{\substack{v \in L_q(I\!R^n) \\ \|v\|_q \neq 0}} \frac{| < F^{-1}\gamma_N Ff, v > |}{\|v\|_q}. \tag{41}$$

To estimate the right-hand side of this formula we use the equality

$$< F^{-1}\gamma_N Ff, v >= (2\pi)^{-n} < \gamma_N Ff, Fv >, \tag{42}$$

giving to it appropriate sense for different values of p.

If $p = 2$ then (42) is the well-known Parseval equality. Using the Cauchy-Bunyakovskij inequality and the relations (41), (42) we get

$$\|F^{-1}\gamma_N Ff\|_2 \leq C\|\gamma_N Ff\|_2 \to 0, \; N \to 0. \tag{43}$$

For $p > 2$ we have $q = \frac{p}{p-1} \in (0, 1)$. Since $v \in L_q$, we can use the Hausdorff-Young lemma saying that the Fourier transform acts continuously from L_q into L_p, $\frac{1}{p} + \frac{1}{q} = 1$ if $1 \leq q \leq 2$. We then have $Fv \in L_p$ and

$$\|Fv\|_p \leq C\|v\|_q. \tag{44}$$

This means that (42) is meaningful for $p > 2$ if $\gamma_N Ff \in L_q$. Since the partition of unity is smooth we have the inclusion $\gamma_N \in C^\infty(G)$. Moreover, $|\gamma_N(\xi)| \leq 1, \; \xi \in G$. This gives us $\gamma_N \in L_\infty$. Thus $\gamma_N Ff \in L_q$. Taking now the inequality (44) into account we have the estimate

$$\|F^{-1}\gamma_N Ff\|_p \leq \|\gamma_N Ff\|_q \|Fv\|_p \leq C\|\gamma_N Ff\|_q \to 0, \; N \to 0.$$

In exactly the same way it can be verified that

$$\|F\gamma_N F^{-1} f\|_p \to 0, \; N \to \infty, \; p \geq 2. \tag{45}$$

Now we show that also $F^{-1}\gamma_N Ff \to 0$ in the norm of L_p for every $p \in (1, 2)$ if $N \to \infty$. We take a fixed number $r \in (1, 2)$. Then for arbitrary $v \in L_{r'}(I\!R^n)$, $r' = r/(r-1) > 2$, by using (45) we have

$$| < F^{-1}\gamma_N Ff, v > | \leq | < f, F\gamma_N F^{-1} v > | \leq \|f\|_r \cdot \|F\gamma_N F^{-1} v\|_{r'} \to 0, \; N \to \infty.$$

So, $\{F\gamma_N F^{-1}f\}_{N=0}^{\infty}$ is a weakly convergent sequence in L_r. Since L_r is a reflexive Banach space, the sequence $\{F\gamma_N F^{-1}f\}_{N=0}^{\infty}$ is bounded in L_r. Using the interpolation inequality (which follows easily from the Hölder inequality)

$$\|f\|_p \leq \|f\|_r^{\lambda} \cdot \|f\|_2^{\mu},\ f \in L_r \cap L_2,\ r < p < 2,\ \frac{1}{p} = \frac{\lambda}{r} + \frac{\mu}{2},\ \lambda + \mu = 1,$$

and taking into account (43) we obtain the estimate

$$\|F^{-1}\gamma_N Ff\|_p \leq \|F^{-1}\gamma_N Ff\|_r^{\lambda} \cdot \|F^{-1}\gamma_N Ff\|_2^{\mu} \leq C\|F^{-1}\gamma_N Ff\|_2^{\mu} \to 0,\ N \to \infty,$$

which gives us the if-assertion of the Lemma.

Necessity. Suppose that $m_n(I\!R^n \setminus G) > 0$. Then there exists a compact set $V \subset I\!R^n \setminus G$ such that $m_n(V) > 0$. Let us take a function $f \in L_p,\ f \neq 0$ and let $\operatorname{supp} Ff = V$. Then for any $v \in \Psi_{G,p}(I\!R^n)$ we have $\operatorname{supp} Ff \cap \operatorname{supp} Fv = \emptyset$. For $p = 2$ this yields

$$\|f - v\|_2^2 = (f - v, f - v)_{L_2} = \|f\|_2^2 + \|v\|_2^2 \geq \|f\|_2^2 > 0.$$

For $1 < p < 2$ we obtain the analogous estimate by using the Hausdorff-Young lemma. In fact,

$$(2\pi)^{\frac{n}{q}}\|f - v\|_p \geq \|Ff - Fv\|_q = (\|Ff\|_q^q + \|Fv\|_q^q)^{1/q} \geq \|Ff\|_q > 0.$$

For $2 < p < \infty$ and for an arbitrary fixed $g \in L_q(I\!R^n),\ \|g\|_q = 1,\ q = p/(p-1)$, we have

$$\|f - v\|_p = \sup_{h \in L_p, h \neq 0} \frac{|(f - v, h)|}{\|h\|_q} \geq |(f - v, g)|.$$

We choose $g \in L_q(I\!R^n)$ (note that $q \in (1, 2)$) satisfying the following conditions:
(i) $\operatorname{supp} Fg = V$;
(ii) $Fg \in H_2^{\beta}(I\!R^n),\ \beta > n(\frac{1}{2} - \frac{1}{p})$;
(iii) $\int_{I\!R^n} f(x)g(x)dx \neq 0$.

It is not difficult to verify that such function exists. According to the Hausdorff-Young lemma the Fourier transform of g belongs to $L_p(I\!R^n)$ and $\|Fg\|_p \leq (2\pi)^{\frac{n}{p}}\|g\|_q$. For any $v \in \Psi_{G,p}(I\!R^n)$ its Fourier transform has a compact support in G by the definition of the space $\Psi_{G,p}(I\!R^n)$. Moreover, the Fourier transform of v is an element of the Sobolev space $H_2^{-\beta}(I\!R^n),\ \beta > n(\frac{1}{2} - \frac{1}{p})$, because $v \in L_p(I\!R^n)$ (see [Hör83], p.242). Hence

$$(g, v) = \int_{R^n} g(x)v(x)dx = \frac{1}{(2\pi)^n} < Fv, Fg >,$$

where $< Fv, Fg >$ means the value of the functional $Fv \in H_2^{-\beta}(\mathbb{R}^n)$ on Fg. It is well-defined due to $Fg \in H_2^{\beta}(\mathbb{R}^n)$. Moreover, $< Fv, Fg > = 0$, since $\operatorname{supp} Fv \cap \operatorname{supp} Fg \subset G \cap V = \emptyset$. Hence, $\|f - v\|_p \geq |(f, g)| > 0$. ■.

Remark 5. *The assertion of the density lemma is also valid for both the Besov spaces* $B^s_{p,q}(\mathbb{R}^n),\ 1 < p < \infty,\ 1 < q < \infty,\ s \in \mathbb{R}$, *and the Triebel-Lizorkin spaces* $F^s_{p,q}(\mathbb{R}^n),\ 1 < p < \infty,\ 1 < q < \infty,\ s \in \mathbb{R}$.

Remark 6. *Results analogous to the density lemma have been obtained earlier by Samko [Sam82], [Sam95] who considered the density problem for the Lizorkin type spaces* Φ_V *in* L_p.

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