Catalan and other numbers – a recurrent theme

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1. Six ways to look at Catalan numbers

The Catalan numbers C_n are next to the binomial coefficients the best studied of all combinatorial counting numbers. The wonderful book by Stanley [28] lists more than 70 instances of enumeration problems which are counted by the sequence (C_n). We will see several of them as we go along.

To start our survey we present six ways to introduce these numbers. Every approach has its own flavor and will be treated in subsequent chapters.

(A) Recursion

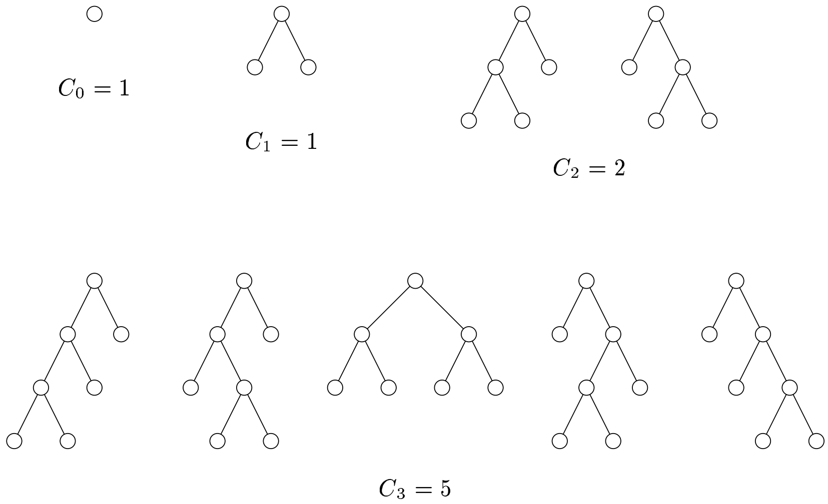
The most natural way, which we call the *combinatorial definition*, is the convolution recursion:

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n \geq 0), \qquad C_0 = 1. \tag{1.1}$$

We obtain as first values:

n	0	1	2	3	4	5	6	7
C_n	1	1	2	5	14	42	132	429

As an example, we immediately find that C_n is the number of rooted binary plane trees with n vertices of degree 2 (by classifying the subtrees at the root):



(B) Representation as an integral

We have

$$C_n = \frac{2^{2n+1}}{\pi} \int_{-1}^{1} x^{2n}(1-x^2)^{1/2} dx. \tag{1.2}$$

How this representation comes about will be explained in the next section. The integral is easily evaluated, resulting in the usual expression

$$C_n = \frac{1}{n+1} \binom{2n}{n}. \tag{1.3}$$

(C) Continued fractions

The ordinary generating function of $(C_0, 0, C_1, 0, C_2, 0, \ldots)$ may be expressed as the following so–called J–fraction:

$$\sum_{n \geq 0} C_n z^{2n} = \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{\cdots}}}} \tag{1.4}$$

We take a brief look at continued fractions in section 3.

(D) Determinants of Hankel matrices

To the sequence (C_n) we consider the two matrices for $n \geq 0$:

$$H_n = \begin{pmatrix} C_0 & C_1 & \dots & C_n \\ C_1 & C_2 & \dots & C_{n+1} \\ \vdots & & \dots & \\ C_n & C_{n+1} & \dots & C_{2n} \end{pmatrix}, \quad \widetilde{H}_n = \begin{pmatrix} C_1 & C_2 & \dots & C_{n+1} \\ C_2 & C_3 & \dots & C_{n+2} \\ \vdots & & \dots & \\ C_{n+1} & C_{n+2} & \dots & C_{2n+1} \end{pmatrix}.$$

H_n and $\widetilde{H}_n$ are called the *Hankel matrices* (of order 0 and 1) of the sequence (C_n). We find

$$\det H_n = \det \widetilde{H}_n = 1 \text{ for all } n. \tag{1.5}$$

Now we may, of course, associate Hankel matrices H_n, $\widetilde{H}_n$ to any given sequence (B_n) of real numbers. It is clear that the sequence

$$\det H_0, \det \widetilde{H}_0, \det H_1, \det \widetilde{H}_1, \dots$$

uniquely determines the original number sequence provided that $\det H_n$ and $\det \widetilde{H}_n$ are $\neq 0$ for all n. Hence, the Catalan numbers (C_n) are the *unique* sequence with $\det H_n = \det \widetilde{H}_n = 1$. We will return to this beautiful characterization in section 8.

(E) Ballot recursion

Consider the infinite lower triangular matrix $B = (b_{n,k})$ defined as follows:

$$\begin{aligned} &b_{0,0} = 1, \quad b_{0,k} = 0 \quad (k > 0) \\ &b_{n,k} = b_{n-1,k-1} + b_{n-1,k} + \dots + b_{n-1,n-1} \quad (n \geq 1)\ . \end{aligned} \tag{1.6}$$

We obtain as first rows and columns (leaving out the 0's):

$n \backslash k$	0	1	2	3	4	5
0	1					
1	1	1				
2	2	2	1			
3	5	5	3	1		
4	14	14	9	4	1	
5	42	42	28	14	5	1

We find for all n

$$\begin{aligned} C_n &= b_{n,0} \\ C_{n+1} &= \sum_{k=0}^{n} b_{n,k} \ . \end{aligned} \tag{1.7}$$

The numbers $b_{n,k}$ are usually called the *ballot numbers* (see e.g. [24]) and are easily computed:

$$b_{n,k} = \tfrac{k+1}{n+1}\binom{2n-k}{n} \ .$$

We will study generalized ballot numbers in some detail in section 9.

(F) 3–term recursion

We come to our last and, for our purposes, most important way to introduce the Catalan numbers. Consider the following 3–term recursion which gives again rise to an infinite lower triangular matrix $A = (a_{n,k})$:

$$\begin{aligned} &a_{0,0} = 1, \quad a_{0,k} = 0 \quad (k > 0) \\ &a_{n,k} = a_{n-1,k-1} + a_{n-1,k+1} \quad (n \geq 1). \end{aligned} \tag{1.8}$$

The first rows and columns look as follows:

$n \backslash k$	0	1	2	3	4	5	6
0	$\underline{1}$						
1	0	1					
2	$\underline{1}$	0	1				
3	0	2	0	1			
4	$\underline{2}$	0	3	0	1		
5	0	5	0	4	0	1	
6	$\underline{5}$	0	9	0	5	0	1

Then $C_n = a_{2n,0}$.

We have presented six rather different ways to introduce the Catalan numbers, but we still have to show that all approaches yield indeed the same sequence (C_n). The fascinating connections between the topics outlined in (A) to (F) will precisely be the content of this paper. For a general background see any of the standard texts such as [16, 24, 27].

2. Recursive matrices and orthogonal systems of polynomials

Our starting point is a general form of (1.8). Given two sequences

$$\sigma = (s_0, s_1, s_2, \ldots), \quad \tau = (t_1, t_2, t_3, \ldots), \quad t_i \neq 0 \text{ for all } i,$$

we define the matrix $A = A^{\sigma,\tau} = (a_{n,k})$ recursively:

$$\begin{aligned} &a_{0,0} = 1, \quad a_{0,k} = 0 \quad (k > 0) \\ &a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \quad (n \geq 1)\ . \end{aligned} \tag{2.1}$$

Let us call $A = A^{\sigma,\tau}$ the σ,τ–matrix. A matrix A is called *recursive* if $A = A^{\sigma,\tau}$ for some pair of sequences σ and τ. The term "recursive matrix" appears in a different context in the literature (see e.g. [5, 6, 25, 26]), but in this paper we always mean $A = A^{\sigma,\tau}$. Given $A = A^{\sigma,\tau} = (a_{n,k})$, the numbers $B_n = B_n^{\sigma,\tau} = a_{n,0}$ are called the *Catalan–like numbers* corresponding to σ, τ.

Looking at (1.8) we see that the sequence $(C_0, 0, C_1, 0, C_2, 0, \ldots)$ corresponds to $\sigma \equiv 0$, $\tau \equiv 1$.

The matrix $A = A^{\sigma,\tau} = (a_{n,k})$ is lower triangular with diagonal equal to 1. Furthermore, we find from (2.1) by induction

$$a_{n+1,n} = s_0 + s_1 + \ldots + s_n. \tag{2.2}$$

Given $\tau = (t_1, t_2, t_3, \ldots)$, we set in the sequel

$$T_n = t_1 t_2 \ldots t_n \quad (n \geq 1), \quad T_0 = 1. \tag{2.3}$$

The following Theorem (which is easily proved by induction) is basic to all that is to come.

Fundamental Theorem. *Given $A = A^{\sigma,\tau} = (a_{n,k})$, then for all m and n*

$$\sum_k a_{m,k} a_{n,k} T_k = a_{m+n,0} = B_{m+n}. \tag{2.4}$$

Let us write (2.4) in a compact matrix form. Define the diagonal matrix T and the infinite Hankel matrix of the sequence (B_n) as follows:

$$T = \begin{pmatrix} T_0 & & & \\ & T_1 & & 0 \\ & 0 & T_2 & \\ & & & \ddots \end{pmatrix}, \quad H = \begin{pmatrix} B_0 & B_1 & B_2 & \ldots \\ B_1 & B_2 & B_3 & \ldots \\ B_2 & B_3 & B_4 & \ldots \\ & \ldots & & \end{pmatrix}.$$

Then (2.4) can be written as

$$ATA^T = H. \tag{2.5}$$

Thus $A = A^{\sigma,\tau}$ satisfies (2.2) and (2.5). Conversely, it is readily seen that recursion (2.5) together with the sequence $(a_{n+1,n})$ *uniquely* determines A. Hence we have the following characterization:

Corollary 1. *$A = (a_{n,k})$ is a recursive matrix if and only if $ATA^T = H$ with $T_n \neq 0$ for all n, $T_0 = 1$. The sequences σ and τ are then given by $s_k = a_{k+1,k} - a_{k,k-1}$, $t_k = T_k / T_{k-1}$.*

If we look at (2.5) and consider the submatrices consisting of rows and columns 0 to n, that is

$$A_n \begin{pmatrix} T_0 & & & \\ & T_1 & & 0 \\ & 0 & \ddots & \\ & & & T_n \end{pmatrix} A_n^T = H_n,$$

we find

$$\det H_n = T_0 T_1 \ldots T_n \neq 0 \tag{2.6}$$

where H_n is the n–th Hankel matrix of the sequence $(B_0, B_1, B_2, \ldots)$. Again the converse holds, and we obtain the following characterization:

Corollary 2. *A sequence (B_n) is Catalan–like if and only if* $\det H_n \neq 0$ *for all n.*

To $A = A^{\sigma,\tau}$ consider the inverse matrix $U = A^{-1} = (u_{n,k})$. U is again lower triangular with diagonal equal to 1, and satisfies the recursion

$$\begin{aligned} &u_{0,0} = 1, \quad u_{0,k} = 0 \quad (k > 0) \\ &u_{n+1,k} = u_{n,k-1} - s_n u_{n,k} - t_n u_{n-1,k} \quad (n \geq 0) \ . \end{aligned} \tag{2.7}$$

Define the polynomial sequence $(p_n(x))$ by

$$p_n(x) = \sum_{k=0}^{n} u_{n,k} x^k, \tag{2.8}$$

then (2.7) translates into the recursive system

$$\begin{aligned} &p_0(x) = 1 \\ &p_{n+1}(x) = (x - s_n)p_n(x) - t_n p_{n-1}(x) \quad (n \geq 0) \ . \end{aligned} \tag{2.9}$$

Hence taking $A = A^{\sigma,\tau}$ and $U = A^{-1}$ together we find

$$\begin{array}{ccc} A = A^{\sigma,\tau} & \Longleftrightarrow & ATA^T = H \\ \Updownarrow & & \Updownarrow \\ p_{n+1}(x) = (x - s_n)p_n(x) - t_n p_{n-1}(x) & & UHU^T = T \end{array}$$

and have thus obtained the equivalence

$$(p_n(x)) \text{ satisfies (2.9) } \Longleftrightarrow UHU^T = T \tag{2.10}$$

for some Hankel matrix H.

It remains to interpret the right–hand side of (2.10) in terms of the sequence $(p_n(x))$, and this leads us to orthogonal systems of polynomials.

A sequence of real polynomials $(p_n(x))$, where $\deg p_n(x) = n$ for all n, is said to form an *orthogonal system* (OPS), if there exists a linear operator $L : \mathbb{R}[x] \longrightarrow \mathbb{R}$ and numbers $T_n \neq 0$ $(n \geq 1)$, $T_0 = 1$, such that

$$L(p_m(x)p_n(x)) = \delta_{m,n} T_n$$

where $\delta_{m,n}$ is the Kronecker symbol.

Suppose $UHU^T = T$, where H is the Hankel matrix of the sequence (B_n). Let the linear operator L be defined by

$$L : x^n \longrightarrow B_n \text{ for all } n \ .$$

Then

$$\begin{aligned} UHU^T = T &\iff \sum_{i,k} u_{m,i} u_{n,k} B_{i+k} = \delta_{m,n} T_n \\ &\iff L((\sum_i u_{m,i} x^i)(\sum_k u_{n,k} x^k)) = \delta_{m,n} T_n \\ &\iff L(p_m(x) p_n(x)) = \delta_{m,n} T_n \text{ for all } m, n, \end{aligned}$$

that is, $(p_n(x))$ forms an OPS. The converse is just as easily established, whence we obtain in view of (2.10) the famous theorem of Favard:

Corollary 3. *A sequence $(p_n(x))$ forms an orthogonal system if and only if $p_{n+1}(x) = (x - s_n)p_n - t_n p_{n-1}(x)$, $p_0(x) = 1$, for some pair of sequences $\sigma = (s_k)$, $\tau = (t_k)$.*

The analytical study of orthogonal systems goes back to Chebychev and Stieltjes and led to the invention of the Stieltjes integral (see [9]). In this context, the sequence $B_n = Lx^n$ is called the *moment sequence of L*. The classical results in this area concern the structure of the operator L, given the OPS $(p_n(x))$: Given $B_n = Lx^n$, let H_n as before be the n–th Hankel matrix. Then

$$\begin{aligned} \det H_n > 0 \text{ for all } n &\implies Lq = \int_{-\infty}^{\infty} q d\psi \\ \det H_n > 0, \det \widetilde{H}_n > 0 \text{ for all } n &\implies Lq = \int_{0}^{\infty} q d\psi \end{aligned}$$

where $\psi(x)$ is bounded and non–decreasing.

Example. Let us analyze our initial sequence (B_n) arising from $\sigma \equiv 0$, $\tau \equiv 1$. We have $T_n = 1$ for all n, and therefore $\det H_n = 1$ by (2.6). The associated OPS satisfies according to (2.9) the recursion

$$p_{n+1} = x p_n(x) - p_{n-1}(x), \quad p_0(x) = 1.$$

Now it is a classical result [21] that the *Chebychev polynomials* $U_n(x) = \frac{\sin(n+1)\vartheta}{\sin\vartheta}$, $x = \cos\vartheta$, satisfy the recursion

$$U_{n+1} = 2xU_n(x) - U_{n-1}(x), \quad U_0(x) = 1.$$

Hence we obtain $p_n(x) = U_n(\frac{x}{2})$. Another classical result says that the orthogonality relation for $(U_n(x))$ is given by

$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = 0 \quad \text{for } m \neq n.$$

Making the sustitution $y = \frac{x}{2}$ and normalizing, we arrive at the operator L:

$$Lq(x) = \frac{2}{\pi}\int_{-1}^{1} q(2x)(1-x^2)^{1/2}dx.$$

Hence the moment sequence (for even index) $B_{2n} = Lx^{2n}$ is precisely as given in (1.2).

3. Recursive matrices and continued fractions

Again this is a time–honored subject and intimately connected with the name Jacobi [9]. In a seminal paper Flajolet [14] studied many combinatorial enumeration problems from this point of view.

Consider a recursive matrix $A = A^{\sigma,\tau}$ associated with the sequences $\sigma = (s_k)$, $\tau = (t_k)$. Let us recall the recursion

$$\begin{array}{lllr} a_{0,0} = 1, & a_{0,k} = 0 & (k > 0) & \\ a_{n,0} = & & s_0 a_{n-1,0} + t_1 a_{n-1,1} & (n \geq 1) \\ a_{n,k} = & a_{n-1,k-1} + & s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}. & \end{array} \tag{3.1}$$

Consider the ordinary generating function of the Catalan–like numbers $B_n = a_{n,0}$:

$$G(z) = \sum_{n\geq 0} B_n z^n.$$

We make the following crucial observation: The first $m+1$ columns $0, 1, \ldots, m$ of A determine the numbers $B_0, B_1, \ldots, B_{2m+1}$.

Consider now the system (3.1) only for the columns $0, 1, \ldots, m$; in other words we set $t_{m+1} = 0$. Let $\widehat{C}_k(z)$ be the generating function of the k–th column of this truncated system.

For an arbitrary generating function $F(z) = \sum_{n \geq 0} F_n z^n$ we define the *shift operator* Δ by

$$\Delta F(z) = \sum_{n \geq 0} F_{n+1} z^n.$$

Clearly,

$$\Delta F(z) = \frac{F(z) - F(0)}{z}. \tag{3.2}$$

Hence the truncated system (3.1) translates into

$$\begin{array}{rll} \Delta \widehat{C}_0(z) = & & s_0 \widehat{C}_0(z) + t_1 \widehat{C}_1(z) \\ \Delta \widehat{C}_k(z) = & \widehat{C}_{k-1}(z) + & s_k \widehat{C}_k(z) + t_{k+1} \widehat{C}_{k+1}(z) \quad (k \geq 1) \ . \end{array}$$

According to (3.2) we find

$$\Delta \widehat{C}_0(z) = \frac{\widehat{C}_0(z) - 1}{z}, \quad \Delta \widehat{C}_k = \frac{\widehat{C}_k(z)}{z} \quad (k \geq 1),$$

and thus

$$\begin{array}{rll} \widehat{C}_0(z) - 1 = & & s_0 z \widehat{C}_0(z) + t_1 z \widehat{C}_1(z) \\ \widehat{C}_k(z) = & z \widehat{C}_{k-1}(z) + & s_k z \widehat{C}_k(z) + t_{k+1} z \widehat{C}_{k+1}(z). \end{array}$$

Written out in detail, we arrive at the following system of linear equations:

$$\begin{array}{rcrcrcl} (1 - s_0 z) \widehat{C}_0 & - & t_1 z \widehat{C}_1 & & & = & 1 \\ -z \widehat{C}_0 & + & (1 - s_1 z) \widehat{C}_1 & - & t_2 z \widehat{C}_2 & = & 0 \\ & & \cdots & & & & \cdots \\ & & & - & z \widehat{C}_{m-1} + (1 - s_m) \widehat{C}_m & = & 0 \end{array} \tag{3.3}$$

Applying Cramer's rule to (3.3) we find

$$\widehat{C}_0(z) = \frac{\det \begin{pmatrix} 1 - s_1 z & -t_2 z & & \\ -z & 1 - s_2 z & & \\ & & \cdots & \\ & & -z & 1 - s_m z \end{pmatrix}}{\det \begin{pmatrix} 1 - s_0 z & -t_1 z & & \\ -z & 1 - s_1 z & & \\ & & \cdots & \\ & & -z & 1 - s_m z \end{pmatrix}} =: \frac{D_1}{D_0}.$$

Developing the determinant D_0 according to the first column gives

$$D_0 = (1 - s_0 z) D_1 - t_1 z^2 D_2,$$

and thus

$$\widehat{C}_0(z) = \frac{D_1}{(1 - s_0 z) D_1 - t_1 z^2 D_2} = \frac{1}{(1 - s_0 z) - t_1 z^2 \frac{D_2}{D_1}} \; . \tag{3.4}$$

Iteration of (3.4) leads therefore to the continued fraction

$$\widehat{C}_0(z) = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{\cdots}{\cdots - \cfrac{t_m z^2}{1 - s_m z}}}}$$

and we *know* from our observation that $\widehat{C}_0(z)$ agrees with $G(z) = \sum B_n z^n$ up to index $2m+1$. Going with m to infinity yields therefore the infinite so–called J–fraction expansion of $G(z)$:

$$G(z) = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{1 - s_2 z - \quad \cdots}}}$$

Looking up once more our initial example $\sigma \equiv 0$, $\tau \equiv 1$, we find precisely the expansion 1.4.

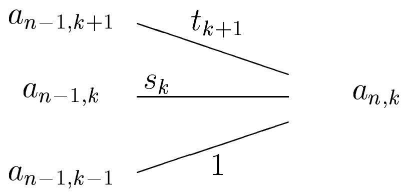
4. Recursive matrices and lattice paths

In his fundamental work [29] Viennot studied orthogonal systems of polynomials – and hence recursive matrices – from the point of view of *weighted lattice paths*. We introduce his basic ideas in this section.

Consider once more a recursive matrix $A = A^{\sigma,\tau} = (a_{n,k})$ given by

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \; .$$

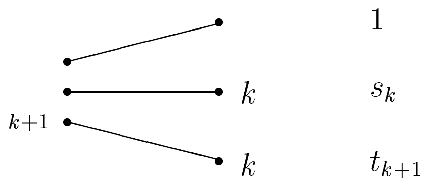
Turning the matrix by $90°$ we associate $a_{n,k}$ with the point (n,k). Hence our recursion looks as follows:



A *Motzkin path* of length n is a lattice path with the following properties:

- it starts at $(0,0)$ and ends at $(n,0)$
- it has steps $(1,1),(1,0),(1,-1)$ (diagonally up, horizontal, diagonally down)
- it never falls below $y=0$.

To the steps we assign weights w as follows:



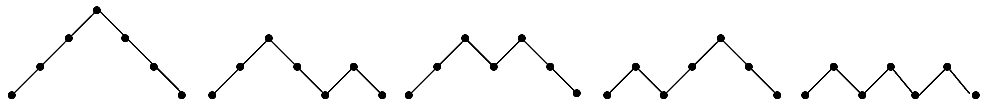
Let $\mathcal{P}_n$ be the set of Motzkin paths of length n. For $P \in \mathcal{P}_n$ we define the *weight* $w(P)$ as

$$w(P) = \prod w\ (\text{steps}).$$

A moment's thought should convince the reader that

$$B_n = \sum_{P \in \mathcal{P}_n} w(P).$$

Example. For our initial example $\sigma \equiv 0$, $\tau \equiv 1$, this interpretation just means that $B_{2n} = C_n$ counts the number of Motzkin paths of length $2n$ without horizontal steps ($\sigma \equiv 0$). These paths are usually called Dyck paths, and their enumeration by means of the Catalan numbers is one of the best–known of all counting results. As an example, for $B_6 = C_3 = 5$ we obtain the 5 paths:



5. Recursive matrices: Umbral calculus

In the last section we discussed some interesting connections of recursive matrices $A = A^{\sigma,\tau}$ to other fields, but how do we compute the Catalan–like numbers $B_n = B_n^{\sigma,\tau}$? Or rather, what do these numbers count? One way (certainly not the only one) to tackle this question is the umbral calculus, so dear to Rota's work. There is a great body of literature on this subject under various names, e.g. [5, 22, 23, 25, 26].

We start once more with the defining recursion

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}. \tag{5.1}$$

We set

$$t_n = q_n u_n \quad (n \geq 1), \quad Q_n = q_1 \dots q_n, \quad Q_0 = 1 \ .$$

Now we consider Q–generating functions

$$G(z) = \sum_{n \geq 0} \frac{G_n}{Q_n} z^n.$$

The (linear) shift operator Δ is defined by

$$\Delta : \begin{array}{l} z^0 \longrightarrow 0 \\ z^n \longrightarrow q_n z^{n-1} \quad (n \geq 1) \ , \end{array}$$

hence

$$\Delta G(z) = \sum_{n \geq 0} \frac{G_{n+1}}{Q_n} z^n \ .$$

To $A = A^{\sigma,\tau}$ let $C_k(z)$ be the Q–generating function of the k–th column, that is

$$C_k(z) = \sum_{n\geq 0} \frac{a_{n,k}}{Q_n} z^n \quad (k \geq 0) \ .$$

The recursion (5.1) therefore translates into

$$\begin{array}{lcrcll} \Delta C_k & = & C_{k-1} + s_k C_k & + & u_{k+1} q_{k+1} C_{k+1} & (k \geq 1) \\ \Delta C_0 & = & a C_0 & + & b q_1 C_1 & \end{array} \tag{5.2}$$

where we set $a = s_0$, $b = u_1$. (The numbers a and b will play a special role.)

Here comes the umbral calculus idea. We consider only solutions of (5.2) of the form

$$C_k(z) = C_0(z) \frac{F(z)^k}{Q_k}, \quad F(0) = 0. \tag{5.3}$$

Hence (5.2) transform with (5.3) into

$$\begin{array}{lcrcll} \Delta(C_0 F^k) & = & q_k C_0 F^{k-1} + s_k C_0 F^k & + & u_{k+1} C_0 F^{k+1} & (k \geq 1) \\ \Delta C_0 & = & a C_0 & + & b C_0 F, & \end{array} \tag{5.4}$$

and the goal is to compute

$$C_0(z) = \sum_{n\geq 0} \frac{B_n}{Q_n} z^n.$$

According to (5.4) we need a multiplication rule for $\Delta(GH)$. To do this we assume the following special case: For any Q–generating functions $G(z)$, $H(z)$

$$\Delta(GH) = (\Delta G) H(qz) + (\Delta H) G \text{ for some } q \in \mathbb{R} \ . \tag{5.5}$$

There are three cases:

(A) $q = 0$: Normalizing $q_1 = 1$, we obtain $q_n = 1$ for all n, resulting in *ordinary generating functions* $G(z) = \sum_{n\geq 0} G_n z^n$.

(B) $q = 1$: Normalizing $q_1 = 1$ we obtain $q_n = n$ for all n, hence $Q_n = n!$, resulting in *exponential generating functions* $G(z) = \sum_{n\geq 0} \frac{G_n}{n!} z^n$.

(C) $q \neq 0, 1$: Normalizing $q_1 = q - 1$, we obtain $q_n = q^n - 1$, hence $Q_n = \prod_{i=1}^{n} (q^i - 1)$, resulting in *$q$–generating functions* $G(z) = \sum_{n \geq 0} \frac{G_n}{\prod_{i=1}^{n} (q^i - 1)} z^n$.

Applying (5.5) to F^k we find

$$\Delta F^k = (\Delta F)(F^{k-1}(z) + F^{k-2}(z)F(qz) + \ldots + F^{k-1}(qz)),$$

which yields for our three calculi:

$$\begin{aligned} &\text{(A)}\ \ q = 0: \quad \Delta F^k = (\Delta F)F^{k-1}, \quad \Delta F = \tfrac{F(z)-F(0)}{z} \\ &\text{(B)}\ \ q = 1: \quad \Delta F^k = (\Delta F)kF^{k-1}, \quad \Delta F = F' \\ &\text{(C)}\ \ q \neq 0, 1: \ \Delta F^k = (\Delta F)(F_{k-1}(z) + \ldots + F^{k-1}(qz)), \quad \Delta F = \tfrac{F(qz)-F(z)}{z}\ . \end{aligned} \tag{5.6}$$

Now we substitute (5.6) into the system (5.4) and try to compute $F(z)$ and $C_0(z)$.

Let us treat the solutions for the three cases separately.

(A) Ordinary generating functions (divided difference calculus)

We obtain

$$\Delta(C_0 F^k) = C_0(\Delta F)F^{k-1},$$

and hence for (5.4) by cancelling

$$\begin{aligned} \Delta F &= 1 + s_k F + u_{k+1}F^2 \quad (k \geq 1) \\ \Delta C_0 &= aC_0 + bC_0 F\ . \end{aligned} \tag{5.7}$$

We see that (5.7) has a solution of the form (5.3) if and only if $s_k = s$ $(k \geq 1)$, $u_k = u$ $(k \geq 2)$. Hence the possible sequences σ and τ are of the form

$$\sigma = (a, s, s, s, \ldots), \quad \tau = (b, u, u, u, \ldots), \tag{5.8}$$

and (5.7) becomes

$$\begin{aligned} \Delta F &= 1 + sF + uF^2 \\ \Delta C_0 &= aC_0 + bC_0 F\ . \end{aligned} \tag{5.9}$$

In terms of the coefficients of $F(z)$ and $C_0(z)$, (5.9) translates into

$$\begin{aligned} F_{n+1} &= sF_n + u\sum_{k=1}^{n-1} F_k F_{n-k} \quad (n \geq 1), \quad F_0 = 0, F_1 = 1 \\ B_{n+1} &= aB_n + b\sum_{k=0}^{n-1} B_k F_{n-k} \quad (n \geq 0), \quad B_0 = 1. \end{aligned} \tag{5.10}$$

This latter system leads now to the desired combinatorial interpretations which we will discuss in detail in the next section.

(B) Exponential generating functions (classical calculus)

A similar analysis leads to the result that the solutions are precisely $s_k = a + ks$, $u_{k+1} = b + ku$ for fixed a, s, b, u. Hence the sequences σ and τ are of the form

$$\sigma = (a + ks), \quad \tau = (k(b + (k-1)u)), \tag{5.11}$$

and the system (5.4) becomes

$$\begin{aligned} F' &= 1 + sF \quad + \quad uF^2 \\ C_0' &= \qquad aC_0 \quad + \quad bC_0F \ . \end{aligned} \tag{5.12}$$

In terms of the coefficients $F(z)$ and $C_0(z)$, (5.12) translates into

$$\begin{aligned} F_{n+1} &= sF_n + u\sum_{k=1}^{n-1} \binom{n}{k} F_k F_{n-k} \quad (n \geq 1), \ F_0 = 0, F_1 = 1 \\ B_{n+1} &= aB_n + b\sum_{k=0}^{n-1} \binom{n}{k} B_k F_{n-k} \quad (n \geq 0), \ B_0 = 1. \end{aligned} \tag{5.13}$$

The system (5.13) leads again to manifold combinatorial interpretations to be studied in the next section.

(C) q–generating functions (q–calculus), $q \neq 0, 1$

Here the solution is incomplete. If the quadratic coefficient of $F(z)$ is $F_2 = 0$, then it can be shown that $F(z) = z$ is the only solution. The case $F_2 \neq 0$ is open. For $F(z) = z$, it is readily seen that $s_k = aq^k$, $u_{k+1} = bq^k$ are the only possibilities. In other words, the sequences σ and τ are of the form

$$\sigma = (aq^k), \quad \tau = (bq^{k-1}(q^k - 1)). \tag{5.14}$$

In terms of the Catalan–like numbers B_n, the system (5.4) translates into

$$B_{n+1} = aB_n + b(q^n - 1)B_{n-1} \quad (n \geq 0), \; B_0 = 1. \tag{5.15}$$

Again we will discuss combinatorial interpretations of (5.15) in the next section.

Before going on to these combinatorial interpretations let us take another quick look at orthogonal systems of polynomials via the umbral calculus.

Let as before $A = (a_{n,k})$, $C_k(z) = \sum\limits_{n\geq 0} \frac{a_{n,k}}{Q_n} z^n$. To state that (5.3) holds, we use the short notation

$$A \longleftrightarrow (C_0, F).$$

The following result is well–known (see [25]):

$$A \longleftrightarrow (C_0, F) \Longleftrightarrow A^{-1} \longleftrightarrow \left(\frac{1}{C_0(\overline{F})}, \overline{F}\right) \tag{5.16}$$

where $\overline{F}$ is the *composition inverse* of F, that is,

$$\overline{F}(F(z)) = F(\overline{F}(z)) = z.$$

As in section 3 set $A^{-1} = (u_{n,k})$ and

$$p_n(x) = \sum_{k=0}^{n} u_{n,k} x^k. \tag{5.17}$$

In umbral calculus, the sequence $(p_n(x))$ is called the *Sheffer sequence* corresponding to $A \longleftrightarrow (C_0, F)$.

Define the infinite (column) vectors $Z = (\frac{z^n}{Q_n})$, $X = (x^n)$. Then by (5.16) and (5.17) we obtain

$$\sum_{n\geq 0} \frac{p_n(x)}{Q_n} z^n = Z^T A^{-1} X = \frac{1}{C_0(\overline{F}(z))} \sum_{k\geq 0} \frac{(x\overline{F}(z))^k}{Q_k}. \tag{5.18}$$

Question. *Given a Q–umbral calculus, which OPS $(p_n(x))$ are Sheffer sequences?*

The answer is given by our analysis of the cases (A) to (C).

(A) Ordinary generating functions

(5.8) and (5.18) yield the result: An OPS $(p_n(x))$ is a Sheffer sequence with

$$\sum_{n\geq 0} p_n(x) z^n = g(z) \frac{1}{1 - x f(z)}$$

if and only if $s_k = s$ $(k \geq 1)$, $t_k = u \quad (k \geq 2)$.

(B) Exponential generating functions

Here (5.11) und (5.18) give a classical result of Meixner [9]: An OPS $(p_n(x))$ is a Sheffer sequence with

$$\sum_{n\geq 0} \frac{p_n(x)}{n!} z^n = g(z) e^{x f(z)}$$

if and only if $s_k = a + ks$, $t_k = k(b + (k-1)u)$.

(C) q–generating functions

With the assumption $F_2 = 0$, we obtain from (5.14) and (5.18): An OPS $(p_n(x))$ is a Sheffer sequence with

$$\begin{aligned}\sum_{n\geq 0} \frac{p_n(x)}{(q^n-1)\dots(q-1)} z^n &= g(z) \sum_{k\geq 0} \frac{(xz)^k}{(q^k-1)\dots(q-1)} \\ &= g(z) \prod_{k\geq 0} \frac{1}{1 + q^k x z}\end{aligned}$$

if and only if $s_k = aq^k$, $t_k = bq^{k-1}(q^k - 1)$.

Example. Let us look at the particularly simple case when $F(z) = z$, thus $A \longleftrightarrow (C_0(z), z)$. This is called the *Appel case* in the literature [22].

(A) Ordinary generating functions

According to (5.9) we obtain

$$\Delta F = 1 = 1 + sz + uz^2.$$

Hence there is no solution since $u \neq 0$.

(B) Exponential generating functions

By (5.12) we obtain

$$F' = 1 = 1 + sz + uz^2,$$

and thus $s = u = 0$, that is, $s_k = a$, $t_k = bk$ by (5.11). Looking again at (5.12) this gives

$$C_0' = aC_0 + bC_0 z,$$

and therefore as solution

$$C_0 = e^{az+\frac{bz^2}{2}}.$$

The corresponding OPS $(p_n(x))$ is given by (5.18):

$$\begin{aligned}\sum_{n\geq 0} \frac{p_n(x)}{n!} z^n &= e^{-az-\frac{bz^2}{2}} \sum_{k\geq 0} \frac{(xz)^k}{k!} \\ &= e^{(-a+x)z-\frac{bz^2}{2}},\end{aligned}$$

and we obtain for $p_n(x)$ by an easy computation the classical *Hermite polynomials* $H_n^{a,b}(x)$

$$H_n^{a,b}(x) = \sum_j (\sum_k \frac{(-1)^{j-k} a^{j-2k} b^k}{(j-2k)! 2^k k!}) n^{\underline{j}} x^{n-j},$$

where $n^{\underline{j}} = n(n-1)\ldots(n-j+1)$ are the falling factorials. In particular, for $a = 0$, hence $s_k = 0$, $t_k = bk$, we find

$$H_n^{0,b}(x) = \sum_{k\geq 0} (-\frac{b}{2})^k \frac{n^{\underline{2k}}}{k!} x^{n-2k}.$$

Example. Let us look again at our initial example $\sigma \equiv 0$, $\tau \equiv 1$. According to (5.7) we have

$$\Delta F = \frac{F}{z} = 1 + F^2,$$

and thus $F = z + zF^2$. A simple computation yields $\overline{F} = \frac{z}{1+z^2}$, and $C_0(\overline{F}) = 1 + z^2$. Hence we infer from (5.18)

$$\sum_{n \geq 0} p_n(x) z^n = \frac{1}{1+z^2} \; \frac{1}{1 - \frac{zx}{1+z^2}} = \frac{1}{1 - xz + z^2}.$$

Comparing coefficients this gives the usual expression for the Chebychev polynomials $U_n(x)$:

$$p_n(x) = \sum_i (-1)^i \binom{n-i}{i} x^{n-2i},$$

and hence

$$U_n(x) = p_n(2x) = \sum_i (-1)^i \binom{n-i}{i} 2^{n-2i} x^{n-2i}.$$

We close this section with a natural problem.

Question: Which recursive matrices A have an inverse $U = A^{-1}$ which is also recursive? Suppose $A = A^{\sigma,\tau}$ and $U = U^{\sigma',\tau'}$. Then it is easily seen that

$$\sigma' = -\sigma.$$

Furthermore, in the Sheffer case $A \longleftrightarrow (C_0, F)$ there are two possibilities:

(a) $\tau' = -\tau$: This implies $\sigma = (s, s, s, \ldots)$, $\tau = (t, 2t, 3t, \ldots)$, and we obtain the *Hermite polynomials* $H_n^{s,t}(x)$ from above.

(b) $\tau' = \tau$: Here we get $\sigma = (s_k)$, $s_k = a + 2mk$, $\tau = (t_k)$, $t_k = k(am + m^2(k-1))$ for some m. The corresponding OPS are the classical *Laguerre polynomials* $L_n^{a,m}(x)$.

Nothing substantial is known in the non–Sheffer case.

6. Catalan–like numbers: Combinatorial interpretations

Starting with the recursion formulae (5.10), (5.13), (5.15) we give a combinatorial interpretation in each case, derive then convolution type formulae, and give a host of examples, including some of the most basic combinatorial counting numbers.

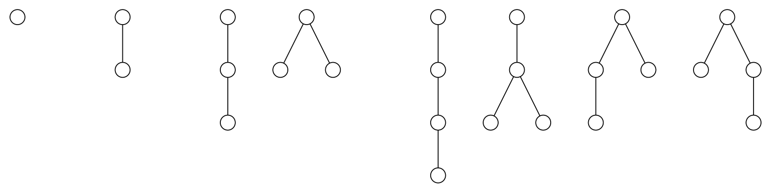
(A) Ordinary generating functions

Given the sequences $\sigma = (a, s, s, \ldots)$, $\tau = (b, u, u, \ldots)$, the Catalan–like numbers $B_n = B_n^{\sigma,\tau}$ are computed by the system (5.10):

$$F_{n+1} = sF_n + u \sum_{k=1}^{n-1} F_k F_{n-k} \quad (n \geq 1), \quad F_0 = 0, F_1 = 1 \tag{6.1}$$

$$B_{n+1} = aB_n + b \sum_{k=0}^{n-1} B_k F_{n-k} \quad (n \geq 0), \quad B_0 = 1. \tag{6.2}$$

Our combinatorial object are *Motzkin trees*, that is, rooted plane trees, in which every vertex has out–degree ≤ 2. The figure shows the trees with $n \leq 4$ vertices.



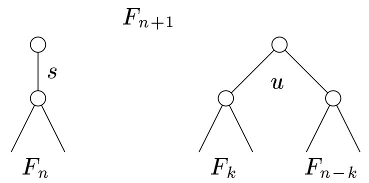
Let $\mathcal{T}_n$ be the set of Motzkin trees with n vertices. For $T \in \mathcal{T}_n$ define the weight

$$w(T) = s^\ell u^m \text{ where } \begin{array}{rcl} \ell & = & \# \text{ vertices of out–degree } 1 \\ m & = & \# \text{ vertices of out–degree } 2. \end{array}$$

Then for F_n in (6.1), we have for $n \geq 2$

$$F_n = \sum_{T \in \mathcal{T}_n} w(T) \quad (F_0 = 0, F_1 = 1) \ .$$

The figure shows the proof:



Now define the refined weight

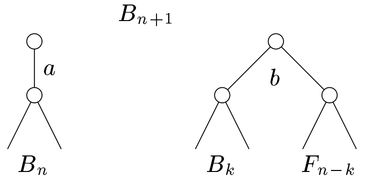
$$W(T) = a^i b^j s^h u^k$$

where i = # vertices of out–degree 1 on left–most branch
j = # vertices of out–degree 2 on left–most branch
h = # vertices of out–degree 1 not on left–most branch
k = # vertices of out–degree 2 not on left–most branch.

Then

$$B_n = \sum_{T \in \mathcal{T}_{n+1}} W(T) \quad (B_0 = 1).$$

The proof is again sketched in the picture:



Example. For $a = b = s = u = 1$ we obtain $B_n = |\mathcal{T}_{n+1}|$, hence $B_n = M_n$ the Motzkin number, which counts all n–edge Motzkin trees [13, 28]. Our initial example $\sigma \equiv 0$, $\tau \equiv 1$ is the case $a = s = 0$, $b = u = 1$. Hence B_n counts the n–edge Motzkin trees with all out–degrees equal to 2, and we obtain precisely the example in section 1.

Now let us look at convolution recursions as in (1.1). From (6.1) and (6.2) the following formula can be derived:

$$\begin{aligned} bB_{n+1} = \;& abB_n + (u-b)\sum_{k=0}^{n-1} B_{k+1}B_{n-k} + (bs - 2au + ab)\sum_{k=0}^{n-1} B_k B_{n-k} \\ & + (a^2u - abs + b^2)\sum_{k=0}^{n-1} B_k B_{n-1-k}\;. \end{aligned}$$

The following cases are especially interesting:

$$\begin{aligned} b = u, a = s: &\qquad B_{n+1} = sB_n + b\sum_{k=0}^{n-1} B_k B_{n-1-k} \\ b = u, a = s-1: &\qquad B_{n+1} = aB_n + \sum_{k=0}^{n-1} B_k B_{n-k} + (b-a)\sum_{k=0}^{n-1} B_k B_{n-1-k} \\ b = 2u, a = s: &\qquad B_n = \frac{1}{2^n}\sum_{k\geq 0}\binom{2k}{k}\binom{k}{n-k} s^{2k-n}(4u - s^2)^{n-k}. \end{aligned}$$

From these formulae or from the combinatorial interpretation we can now derive a number of interesting examples:

$\tau \equiv 1$ $(b = u = 1)$:

a	s		
0	0	$(C_0, 0, C_1, 0, C_2, 0, C_3, \ldots)$	Catalan
0	1	$(R_0, R_1, R_2, R_3, R_4, \ldots)$	Riordan [7]
		$= (1, 0, 1, 1, 3, 6, 15, \ldots)$	
0	2	$(F_0, F_1, F_2, F_3, \ldots)$	Fine [12]
		$= (1, 0, 1, 2, 6, 18, 57, \ldots)$	
1	0	$\binom{n}{\lfloor n/2 \rfloor}$	middle binomials
		$= (1, 1, 2, 3, 6, 10, 20, \ldots)$	
1	1	(M_n)	Motzkin
		$= (1, 1, 2, 4, 9, 21, 51, \ldots)$	
1	2	(C_n)	Catalan
		$= (1, 1, 2, 5, 14, 42, 132, \ldots)$	
2	2	(C_{n+1})	Catalan
3	2	$\binom{2n+1}{n}$	central binomials
3	3	(H_n)	restricted hexagonals [18]

$b = 2, u = 1$:

$a = s = 1$:	$(1, 1, 3, 7, 19, 51, \ldots)$	trinomials
$a = s = 2$:	$\binom{2n}{n}$	binomials

$b = 2, u = 2$:

$a = 2, s = 3$:	$(1, 2, 6, 22, 90, 394, \ldots)$	Schröder

Of course, there are a host of other relations that can be deduced between the various numbers, such as $\text{Schröder}_n = \sum_k \binom{2n-k}{k} C_{n-k}$, $M_n = R_n + R_{n+1}$, or $C_n = 2F_n + F_{n-1}$.

(B) Exponential generating functions

Given $\sigma=(a+ks)$, $\tau=(k(b+(k-1)u))$, the Catalan–like numbers $B_n=B_n^{\sigma,\tau}$ are given by the system (5.13):

$$F_{n+1}=sF_n+u\sum_{k=1}^{n-1}\binom{n}{k}F_kF_{n-k}\quad (n\geq 1), F_0=0, F_1=1 \tag{6.3}$$

$$B_{n+1}=aB_n+b\sum_{k=0}^{n-1}\binom{n}{k}B_kF_{n-k}\quad (n\geq 0), B_0=1. \tag{6.4}$$

We distinguish two cases:

(B1) $u=0$: Here $s_k=a+ks$, $t_k=bk$, and (6.3) yields $F_n=s^{n-1}(n\geq 1)$, $F_0=0$.

In this case the combinatorial object constitutes *partitions* of an n–set. The figure shows the partitions up to $n=4$.

$$\begin{array}{cccccc}
1 & 12 & 123 & 1234 & 12-34 & 2-3-14\\
 & 1-2 & 12-3 & 123-4 & 13-24 & 1-23-4\\
 & & 2-13 & 3-124 & 23-14 & 1-3-24\\
 & & 1-23 & 2-134 & 12-3-4 & 1-2-34\\
 & & 1-2-3 & 1-234 & 2-13-4 & 1-2-3-4
\end{array}$$

The *Bell number* Bell_n counts the total number of partitions.

Let $\mathcal{P}_n$ be the partitions of an n–set. For $P\in\mathcal{P}_n$ define the weight

$$W(P)=a^{n_1}b^{n_2}s^{n-n_1-2n_2}$$

where $n_1 = \#$ singleton blocks
$n_2 = \#$ blocks of size ≥ 2.

Then it is readily verified that

$$B_n=\sum_{P\in\mathcal{P}_n}W(P).$$

Hence we obtain the following examples:

$a=b=s=1:$	$B_n = \mathrm{Bell}_n$	Bell
$\sigma=(1,2,3,\ldots)$	$=(1,1,2,5,15,52,203,\ldots)$	
$\tau=(1,2,3,\ldots)$		
$a=b,\ s=1:$	$B_n = \sum\limits_{k=0}^{n} S_{n,k}a^k$	Stirling polynomial
$\sigma=(a,a+1,a+2,\ldots)$	where $S_{n,k}$ are the Stirling	
$\tau=(a,2a,3a,\ldots)$	numbers of the second kind	
$a=b=1, s=0:$	$B_n = \#$ involutions	involutions
$\sigma=(1,1,1,\ldots)$	$=(1,1,2,4,10,26,76,\ldots)$	
$\tau=(1,2,3,\ldots)$		
$a=0, b=1, s=0$:	$B_n = 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$	fixpoint–free
$\sigma=(0,0,0,\ldots)$	or 0	involutions
$\tau=(1,2,3,\ldots)$	$=(1,0,1,0,3,0,15,\ldots)$	

(B2) $u \neq 0$: The combinatorial objects are *permutations* of $\{1,2,\ldots,n\}$. Consider a permutation $\pi=\pi_1\pi_2\ldots\pi_n$, and set $\pi_0=\pi_{n+1}=n+1$. The weight $w(\pi)$ is defined as

$$w(\pi)=s^{\ell}u^{m},$$

where $\ell \;=\; \#$ double rises $\pi_{i-1}<\pi_i<\pi_{i+1}$

$m \;=\; \#$ local maxima.

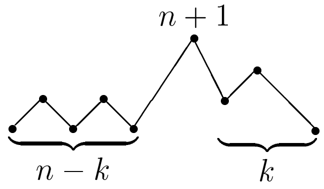
Let $\mathcal{S}_n$ be the set of permutations of $\{1,2,\ldots,n\}$ *without double falls* (including $\pi_0>\pi_1>\pi_2$). Then

$$F_n=\sum_{\pi\in\mathcal{S}_n} w(\pi).$$

The proof is suggested by the following figure:

$\pi_{n+1}=n+1 \quad\Longrightarrow\quad sF_n$ (recall $\pi_{n+2}=n+2$)

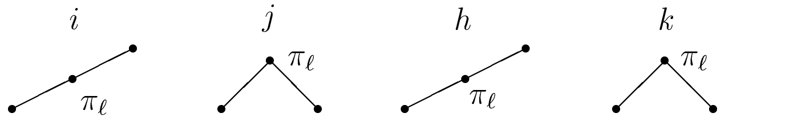
$\pi_{n+1}\neq n+1 \quad\Longrightarrow\quad u\sum\limits_{k=1}^{n-1}F_kF_{n-k}$



Now we consider the refined weight for $\pi \in \mathcal{S}_n$:

$$W(\pi) = a^i b^j s^h u^k,$$

where $\pi_0 = 0$, $\pi_{n+1} = n+1$.

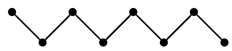


$\pi_i < \pi_\ell (i < \ell)$ $\quad$ $\pi_i < \pi_\ell (i < \ell)$ $\quad$ $\exists i < \ell, \pi_i > \pi_\ell$ $\quad$ $\exists i < j : \pi_i > \pi_\ell$

Then

$$B_n = \sum_{\pi \in \mathcal{S}_n} W(\pi).$$

The proof is similar to the previous one, by considering the cases $\pi_{n+1} = n+1$ or $\pi_{n+1} \neq n+1$. Again we obtain some very interesting examples.

$a = s = 0,\ b = u = 1$:	$B_n =$ Secant numbers	alternating
$\sigma \equiv 0$	$= (1, 0, 1, 0, 5, 0, 61, \ldots)$	permutations
$\tau = (k^2)$		
$a = s = b = u = 1:$	$B_n = \lvert\mathcal{S}_n\rvert$	without double falls
$\sigma = (k+1)$	$= (1, 1, 2, 5, 17, 70, 349, \ldots)$	
$\tau = (k^2)$		

For $s = 2$, $u = 1$, we readily obtain $F_n = n!$ $(n \geq 1)$ in (6.3), and hence

$$B_{n+1} = aB_n + b \sum_{k=0}^{n-1} \binom{n}{k} B_k (n-k)!.$$

If we define a new weight $\overline{W}(\pi)$ of any permutation π of $\{1, 2, \ldots, n\}$ by

$$\overline{W}(\pi) = a^v b^w$$

where
$$\begin{aligned} v &= \#\text{ fixpoints} \\ w &= \#\text{ cycles of length } \geq 2, \end{aligned}$$

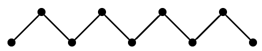
then it follows that

$$B_n = \sum_{\pi} \overline{W}(\pi).$$

From this we deduce the following examples:

$a = b = 1, s = 2, u = 1$:	$B_n = n!$	all permutations
$\sigma = (2k+1)$ $\tau = (k^2)$	$= (1, 1, 2, 6, 24, 120, 720, \ldots)$	
$a = 0,\ b = 1,\ s = 2,\ u = 1$:	$B_n = D_n = \#$ derangements	fixpoint–free
$\sigma = (2k)$	$= (1, 0, 1, 2, 9, 44, 265, \ldots)$	
$\tau = (k^2)$		
$a = b,\ s = 2,\ u = 1$:	$B_n = \sum\limits_k s_{n,k} a^k$	Stirling polynomial
$\sigma = (a + 2k)$ $\tau = (k(a + k - 1))$	where $s_{n,k}$ are the Stirling numbers of the first kind	

When $a = s$ and $b = 2u$, then (6.3), (6.4) yield $B_n = F_{n+1}$. From this, we deduce as examples:

$a = s = 0, b = 2, u = 1$:	$B_n =$ Tangent numbers	alternating
$\sigma \equiv 0$	$= (1, 0, 2, 0, 16, 0, 272, \ldots)$	permutations
$\tau = (k(k+1))$		

Another case which is easily analyzed is $s = 2m,\ u = m^2$. We obtain from (6.3), $F_n = m^{n-1} n!$ and from (6.4)

$$B_{n+1} = (mn + a)B_n + (b - am)nB_{n-1}.$$

In particular, this gives for $m = 1$ again

$$a = b,\ s = 2,\ u = 1:\quad \begin{array}{rcl} B_n & = & a^{\overline{n}} = a(a+1)\ldots(a+n-1) \\ & = & \sum\limits_k s_{n,k} a^k. \end{array}$$

and for

$$\begin{array}{ll} a = b = 2,\ s = 2,\ u = 1: & B_n = (n+1)! \\ \sigma = (2 + 2k),\ \tau = (k(k+1)) & \end{array}$$

Finally, for $a = b = u,\ s = u + 1$, we find $B_n = uF_n$, and from this

$$B_n = \sum_k A_{n,k} u^k \quad \text{(Eulerian polynomial)},$$

where $A_{n,k}$ is the number of permutations of $\{1, \ldots, n\}$ with $k-1$ descents.

(C) q–generating functions

Given the sequences $\sigma = (aq^k)$, $\tau = (bq^{k-1}(q^k - 1))$, we have the recursion (5.15):

$$B_{n+1} = aB_n + b(q^n - 1)B_{n-1} \ . \tag{6.5}$$

What do these Catalan–like numbers count? The nicest result is obtained in the case $a = b + 1$. We claim that for $a = b + 1$

$$B_n = \sum_k \binom{n}{k}_q b^k \quad \textit{Galois polynomial,} \tag{6.6}$$

where $\binom{n}{k}_q = \frac{(q^n-1)\ldots(q^{n-k+1}-1)}{(q^k-1)\ldots(q-1)}$ are the *Gaussian coefficients*. In particular, when q is a prime power, then $\binom{n}{k}_q$ counts the number of k–dimensional subspaces of an n–dimensional vector space over $GF(q)$.

Here is a proof of (6.6) à la Rota: Let

$$\begin{aligned} g_k(x) &= (x-1)(x-q)\cdots(x-q^{k-1}) \quad (k \geq 1) \\ g_0(x) &= 1 \end{aligned} \tag{6.7}$$

be the *Gaussian polynomials*.

Denote as before by Δ the shift operator of the q–calculus, that is

$$\Delta x^n = \begin{cases} (q^n - 1)x^{n-1} & (n \geq 1) \\ 0 & (n = 0) \ . \end{cases}$$

It is immediately verified that

$$\Delta g_k(x) = (q^k - 1)g_{k-1}(x) \quad (k \geq 1) \ . \tag{6.8}$$

Now we define the linear operator $L : I\!R[x] \longrightarrow I\!R$ by

$$L : g_k(x) \longrightarrow b^k \ . \tag{6.9}$$

From the well–known relation $x^n = \sum\limits_k \binom{n}{k}_q g_k(x)$ it follows that

$$Lx^n = \sum_k \binom{n}{k}_q b^k.$$

So we have to show
$$Lx^n = B_n$$
or, in other words, that Lx^n satisfies the recursion (6.5), that is,
$$Lx^{n+1} = (b+1)Lx^n + b(q^n - 1)Lx^{n-1}.$$
From (6.7) we have
$$g_{n+1}(x) = xg_n(x) - q^n g_n(x),$$
hence by (6.9)
$$\begin{aligned} Lxg_n(x) &= Lg_{n+1}(x) + q^n Lg_n(x) \\ &= b^{n+1} + q^n b^n \\ &= b^n(b+1) + (q^n - 1)b^n. \end{aligned} \tag{6.10}$$
Furthermore by (6.8),
$$L\Delta g_n(x) = L(q^n - 1)g_{n-1}(x) = (q^n - 1)b^{n-1},$$
and thus by (6.10)
$$Lxg_n(x) = (b+1)Lg_n(x) + bL\Delta g_{n(x)}. \tag{6.11}$$
Since the Gaussian polynomials form a basis, (6.11) holds for *all* polynomials $p(x)$:
$$Lxp(x) = (b+1)Lp(x) + bL\Delta p(x).$$
Setting $p(x) = x^n$, we thus infer
$$Lx^{n+1} = (b+1)Lx^n + b(q^n - 1)Lx^{n-1},$$
as claimed.

As example we obtain for $a = 2$, $b = 1$, that is, the sequences $\sigma = (2q^k)$, $\tau = (q^{k-1}(q^k - 1))$:
$$B_n = \sum_{k=0}^{n} \binom{n}{k}_q \quad \text{Galois number,}$$
which for q a prime power is the total number of subspaces of an n–dimensional vector space over $GF(q)$.

To end this section we return to the general case and derive some binomial relations between Catalan–like numbers which unify many known instances.

Let $A = A^{\sigma,\tau} = (a_{n,k})$ be a recursive matrix given as in (5.1):

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}. \tag{6.12}$$

Denote by $P = \left(\binom{n}{k}\right)$ the infinite Pascal matrix, having $\binom{n}{k}$ as its $(n,k)-$entry. By checking (6.12), the following two results are readily seen:

Let $A = A^{\sigma,\tau}$, $A' = PA$, *then* $A' = A^{\sigma+1,\tau}$, *where* $\sigma + 1 = (s_0 + 1, s_1 + 1, s_2 + 1, \ldots)$. (6.13)

Let $A = A^{\sigma,\tau} = (a_{n,k})$, $B = (b_{n,k})$ *with* $b_{n,k} = (-1)^{n-k} a_{n,k}$, *then* $B = A^{-\sigma,\tau}$, *where* $-\sigma = (-s_0, -s_1, -s_2, \ldots)$. (6.14)

Iterating (6.13) we have the following relation between Catalan–like numbers B_n for σ, τ and a constant λ:

$$B_n^{\sigma+\lambda,\tau} = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} B_k^{\sigma,\tau} \tag{6.15}$$

$$B_n^{-\sigma,\tau} = (-1)^n B_n^{\sigma,\tau}. \tag{6.16}$$

We can now look up our list of examples and apply these formulae. Here are a few examples:

$\sigma \equiv 0, \tau \equiv 1, \lambda = 1$: $\quad M_n = \sum_k \binom{n}{2k} C_k$

$\sigma \equiv 1, \tau \equiv 1, \lambda = 1$: $\quad C_{n+1} = \sum_k \binom{n}{k} M_k$.

Comparing $\sigma \equiv 2$, $\tau = (2, 1, 1, \ldots)$ and $-\sigma, \tau$, we obtain

$$\binom{2n}{n} = \sum_k (-1)^k \binom{n}{k} 4^{n-k} \binom{2k}{k}.$$

Comparing $\sigma = (3, 2, 2, \ldots)$, $\tau \equiv 1$, $\sigma' = (-1, -2, -2, \ldots)$, $\tau' \equiv 1$, yields

$$\binom{2n+1}{n} = \sum_k (-1)^k \binom{n}{k} 4^{n-k} C_k.$$

Finally, comparing $\sigma = (1, 2, 2, \ldots)$, $\tau \equiv 1$, and $\sigma' = (-1, 0, 0, \ldots)$, $\tau' \equiv 1$, we find

$$C_n = \sum_k (-1)^k \binom{n}{k} 2^{n-k} \binom{k}{\lfloor k/2 \rfloor},$$

and comparing $\sigma = (1, 1, 1, \ldots)$, $\tau \equiv 1$, and $\sigma' = (-2, -2, -2, \ldots)$, $\tau' \equiv 1$, we have

$$M_n = \sum_k (-1)^k \binom{n}{k} 3^{n-k} C_{k+1}.$$

For more on this topic the reader is referred to [2].

7. Non–Sheffer Sequences

In the case when the basic recursion (5.1) does not satisfy the Sheffer–relation (5.3), only a few results are known. These results are of two types: Either new sequences $\widehat{\sigma}, \widehat{\tau}$ are constructed out of a given pair σ, τ (the binomial formulae (6.15) are examples), or the Catalan–like numbers B_n satisfy certain simple recursions.

A. As an illustration of the first type, we have the following result. Suppose we are given $\sigma = (0, 0, 0, \ldots)$, $\tau = (t_k)$ arbitrary. Then, for the Catalan–like numbers $B_n = B_n^{\sigma,\tau}$ we clearly have

$$B_{2n+1} = 0 \quad (n \geq 0).$$

In the literature this is called the *symmetric* case.

Define the sequences $\widehat{\sigma} = (\widehat{\sigma}_k)$, $\widehat{\tau} = (\widehat{\tau}_k)$ by

$$\begin{aligned} &\widehat{s}_0 = t_1, \quad \widehat{s}_1 = t_2 + t_3, \ldots, \quad \widehat{s}_k = t_{2k} + t_{2k+1}, \ldots \\ &\widehat{t}_1 = t_1 t_2, \quad \widehat{t}_2 = t_3 t_4, \ldots, \quad \widehat{t}_k = t_{2k-1} t_{2k}, \ldots \end{aligned} \tag{7.1}$$

Then it is easily checked that for $\widehat{B}_n = B_n^{\widehat{\sigma},\widehat{\tau}}$ we have

$$\widehat{B}_n = B_{2n}. \tag{7.2}$$

Examples

i) $\sigma \equiv 0$, $\tau \equiv 1$ gives $B_{2n} = C_n$ (Catalan), and we obtain by (7.1) once more that $\widehat{\sigma} = (1, 2, 2, \ldots)$, $\widehat{\tau} \equiv 1$, yields $\widehat{B}_n = C_n$.

ii) $\sigma \equiv 0$, $\tau = (a, b, a, b, a, b, \ldots)$. Here we have $\widehat{\sigma} = (a, a+b, a+b, \ldots)$, $\widehat{\tau} = (ab, ab, ab, \ldots)$. It is readily seen that

$$\widehat{B}_n = B_n^{\widehat{\sigma},\widehat{\tau}} = \sum_{k=0}^{n-1} N(n,k) a^{n-k} b^k$$

where

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

are the *Narayana numbers.* In particular, the case $a = 1$, that is, $\widehat{\sigma} = (1, b+1, b+1, \ldots)$, $\widehat{\tau} = (b, b, b, \ldots)$ yields the Narayana polynomial

$$\widehat{B}_n = \sum_{k=0}^{n-1} N(n,k) b^k.$$

Since $a = b = 1$ gives $\widehat{\sigma} = (1, 2, 2, \ldots)$, $\widehat{\tau} \equiv 1$, we obtain the well–known relation $C_n = \sum\limits_{k=0}^{n-1} N(n,k)$.

This approach can also be used to derive some binomial identities. As an example, consider $\sigma' = (1, 1, 1, \ldots)$, $\tau' = (2, 1, 1, \ldots)$. The Catalan–like number B'_n is the trinomial number, and we know from section 6(A)

$$B'_n = \frac{1}{2^n}\sum_k \binom{2k}{k}\binom{k}{n-k} 3^{n-k}.$$

Now we want to show that also

$$B'_n = \sum_k \binom{n}{2k}\binom{2k}{k}.$$

To prove this, consider $\sigma \equiv 0$, $\tau = (2, 1, 1, \ldots)$. The associated sequences $\widehat{\sigma}, \widehat{\tau}$ are then by (7.1)

$$\widehat{\sigma} = (2, 2, 2, \ldots), \quad \widehat{\tau} = (2, 1, 1, \ldots),$$

which according to our list in section 6 gives

$$B_{2n} = \widehat{B}_n = \binom{2n}{n}, \quad B_{2n+1} = 0.$$

Applying the binomial formula (6.15) to σ, τ we obtain for $\sigma' = \sigma + 1$, $\tau' = \tau$, precisely

$$B'_n = \sum_k \binom{n}{2k}\binom{2k}{k}.$$

B. Here is a simple but important illustration of the second type of results. Suppose the sequence (B_n) is given by the following recursion:

$$\begin{aligned} B_{2n+2} &= \tfrac{\alpha n+\beta}{\gamma n+\delta} B_{2n}, \quad B_0 = 1, \\ B_{2n+1} &= 0 \quad \text{for all } \; n. \end{aligned} \tag{7.3}$$

Then it can be shown that (B_n) is Catalan–like if and only if for all k

$$\alpha k + \beta \neq 0, \gamma k + \delta \neq 0, \alpha\delta - \beta\gamma + k\alpha\gamma \neq 0.$$

The associated sequences are then $\sigma \equiv 0$, $\tau = (t_k)$ with

$$\begin{aligned} t_{2k} &= k\tfrac{(\alpha\delta-\beta\gamma)+(k-1)\alpha\gamma}{((2k-2)\gamma+\delta)((2k-1)\gamma+\delta)} \; (k \geq 1) \\ t_{2k+1} &= \tfrac{(k\alpha+\beta)((k-1)\gamma+\delta)}{((2k-1)\gamma+\delta)(2k\gamma+\delta)} \; (k \geq 1), \; t_1 = \tfrac{\beta}{\delta}. \end{aligned} \tag{7.4}$$

When $\alpha = 2\beta$, we may combine (7.4) into a single expression

$$t_k = \frac{k\beta(\delta + (k-3)/2\gamma)}{((k-2)\gamma+\delta)((k-1)\gamma+\delta)} \; (k \geq 2), \; t_1 = \frac{\beta}{\delta} \; . \tag{7.5}$$

There are two cases of (7.5) which have been studied intensively [9].

Consider $\alpha = 2$, $\beta = 1$, $\gamma = 2$, $\delta = 2\lambda + 2$ with $\lambda \notin \{0, -1, -2, \ldots, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}$.

Here

$$t_k = \frac{k(k+2\lambda-1)}{(2k+2\lambda-2)(2k+2\lambda)} \quad (k \geq 1) \; .$$

i) For $\lambda \in I\!N$ this gives by (7.3)

$$B_{2n} = \frac{1}{4^n}\frac{\binom{2n}{n}}{\binom{n+\lambda}{n}},$$

and, in particular, for $\lambda = 1$

$$B_{2n} = \frac{1}{4^n} C_n.$$

ii) For $\lambda = \frac{2m-1}{2}$, $m \in I\!N$, this gives

$$B_{2n} = \frac{\binom{m+n}{m}}{\binom{2m+2n}{2m}},$$

and, in particular, for $\lambda = \frac{1}{2}$

$$B_{2n} = \frac{1}{2n+1}.$$

The associated orthogonal polynomial systems are for arbitrary λ the *Gegenbauer polynomials*, and for $\lambda = \frac{1}{2}$ the *Legendre polynomials.*

Another classical case is $\alpha = 2$, $\beta = 2b+2$, $\gamma = 1$, $\delta = a+b+2$ which leads to the *Jacobi polynomials.*

C. Another simple recursion is

$$\begin{aligned} B_{2n+2} &= c^n B_{2n}, \;\; B_0 = 1, \\ B_{2n+1} &= 0\ . \end{aligned}$$

The sequence (B_n) is Catalan–like if and only if $c \neq 1$, in which case we obtain

$$\begin{aligned} t_{2k} &= c^{2k-1} - c^{k-1} \quad && (k \geq 1) \\ t_{2k+1} &= c^{2k} && (k \geq 0) \end{aligned}$$

with $B_{2n} = c^{\binom{n}{2}}$.

D. Let us, finally, look at the convolution recursion

$$\begin{aligned} B_{2n} &= \tfrac{n+1}{2} \sum_k \binom{2n-1}{2k} \tfrac{B_{2k}}{k+1} \tfrac{B_{2n-2-2k}}{n-k}, \;\; B_0 = 1, \\ B_{2n+1} &= 0. \end{aligned}$$

The sequence (B_{2n}) we obtain are the *Genocchi numbers* $B_{2n} = G_{2n}$

$$(G_n) = (1, 0, 1, 0, 3, 0, 17, 0, 155, 0, 2073, \ldots).$$

The associated sequence $\tau = (t_k)$ is given by

$$\begin{array}{lcll} t_{2k} & = & k(k+1) & (k \geq 1) \\ t_{2k+1} & = & (k+1)^2 & (k \geq 0). \end{array}$$

The generating function of the Genocchi numbers is given by

$$G(z) = \sum_{n \geq 0} \frac{G_n z^n}{n!} = (ztg(\frac{z}{2}))''.$$

8. Determinants of Hankel matrices

Let us now discuss the characterization of Catalan–like sequences by means of the determinants of the Hankel matrices H_n and $\widetilde{H}_n$, as introduced in section 1.

Let (B_n) be the Catalan–like sequence associated with $\sigma = (s_k)$, $\tau = (t_k)$, and recursive matrix $A = (a_{n,k})$. We consider the first three Hankel matrices of order $0, 1$ and 2:

$$H_n = (B_{i+j}),\ \widetilde{H}_n = (B_{i+j+1}),\ \widetilde{\widetilde{H}}_n = (B_{i+j+2})$$

for $0 \leq i, j \leq n$.

The key is the fundamental theorem of section 2:

$$ATA^T \ = \ H \tag{8.1}$$

where $T = \begin{pmatrix} T_0 & & & \\ & T_1 & & 0 \\ & 0 & T_2 & \\ & & & \ddots \end{pmatrix}$, $T_n = t_1 \ldots t_n$, $T_0 = 1$, and $H = (B_{i+j})$.

We further consider the infinite Hankel matrices

$$\widetilde{H} = (B_{i+j+1}),\ \widetilde{\widetilde{H}} = (B_{i+j+2}).$$

Denote by ΔA the shifted matrix

$$\Delta A = \begin{pmatrix} a_{10} & a_{11} & a_{12} & \ldots \\ a_{20} & a_{21} & a_{22} & \ldots \\ a_{30} & a_{31} & a_{32} & \ldots \\ & & \ldots & \end{pmatrix}$$

and by J the infinite matrix

$$J = \begin{pmatrix} s_0 & 1 & & & \\ t_1 & s_1 & 1 & & 0 \\ & t_2 & s_2 & 1 & \\ & 0 & \ddots & \ddots & \ddots \end{pmatrix}.$$

The recursion

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}$$

can thus be written in matrix form

$$\Delta A \;=\; A\, J. \tag{8.2}$$

Looking at (8.1) we find

$$(\Delta A) T A^T = \widetilde{H} \tag{8.3}$$

$$(\Delta A) T (\Delta A)^T = \widetilde{\widetilde{H}}\ . \tag{8.4}$$

Now we restrict the matrix equations to the rows and columns up to n. For an infinite matrix $M = (m_{ij})$ we denote by $M_n = (m_{ij})$, $0 \le i,j \le n$, its n–th principal minor.

Since A_n is lower triangular with 1's in the main diagonal, we have $\det A_n = 1$ and obtain with (8.1) our first result already mentioned in section 2:

Result 1. $\det H_n = \Pi_n := T_0 T_1 \ldots T_n$.

Looking at (8.2) we have

$$\det(\Delta A)_n = \det J_n$$

which gives bei (8.3):

Result 2. $\det \widetilde{H}_n = d_n \Pi_n$, *where* $d_n = \det J_n$.

From the set-up of J the d_n's are determined by the recursion

$$\begin{aligned} d_{-1} =&\ 1,\ d_0 = s_0 \\ d_n =&\ s_n d_{n-1} - t_n d_{n-2}\ (n \ge 1). \end{aligned} \tag{8.5}$$

Finally, we look at (8.4). Restricting to the n–th principal minor this gives for $i+j<2n$

$$\sum_{k=0}^{n} a_{i+1,k} T_k a_{j+1,k} = B_{i+j+2},$$

and for $i=j=n$, since the index runs only up to n,

$$\sum_{k=0}^{n} a_{n+1,k} T_k a_{n+1,k} = B_{2n+2} - T_{n+1}.$$

Hence

$$(\Delta A)_n \begin{pmatrix} T_0 & & 0 \\ & \ddots & \\ 0 & & T_n \end{pmatrix} (\Delta A)_n^T = \begin{pmatrix} B_2 & B_3 & \ldots & B_{n+2} \\ & & \ldots & \\ B_{n+2} & B_{n+3} & \ldots & B_{2n+2} - T_{n+1} \end{pmatrix},$$

and thus

$$(\det J_n)^2 \Pi_n = \det \widetilde{\widetilde{\widetilde{H}}}_n - T_{n+1} \det \widetilde{\widetilde{\widetilde{H}}}_{n+1}.$$

Solving this last recursion we obtain with d_n as in (8.5):

Result 3. $$\det \widetilde{\widetilde{\widetilde{H}}}_n = \Pi_{n+1} \sum_{j=-1}^{n} \frac{dj^2}{T_{j+1}}. \tag{8.6}$$

Examples. For $\tau \equiv 1$ we have $T_n = 1$ and $\Pi_n = 1$ for all n, and the formulae read:

$$\det H_n = 1, \;\; \det \widetilde{H}_n = d_n, \;\; \det \widetilde{\widetilde{\widetilde{H}}}_n = \sum_{j=-1}^{n} d_j^2. \tag{8.7}$$

To apply (8.7) we have to solve (8.5):

i) $\sigma = (1,2,2,\ldots)$, $B_n = C_n$. Here (8.5) yields $d_n = 1$ for all n and we have proved our initial characterization: The Catalan numbers C_n are the unique sequence with $\det H_n = \det \widetilde{H}_n = 1$ for all n. Furthermore, $\det \widetilde{\widetilde{\widetilde{H}}}_n = n+2$.

ii) $\sigma \equiv 1$, $B_n = M_n$ (Motzkin). We find (see [1]):

$$\det \widetilde{H}_n = \begin{cases} 1 & n \equiv 0,5 \pmod 6 \\ 0 & n \equiv 1,4 \pmod 6 \\ -1 & n \equiv 2,3 \pmod 6 \end{cases}.$$

Furthermore, $\det \widetilde{\widetilde{H}}_n = \lceil \frac{2n+4}{3} \rceil$.

iii) $\sigma = (0, 2, 2, 2, \ldots)$, $B_n = F_n$ (Fine). Here $\det \widetilde{H}_n = -n$.

iv) $\sigma = (3, 2, 2, \ldots)$, $B_n = \binom{2n+1}{n}$. We obtain $\det \widetilde{H}_n = 2n + 3$.

v) $\sigma = (3, 3, 3, \ldots)$, $B_n =$ hexagonal number. This gives $\det \widetilde{H}_n = f_{2n+4}$ (Fibonacci number).

Let us, finally, look at a few examples with $\tau \not\equiv 1$.

vi) $\sigma = (1, 2, 3, \ldots)$, $\tau = (1, 2, 3, \ldots)$, $B_n =$ Bell number. We have $T_n = n!$, and thus $\Pi_n = n!! := \prod_{k=0}^{n} (k!)$. Recursion (8.5) yields $d_n = 1$ for all n, and we obtain (see [3]):

$$\det H_n = n!!, \ \ \det \widetilde{H}_n = n!!.$$

vii) $\sigma = (1, 3, 5, 7, \ldots)$, $\tau = (1, 4, 9, 16, \ldots)$, $B_n = n!$. Here $T_n = n!^2$, $\Pi_n = (n!!)^2$, and (8.5) gives $d_n = (n + 1)!$. Hence

$$\det H_n = (n!!)^2, \ \ \det \widetilde{H}_n = (n!!)^2 (n + 1)!.$$

This survey would not be complete without a short discussion of the beautiful result of Gessel–Viennot [15] connecting determinants to non–intersecting path systems (see also [20]).

We first state the Gessel–Viennot result and apply it then to our situation. Let $G = (V, E)$ be a directed acyclic graph, where every edge $e \in E$ carries a (real) weight $w(e)$. We extend these weights to arbitrary directed paths P by setting

$$w(P) = \prod_{e \in P} w(e).$$

For $u, v \in V$ let $\mathcal{P}(u, v)$ be the set of directed paths from u to v.

Choose any two sets of $n+1$ vertices $\mathcal{U} = \{u_0, u_1, \ldots, u_n\}$, $\mathcal{V} = \{v_0, v_1, \ldots, v_n\}$, where the sets $\mathcal{U}$ and $\mathcal{V}$ need not be disjoint. A *path system* from $\mathcal{U}$ to $\mathcal{V}$ is a permutation σ of $\{0, 1, \ldots, n\}$ together with $n + 1$ directed paths $P_{0,\sigma(0)} \in \mathcal{P}(u_0, v_{\sigma(0)}), \ldots, P_{n,\sigma(n)} \in \mathcal{P}(u_n, v_{\sigma(n)})$, and the path system is said to be *vertex–disjoint* if two distinct paths never share a vertex.

Now associate to $\mathcal{U}$ and $\mathcal{V}$ the *path matrix* $M = (m_{ij})$ where

$$m_{ij} = \sum_{P \in \mathcal{P}(u_i, v_j)} w(P) \ .$$

Here is the theorem of Gessel–Viennot:

$$\det M = \sum_{\mathcal{VD}} (\text{sign } \sigma) w(P_{0,\sigma(0)}) \dots w(P_{n,\sigma(n)}) \tag{8.8}$$

where the sum is extended over all vertex-disjoint path systems from $\mathcal{U}$ to $\mathcal{V}$.

In our situation we are given as usual $\sigma = (s_k)$, $\tau = (t_k)$ and convert the recursion

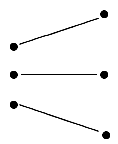
$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}$$

into a weighted graph as in section 4.

In other words, we consider the lattice graph $G = G^{\sigma,\tau} = (V, E)$ in the upper half–plane, that is,

$$V = \{(x, y) : x, y \in \mathbb{Z}, y \geq 0\}$$

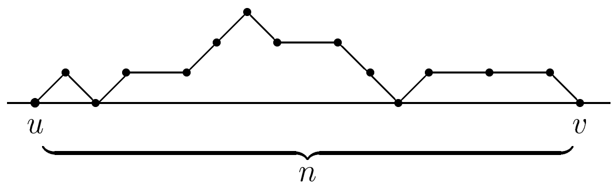
with the edges going east, north–east or south–east. The weights are

E :  weight

	1	
weight	s_k	$y = k$
	t_{k+1}	$y = k+1$ to $y = k$.

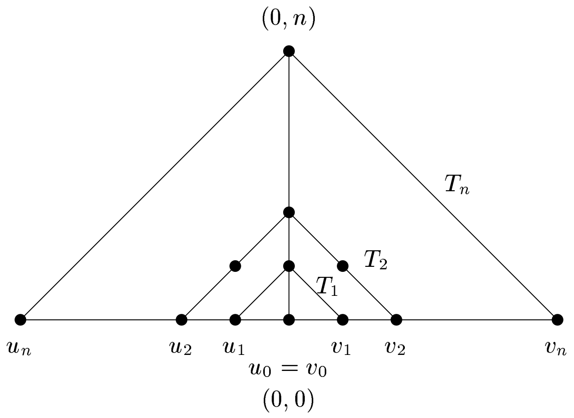
Then as in section 4 we find (see the figure) that for any vertices u, v of G on the x–axis which are n steps apart

$$m(u, v) = \sum_{P \in \mathcal{P}(u,v)} w(P) = B_n.$$



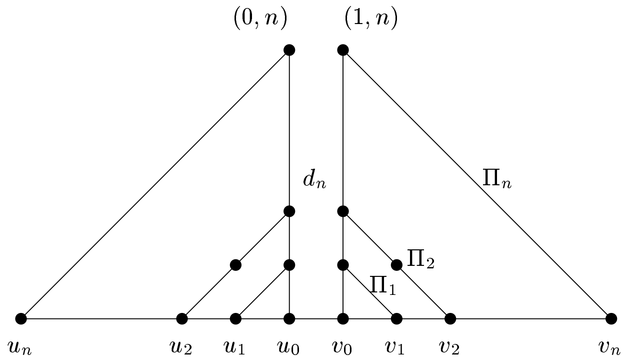
With the theorem of Gessel–Viennot we can now reprove our results on the determinants of H_n, $\widetilde{H}_n$ and $\widetilde{\widetilde{H}}_n$ at one glance (“proof by picture”).

Result 1.



In this case $M = H_n$. There is clearly only one vertex-disjoint system with $\sigma = \mathrm{id}$, as depicted in the figure. Hence $\det H_n = T_0 T_1 \dots T_n = \Pi_n$.

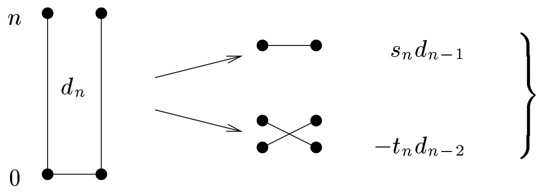
Result 2.



We have $M = \widetilde{H}_n$. The vertex-disjoint systems are forced to take the paths outside the strip between u_0 and v_0 as in result 1. Hence

$$\det \widetilde{H}_n = d_n \Pi_n.$$

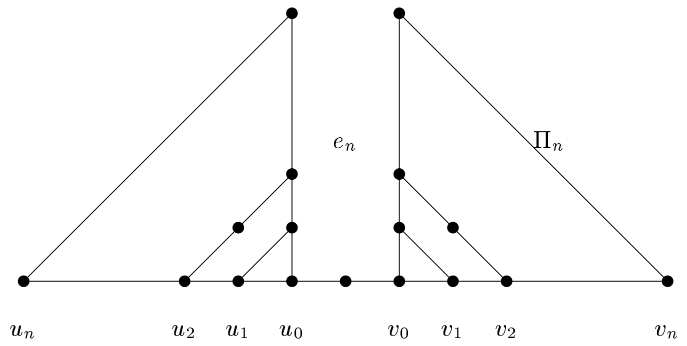
To compute d_n recursively, we classify the systems according to the top as in the following figure, and arrive precisely at the recursion (8.5).



$$d_n = s_n d_{n-1} - t_n d_{n-2}$$

Note that we have sign $\sigma = -1$ in the second case, since the edges cross.

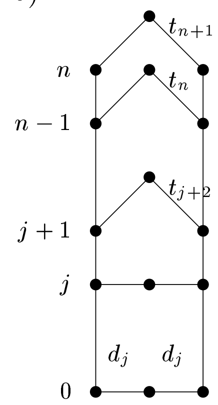
Result 3.



We have $M = \widetilde{\widetilde{H}}_n$, and it again suffices to consider the strip between u_0 and v_0, that is

$$\det \widetilde{\widetilde{H}}_n = e_n \Pi_n.$$

Classifying the vertex-disjoint systems inside the strip as suggested by the following figure, we obtain (8.6).



Indeed,

$$e_n = \sum_{j=-1}^{n} d_j^2 (t_{j+2} \cdots t_{n+1}) = T_{n+1} \sum_{j=-1}^{n} \frac{d_j^2}{T_{j+1}},$$

and thus

$$\det \widetilde{\widetilde{H}}_n = \Pi_{n+1} \sum_{j=-1}^{n} \frac{d_j^2}{T_{j+1}}.$$

9. Ballot Numbers

We come to our last characterization of the sequence (C_n) of Catalan numbers, by means of a *ballot table.* Let us repeat the set–up. Consider the recursive system

$$\begin{aligned} &b_{0,0} = 1, \quad b_{0,k} = 0 \qquad (k > 0) \\ &b_{n,k} = b_{n-1,k-1} + b_{n-1,k} + \ldots + b_{n-1,n-1} \quad (n \geq 1) \ . \end{aligned} \tag{9.1}$$

Then

$$b_{n,0} = C_n \quad \text{and} \quad \sum_{k=0}^{n} b_{n,k} = C_{n+1}. \tag{9.2}$$

(Of course, we still have to prove this.)

Thus the sequence (C_n) can be defined via (9.1). But, on the other hand, the sequences (C_n) and (C_{n+1}) are, as we know, also given via 3–term recursions

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1}, \tag{9.3}$$

namely $\sigma = (1, 2, 2, \ldots)$, $\tau \equiv 1$ resp. $\sigma = (2, 2, 2, \ldots)$, $\tau \equiv 1$.

It is the purpose of this chapter to exhibit some general connections between ballot tables on the one hand and recursive matrices on the other.

We define a *generalized ballot table* as follows: Let $\mu = (m_0, m_1, m_2, \ldots)$ be a sequence of real numbers. We associate to μ the infinite lower triangular matrix $B^\mu = (b_{n,k})$ by setting

$$\begin{aligned} &b_{0,0} = 1, \ b_{0,k} = 0 \quad (k > 0) \\ &b_{n,k} = b_{n-1,k-1} + m_k b_{n-1,k} + b_{n-1,k+1} + \ldots + b_{n-1,n-1} \quad (n \geq 1). \end{aligned} \tag{9.4}$$

B^μ is called the *ballot table* corresponding to μ. We are interested in the *sum* $S_n^\mu = \sum\limits_{k=0}^{n} b_{n,k}$, and the element in column *zero*, $Z_n^\mu = b_{n,0}$.

Hence for our example (still to be proved) $\mu \equiv 1$:

$$S_n^\mu = C_{n+1}, \ Z_n^\mu = C_n,$$

that is, the *sum* corresponds to $\sigma = (2, 2, 2, \ldots)$, $\tau \equiv 1$, and the zero–element to $\sigma = (1, 2, 2, \ldots)$, $\tau \equiv 1$.

Generalizing this example we use the following short notation:

$\mu \stackrel{S}{\longleftrightarrow} {\sigma \atop \tau}$ means that $S_n^\mu = B_n^{\sigma,\tau}$ for all n where S_n^μ and $B_n^{\sigma,\tau}$ are defined according to (9.4) and (9.3),

$\mu \stackrel{Z}{\longleftrightarrow} {\sigma \atop \tau}$ means that $Z_n^\mu = B_n^{\sigma,\tau}$ for all n.

The following theorem is basic [4]: *For any* $\alpha \in I\!R$.

$$\begin{aligned} \text{If} \quad \mu \stackrel{S}{\longleftrightarrow} {\sigma \atop \tau} \quad \text{then} \quad (\alpha,\alpha,\mu) &\stackrel{S}{\longleftrightarrow} {(\alpha+1,\sigma) \atop (1,\tau)} \\ (\alpha,\mu) &\stackrel{Z}{\longleftrightarrow} {(\alpha,\sigma) \atop (1,\tau)} . \end{aligned} \tag{9.5}$$

Iterating (9.5) we obtain as immediate consequences:

$$\mu = (m_0, m_0, m_1, m_1, m_2, m_2, \ldots) \stackrel{S}{\longleftrightarrow} {\sigma = (m_0+1, m_1+1, m_2+1, \ldots) \atop \tau = (1,1,1,\ldots)}$$

and, in particular

$$\mu = (a, a, s, s, s, s, \ldots) \stackrel{S}{\longleftrightarrow} {\sigma = (a+1, s+1, s+1, \ldots) \atop \tau = (1,1,1,\ldots).}$$

Furthermore,

$$\mu = (m_0, m_1, m_1, m_2, m_2, \ldots) \stackrel{Z}{\longleftrightarrow} {\sigma = (m_0, m_1+1, m_2+1, \ldots) \atop \tau = (1,1,1,\ldots)}$$

and, in particular

$$\mu = (a, s, s, s, \ldots) \stackrel{Z}{\longleftrightarrow} {\sigma = (a, s+1, s+1, \ldots) \atop \tau = (1,1,1,\ldots)\ .}$$

Looking at our list of Catalan–like numbers in section 6(A), we arrive at the following interesting cases, where τ is always $\tau \equiv 1$.

$$\begin{array}{llll}
\mu = (1,1,1,\ldots) \stackrel{S}{\longleftrightarrow} & \sigma = (2,2,2,\ldots) & S_n = C_{n+1} & \text{(Catalan)} \\
\mu = (0,0,0,\ldots) \stackrel{S}{\longleftrightarrow} & \sigma = (1,1,1,\ldots) & S_n = M_n & \text{(Motzkin)} \\
\mu = (1,1,1,\ldots) \stackrel{Z}{\longleftrightarrow} & \sigma = (1,2,2,\ldots) & Z_n = C_n & \text{(Catalan)} \\
\mu = (0,0,0,\ldots) \stackrel{Z}{\longleftrightarrow} & \sigma = (0,1,1,\ldots) & Z_n = R_n & \text{(Riordan)} \\
 & & R_n = S_{n-1} - Z_{n-1} & \\
\mu = (0,1,1,\ldots) \stackrel{Z}{\longleftrightarrow} & \sigma = (0,2,2,\ldots) & Z_n = F_n & \text{(Fine)} \\
 & & F_n = S_{n-1} - Z_{n-1} & \\
\mu = (1,0,0,\ldots) \stackrel{Z}{\longleftrightarrow} & \sigma = (1,1,1,\ldots) & Z_n = M_n & \text{(Motzkin)} \\
 & & M_n = S_{n-1}\ . &
\end{array}$$

In particular, this finally proves our claim for $\mu \equiv 1$: $S_n = C_{n+1}$ and $Z_n = C_n$.

This last table, containing the Catalan, Motzkin, Riordan and Fine numbers, partly explans the ubiquity of these numbers in counting problems via the following approach.

Consider the usual ballot recursion (9.1)

$$b_{n,k} = b_{n-1,k-1} + b_{n-1,k} + \ldots + b_{n-1,n-1}.$$

Suppose we are given sets $\mathcal{S}_{n,k}$ depending on two parameters n and k, $0 \leq k \leq n$ with $\mathcal{S}_n = \bigcup\limits_{k=0}^{n} \mathcal{S}_{n,k}$, $|\mathcal{S}_0| = 1$. Assume that the following holds for all $n \geq 1$:

(A) There are maps $\varphi_{i,j} : \mathcal{S}_{n-1,i} \longrightarrow \mathcal{S}_{n,j}$ for all pairs (i,j) with $0 \leq j \leq i+1 \leq n$.

(B) Any $P \in \mathcal{S}_{n,k}$ is the image under exactly one map $\varphi_{i,k}$; let us denote this pre–image by ψP.

Then, clearly, the numbers $b_{n,k} = |\mathcal{S}_{n,k}|$ satisfy recursion (9.1), and hence $|\mathcal{S}_n| = C_{n+1}$.

Now we obtain further interpretations if the ballot sequence $\mu = (m_k)$ consists of 0's and 1's only, because $m_k = 1$ resp. 0 simply corresponds to the fact that the map $\varphi_{k,k} : \mathcal{S}_{n-1,k} \longrightarrow \mathcal{S}_{n,k}$ is allowed resp. forbidden. But according to our list, these sequences μ lead precisely to our four number sequences (C_n), (M_n), (R_n) and (F_n).

For convenience, we list again the first few numbers:

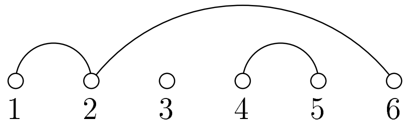
$$\begin{array}{rcl} C_n & = & (1,1,2,5,14,42,132,\ldots) \\ M_n & = & (1,1,2,4,9,21,51,\ldots) \\ R_n & = & (1,0,1,1,3,6,15,\ldots) \\ F_n & = & (1,0,1,2,6,18,57,\ldots)\ . \end{array}$$

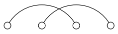
Let us illustrate this "ballot approach" to enumeration problems with three classical examples [28]. We just give the sets $\mathcal{S}_{n,k}$ and the maps $\varphi_{i,j}$ and ψ; the proofs are then straightforward.

Examples

1. Non–crossing partitions

Let $\mathcal{S}_n$ be the set of non–crossing set partitions of $\{1, 2, \ldots, n+1\}$, where we draw a partition on a horizontal line, with arcs (above the line) connecting the elements in a block. For example

$126 - 3 - 45$ corresponds to 

Non–crossing means that crossing arcs  are forbidden.

For a partition P we denote by P_i the block containing i. Now we set

$$\mathcal{S}_{n,k} = \{P \in \mathcal{S}_n : |P_{n+1}| = k + 1\}.$$

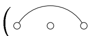
(A) The map $\varphi_{i,j} : \mathcal{S}_{n-1,i} \longrightarrow \mathcal{S}_{n,j}$ is defined as follows. Let $P \in \mathcal{S}_{n-1,i}$. Take the j smallest elements in P_n and merge them with $n+1$ to form a new block. This gives the partition $\varphi_{i,j}(P) \in \mathcal{S}_{n,j}$. For example

$$\varphi_{2,1}(126 - 3 - 45) = 26 - 3 - 45 - 17.$$

(B) Let $P \in \mathcal{S}_{n,k}$. Delete $n+1$ and merge $P_{n+1} \setminus n+1$ with P_n; this gives $\psi\pi \in \mathcal{S}_{n-1}$. For example,

$$\psi(126 - 3 - 45) = 3 - 1245.$$

Interpretating the various μ leading to our counting sequences we obtain the following well–known results:

$$\begin{array}{lcl}
C_n & = & \#\text{ non–crossing partitions of } \{1,2,\ldots,n\} \\
M_{n-1} & = & \#\text{ non–crossing partitions of } \{1,2,\ldots,n\} \\
 & & \quad \text{without singleton blocks except possi-} \\
 & & \quad \text{bly } \{n\} \\
R_n & = & \#\text{ non–crossing partitions of } \{1,2,\ldots,n\} \\
 & & \quad \text{without singleton blocks} \\
F_n & = & \#\text{ non–crossing partitions of } \{1,2,\ldots,n\} \\
 & & \quad \text{without singleton seen from above} \\
M_n & = & \#\text{ non–crossing partitions of } \{1,2,\ldots,n\} \\
 & & \quad \text{without singletons seen from below but} \\
 & & \quad \text{not from above (} \end{array}$$  is forbidden).

As an example, we list the $F_4 = 6$ partitions of $\{1, 2, 3, 4\}$ without singletons seen from above:

2. Permutations

Let $\mathcal{S}_n$ be the set of permutations $\pi = \pi_1 \ldots \pi_{n+1}$ of $\{1, 2, \ldots, n+1\}$ with no subsequence $\pi_i \pi_j \pi_k$ $(i < j < k)$ such that $\pi_j < \pi_k < \pi_i$. We say shortly that the *pattern* 312 is forbidden. Set $\mathcal{S}_{n,k} = \{\pi \in \mathcal{S}_n : \pi_1 = k + 1\}$. The maps are then:

(A) Let $\pi_1 = i + 1, \pi_2 \pi_3 \ldots \pi_n \in \mathcal{S}_{n-1,i}$, then $\pi' = \pi'_1 \ldots \pi'_{n+1} = \varphi_{i,j}(\pi)$ is given as follows: $\pi'_1 = j + 1$, and for $k \geq 2$ we have $\pi'_k = \pi_{k-1}$ if $\pi_{k-1} < j + 1$ resp. $\pi'_k = \pi_{k-1} + 1$ if $\pi_{k-1} \geq j + 1$.

(B) For $\pi = k + 1, \pi_2 \ldots \pi_{n+1} \in \mathcal{S}_{n,k}$, set $\psi\pi = \pi'_1 \ldots \pi'_n \in \mathcal{S}_{n-1}$ where $\pi'_i = \pi_{i+1}$ if $\pi_{i+1} < k + 1$ resp. $\pi'_i = \pi_{i+1} - 1$ if $\pi_{i+1} > k + 1$.

This gives

C_n	$=$	# permutations of $\{1, \ldots, n\}$ without a 312–pattern
M_{n-1}	$=$	# permutations of $\{1, \ldots, n\}$ without a 312–pattern such that if $\pi_i < \pi_{i+1}$, then there exists $j > i + 1$ with $\pi_i < \pi_j < \pi_{i+1}$
R_n	$=$	# permutations of $\{1, \ldots, n\}$ as in the previous line and with $\pi_1 \geq 2$
F_n	$=$	# permutations of $\{1, \ldots, n\}$ without a 312–pattern and without a splitting element π_k, meaning $\pi_k = k$ and $\pi_i < \pi_k$ $(i < k)$ and $\pi_i > \pi_k$ $(i > k)$.

Let us list the $F_4 = 6$ permutations with this last property:

2143, 2341, 2431, 3241, 3421, 4321

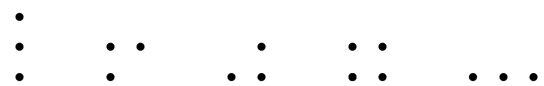
3. Parallelogram Animals

This example has been studied in many papers, e.g. [8, 10, 11, 17]. An animal P consists of rows $r_0, r_1, \ldots, r_t$ of lattice points such that

a) the points in row i have integral coordinates $(f_i, i), (f_i+1, i), \ldots, (\ell_i, i)$ with $f_0 = 0$

b) $f_i \leq f_{i+1} \leq \ell_i \leq \ell_{i+1}$ for all i.

The number $n = \ell_t + t$ is called the *site perimeter* of P. The figure shows the five animals with site perimeter 2:



Let $\mathcal{S}_n$ be the set of parallelogram animals with site perimeters n, and

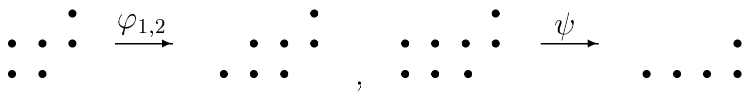
$$\mathcal{S}_{n,k} = \{P \in \mathcal{S}_n : \ell_0(P) = k\}$$

The maps $\varphi_{i,j}$ and ψ are given as follows:

(A) Let $P \in \mathcal{S}_{n-1,i}$. If $j = i+1$, add a point to the bottom line, moving P one step to the right. If $j \leq i$, add a new bottom row of $j+1$ points, moving P one step up.

(B) Let $P \in \mathcal{S}_{n,k}$. If $f_1 > 0$, delete the left–most point on the bottom line. If $f_1 = 0$, delete the bottom line.

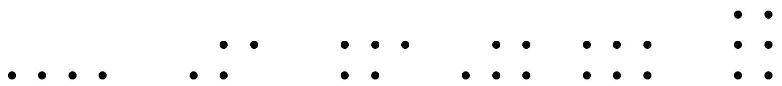
The figure shows the operations:



We obtain

C_{n+1}	$=$	# animals with site perimeter n
M_n	$=$	# animals with site perimeter n and $\ell_0 < \ell_1 < \ldots < \ell_t$ ("no walls")
R_{n+1}	$=$	# animals with site perimeter n, without walls and singleton rows
F_{n+1}	$=$	# animals with site perimeter n without singleton rows
M_{n+1}	$=$	# animals with site perimeter n, without walls, except singleton rows.

The figure shows the $F_4 = 6$ corresponding animals with $n = 3$:



There are numerous *weighted* versions (see [4, 8]). Let us discuss just one example for parallelogram animals. Consider the set $\mathcal{S}_n$ of animals with site perimeter n. For $P \in \mathcal{S}_n$ set

$$\begin{array}{rcll} b(P) & = & \ell_0 & \text{(base)} \\ w(P) & = & \#\{i : \ell_i = \ell_{i+1}\} & \text{(wall)} \\ h(P) & = & t & \text{(height)} \end{array}$$

and define the *weight* $W(P)$ of P

$$W(P) = \alpha^{b(P)} \beta^{w(P)} \gamma^{h(P)}.$$

Then the following can be proved [4]:

$$\sum_{P \in \mathcal{S}_n} W(P) = B_n^{\sigma,\tau} \;\; \textit{where} \;\; \begin{array}{l} \sigma = (\alpha + \beta\gamma, 1 + \beta\gamma, 1 + \beta\gamma, \ldots) \\ \tau = (\gamma, \gamma, \gamma, \ldots) \; . \end{array}$$

In particular, for $\alpha = 0$, $\beta = 1$, we obtain

$$\sum_{P \in \mathcal{S}_n, b(P)=0} \gamma^{h(P)} = B_n^{\sigma,\tau} \text{ with } \begin{array}{l} \sigma = (\gamma, \gamma + 1, \gamma + 1, \ldots) \\ \tau = (\gamma, \gamma, \gamma, \ldots). \end{array}$$

These sequences σ, τ have already appeared in section 7, and we find the Narayana polynomial

$$B_n^{\sigma,\tau} = \sum_{k=0}^{n-1} N(n,k) \gamma^{n-k},$$

and with $b(P) = 0$

$$\sum_{P \in \mathcal{S}_n} \gamma^{h(P)} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \binom{n+1}{k+1} \gamma^k.$$

This gives the famous result (see e.g. [8]) that the number of animals which fit within a rectangle dimension $a \times b$ points (but not a smaller one) equals

$$\frac{1}{a+b-1}\binom{a+b-1}{a}\binom{a+b-1}{b}.$$

For $a = 2$, $b = 3$ we obtain the following six animals:

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