NARROW OPERATORS AND RICH SUBSPACES OF BANACH SPACES WITH THE DAUGAVET PROPERTY

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Abstract. Let X be a Banach space. We introduce a formal approach which seems to be useful in the study of those properties of operators on X which depend only on the norms of the images of elements. This approach is applied to the Daugavet equation for norms of operators; in particular we develop a general theory of narrow operators and rich subspaces of spaces X with the Daugavet property previously studied in the context of the classical spaces $C(K)$ and $L_1(\mu)$.

1. Introduction

Following [13] we say that a Banach space X has the *Daugavet property* if for every operator $T\colon X \to X$ of rank 1 the *Daugavet equation*

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{1.1}$$

is fulfilled. It is known that then every weakly compact operator, even every strong Radon-Nikodým operator, and every operator not fixing a copy of ℓ_1 satisfies (1.1) as well ([13], [21]). Incidentally, this shows that our definition of the Daugavet property is equivalent to the ones which have been proposed in [1] and [11]. Classical results due to Daugavet [5], Lozanovskii [15], and Foiaş, Singer and Pełczyński [7] state that $C(K)$, $L_1(\mu)$ and $L_\infty(\mu)$ have the Daugavet property provided that K is perfect and μ is non-atomic. Recently, corresponding results in the non-commutative setting were obtained by Oikhberg [16].

The papers [13] and [21] study Banach spaces with the Daugavet property from a structural point of view; for example it is shown that such a space never embeds into a space having an unconditional basis, and it contains (many) subspaces isomorphic to ℓ_1. Also, hereditary properties of the Daugavet property are established there.

Returning to the classical spaces $C(K)$ and $L_1(\mu)$ we mention that a different approach to (1.1) on these spaces was launched earlier in [12] and [17]. These papers study a duality between certain operators, called *narrow operators*, and certain subspaces, called *rich subspaces*, of such spaces. (For

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the definitions, which differ in the two cases, see Section 2.) One of the key features of this approach is that the concept of a narrow operator on $C(K)$ or $L_1(\mu)$, which makes sense for operators from these spaces into an arbitrary range space, only depends on the values $\|Tx\|$, but not on the images Tx themselves.

The idea of the present paper is to introduce narrow operators and rich subspaces in general. In Section 2 we propose a formalism in order to deal with those properties of an operator which depend only on the norms of the images of elements. We define corresponding equivalence classes and their formal sums and differences, which is reminiscent of certain procedures in the theory of operator ideals. Then, in Section 3 we introduce and study narrow operators on Banach spaces with the Daugavet property. We show, in particular, that strong Radon-Nikodým operators are narrow and that narrow operators mapping X to itself satisfy (1.1). In Section 4 we prove that operators not fixing a copy of ℓ_1 are narrow, thus extending a result from [21]. To do so we need an extension of a theorem due to Rosenthal characterising separable Banach spaces that fail to contain isomorphic copies of ℓ_1 (Theorem 4.3), which seems to be of independent interest.

Section 5 deals with rich subspaces. As in the classical case of $C(K)$ or $L_1(\mu)$, a closed subspace $Y \subset X$ is called rich if the quotient map $q\colon X \to X/Y$ is narrow. One of the main results here is that the Daugavet property passes to rich subspaces, which leads to new hereditary properties. We also study a related class of subspaces which we term wealthy. What looks like a quibble of words is another main result from Section 5: a subspace is rich if and only if it is wealthy. In fact, we also need to deal with a slightly more general class of operators called strong Daugavet operators. It turns out that there are strong Daugavet operators which are not narrow; an example to this effect is presented in Section 6.

As for notation, we denote the closed unit ball of a Banach space by $B(X)$ and its unit sphere by $S(X)$. The slice of $B(X)$ determined by a functional $x^* \in S(X^*)$ and $\varepsilon > 0$ is the set

$$S(x^*, \varepsilon) = \{x \in B(X)\colon x^*(x) \geq 1 - \varepsilon\}.$$

We shall repeatedly make use of the following characterisation of the Daugavet property in terms of slices or weakly open sets from [13, Lemma 2.2] and [21, Lemma 2.2] respectively.

Lemma 1.1. *The following assertions are equivalent:*

(i) *X has the Daugavet property.*

(ii) *For every $x \in S(X)$, $\varepsilon > 0$ and every slice S of $B(X)$ there exists some $y \in S$ such that $\|x + y\| > 2 - \varepsilon$.*

(iii) *For every $x \in S(X)$, $\varepsilon > 0$ and every nonvoid relatively weakly open subset U of $B(X)$ there exists some $y \in U$ such that $\|x+y\| > 2-\varepsilon$.*

Actually, this lemma characterises *Daugavet pairs* (Y, X), meaning a Banach space X and a subspace $Y \subset X$ such that

$$\|J + T\| = 1 + \|T\|$$

for every operator from Y into X of rank 1; here J denotes the canonical embedding map. The only modification to be made in the formulation of Lemma 1.1 is that S and U refer to slices and subsets of $B(Y)$.

Finally we mention that all the Banach spaces in this paper are tacitly assumed to be real.

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2. The semigroup $\mathscr{OP}(X)$

Throughout the paper the symbol X will be used for a fixed Banach space, the symbols T, T_i etc. for bounded linear operators, acting from X to some other Banach space (not necessarily the same one).

Definition 2.1. We say that two operators T_1 and T_2 are *equivalent* (in symbols $T_1 \sim T_2$) if $\|T_1 x\| = \|T_2 x\|$ for every $x \in X$. A class $\mathscr{M}$ of operators is said to be *admissible* if for every $T \in \mathscr{M}$ all the members of the equivalence class of T also belong to $\mathscr{M}$.

In other words, the operators T_1 and T_2 are equivalent if there is an isometry $U\colon T_1(X) \to T_2(X)$ such that $T_2 = UT_1$. For example, the classes of finite-rank operators, compact operators, weakly compact operators, operators bounded from below are admissible; surjections, isomorphisms, projections are examples of non-admissible operator classes.

Definition 2.2. We say that $T_1 \le T_2$ if $\|T_1 x\| \le \|T_2 x\|$ for every $x \in X$. A class $\mathscr{M}$ of operators forms an *order ideal* if for every $T \in \mathscr{M}$ every operator $T_1 \le T$ also belongs to $\mathscr{M}$.

In other words, $T_1 \le T_2$ if there is a bounded operator $U\colon T_2(X) \to T_1(X)$ of norm ≤ 1 such that $T_1 = UT_2$. Order ideals are clearly admissible. The classes of finite-rank operators, compact operators, weakly compact operators are order ideals.

Definition 2.3. A sequence (T_n) of operators is said to be $\sim$*convergent* to an operator T if $\|T_n x\| \to \|Tx\|$ uniformly on $B(X)$. In terms of $\sim$convergence we define the notions of a $\sim$closed set of operators, $\sim$closure, etc. in a natural way.

Of course, the $\sim$limit of a sequence is not unique, but it is unique up to equivalence of operators.

For example, the class $\mathscr{F}(X)$ of finite-rank operators on an infinite-dimensional space X is not $\sim$closed: its $\sim$closure contains all compact operators. Indeed, let $T\colon X \to Y$ be compact. Then, for the canonical isometry U from Y into $C(B(Y^*))$, $T_1 := UT$ is compact, too, and by definition $T_1 \sim T$. Since $C(B(Y^*))$ has the approximation property, T_1 can be approximated by finite-rank operators in the above sense.

In fact, the $\sim$closure of $\mathscr{F}(X)$ coincides with the class $\mathscr{C}(X)$ of all compact operators since $\mathscr{C}(X)$ is $\sim$closed. To see this suppose that (T_n) is a $\sim$convergent sequence of compact operators on X with limit T. Let (x_n) be a bounded sequence in X; using a diagonal procedure one can find a subsequence (x'_n) such that $(T_k x'_n)_n$ is convergent for each k. But $\|T_k x\| \to \|Tx\|$ uniformly on bounded sets as $k \to \infty$; hence (Tx'_n) is a Cauchy sequence and thus convergent.

Definition 2.4. Let $\mathscr{N}$ be a collection of subsets in X. We define a class of operators $\mathscr{N}^\sim$ as follows: $T \in \mathscr{N}^\sim$ if for every $A \in \mathscr{N}$, T is unbounded from below on A; i.e.,

$$\forall \varepsilon > 0\ \exists x \in A\colon\ \|Tx\| \le \varepsilon.$$

Evidently, $\mathscr{N}^\sim$ is a $\sim$closed order ideal, and it is *homogeneous* in the sense that $\lambda T \in \mathscr{N}^\sim$ whenever $\lambda \in \mathbb{R}$ and $T \in \mathscr{N}^\sim$. For example, if $\mathscr{N} = \{S(X)\}$, then $\mathscr{N}^\sim = \mathscr{UB}(X)$, the class of operators that are *unbounded from below* which is defined by

$$T \in \mathscr{UB}(X) \iff \inf\{\|Tx\|\colon \|x\| = 1\} = 0.$$

A significant example for us is the class of all *C-narrow operators* on the space $C(K)$. This class was introduced in [12] as the class of those operators $T\colon C(K) \to Y$ which are unbounded from below on the unit sphere of each subspace $J_F := \{f \in C(K)\colon f|_F = 0\}$, where F is a proper closed subset of K. To put it another way, if $\mathscr{N}$ denotes the collection of these unit spheres, then the class of C-narrow operators is just $\mathscr{N}^\sim$.

Another important example is the class of all *L_1-narrow operators* on the space $L_1 = L_1(\Omega, \Sigma, \mu)$. An operator $T\colon L_1 \to Y$ is called L_1-narrow if for each $B \in \Sigma$ and each $\varepsilon > 0$ there is a function vanishing off B and taking only the values -1 and 1 on B such that $\|Tf\| \le \varepsilon$. In other words, if $\mathscr{N}$ now denotes the collection of these sets of functions, then the class of L_1-narrow operators is again $\mathscr{N}^\sim$. L_1-narrow operators were formally introduced in [17], but the complement of this class was studied earlier by Ghoussoub and Rosenthal who called non-L_1-narrow operators *norm-sign-preserving*. An operator is not L_1-narrow if and only if it is not a sign-embedding on any $L_1(B)$-subspace ([9], [19], [20]).

We caution the reader that in [12] and [17] only the term "narrow" is used. In this paper we prefer to speak of C- and L_1-narrow operators in order not to mix up these notions with our concept of a narrow operator in Section 3; cf., however, Theorem 3.7.

We now define $\mathscr{OP}(X)$ as the class of all operators on X with the convention that equivalent operators will be identified. Hence $\mathscr{OP}(X)$ is actually a collection of equivalence classes, and in fact it is a set. Namely, for an operator T on X its equivalent class can be identified with the seminorm $x \mapsto \|Tx\|$, and the collection of seminorms on X is clearly a set. Thus, admissible families of operators can be identified with subsets of $\mathscr{OP}(X)$, and it makes sense to write $T \in \mathscr{OP}(X)$ or $\mathscr{M} \subset \mathscr{OP}(X)$.

We now introduce addition and subtraction on $\mathscr{OP}(X)$. If $T_1\colon X \to Y_1$ and $T_2\colon X \to Y_2$ are two operators, define

$$T_1 \tilde{+} T_2\colon X \to Y_1 \oplus_1 Y_2, \quad x \mapsto (T_1 x, T_2 x);$$

i.e.,

$$\|(T_1 \tilde{+} T_2)x\| = \|T_1 x\| + \|T_2 x\|.$$

Definition 2.5. If $\mathscr{M}_1, \mathscr{M}_2 \subset \mathscr{OP}(X)$ are non-empty, then their $\sim$*sum* is defined by $\mathscr{M}_1 \tilde{+} \mathscr{M}_2 = \{T_1 \tilde{+} T_2\colon T_1 \in \mathscr{M}_1,\ T_2 \in \mathscr{M}_2\}$. Their $\sim$*difference* is defined by $\mathscr{M}_2 \tilde{-} \mathscr{M}_1 = \{T \in \mathscr{OP}(X)\colon T \tilde{+} T_1 \in \mathscr{M}_2 \text{ whenever } T_1 \in \mathscr{M}_1\}$.

The operation $\tilde{+}$ is a commutative and associative operation on $\mathscr{OP}(X)$, and we have $0 \in \mathscr{M}_2 \tilde{-} \mathscr{M}_1$ if and only if $\mathscr{M}_1 \subset \mathscr{M}_2$.

Let us give some examples.

Example 2.6. Let K be a compact Hausdorff space and let $\mathscr{MUB}(C(K))$ denote the class of operators equivalent to some multiplication operator M_h: $f \mapsto hf$ on $C(K)$ which is unbounded from below; i.e., where h has a zero. Then $\mathscr{UB}(C(K)) \tilde{-} \mathscr{MUB}(C(K))$ consists exactly of the C-narrow operators described above.

Proof. Let $T\colon C(K) \to Y$ be C-narrow. If h has a zero, we have to show that, given $\varepsilon > 0$, there is some $f \in S(C(K))$ such that both $\|Tf\| \le \varepsilon$ and $\|hf\|_\infty \le \varepsilon$. Now, if $F = \{|h| \ge \varepsilon\}$, which is a proper subset of K, and $f \in S(J_F)$ such that $\|Tf\| \le \varepsilon$, then $\|hf\|_\infty \le \varepsilon$ as well.

Conversely, if a closed proper subset $F \subset K$ is given, pick some $h \in S(C(K))$ such that $h = 1$ on F, $h = 0$ off a neighbourhood V of F. If $\|f\|_\infty \le 1$, $\|Tf\| \le \varepsilon$ and $\|hf\|_\infty \le \varepsilon$, then in particular $|f| \le \varepsilon$ on F. Hence it is possible to replace f by a function $g \in S(J_F)$ such that $\|Tg\| \le 2\varepsilon$, which proves that T is C-narrow. $\square$

For our next example recall that an operator on X is a left semi-Fredholm operator if its kernel is finite-dimensional and its range is closed, and it is strictly singular if it is unbounded from below on (the unit sphere of) each infinite-dimensional subspace of X.

Example 2.7. The class $\mathscr{UB}(X) \tilde{-} \mathscr{F}(X)$ consists of all operators that are not left semi-Fredholm operators; $\mathscr{UB}(X) \tilde{-} (\mathscr{UB}(X) \tilde{-} \mathscr{F}(X))$ consists of all strictly singular operators.

Proof. Let us denote $\mathscr{G}(X) = \mathscr{UB}(X) \tilde{-} \mathscr{F}(X)$ and $\mathscr{H}(X) = \mathscr{UB}(X) \tilde{-} \mathscr{G}(X)$.

If T is a left semi-Fredholm operator, then, since $\ker T$ is complemented by a finite-codimensional subspace $Y \subset X$, $T|_Y$ is bounded from below, because T acts as an isomorphism from Y onto $T(X)$. On the other hand, if

$T|_Y$ is bounded from below on some finite-codimensional subspace $Y \subset X$, then $T(Y)$ and, consequently, $T(X)$ must be closed, and $\ker T$ is finite-dimensional, since otherwise $Y \cap \ker T \neq \{0\}$. This shows that T is not a left semi-Fredholm operator if and only if

(a) $T|_Y$ is not bounded from below on any finite-codimensional subspace $Y \subset X$.

Now, if T satisfies (a), $F \in \mathscr{F}(X)$ and $Y = \ker F$, then $T \tilde{+} F \in \mathscr{UB}(X)$, i.e., $T \in \mathscr{G}(X)$. Conversely, if $T \in \mathscr{G}(X)$, $Y \subset X$ is finite-codimensional and q: $X \to X/Y$ is the quotient map, then, since $T \tilde{+} q \in \mathscr{UB}(X)$, T satisfies (a).

Thus, we have shown the announced characterisation of $\mathscr{G}(X)$ and, moreover, we have shown that (a) provides another characterisation of $\mathscr{G}(X)$. It follows from (a) that $T \in \mathscr{G}(X)$ if and only if

(b) for every $\varepsilon > 0$ there exists an infinite-dimensional subspace $Z \subset X$ such that $\|T|_Z\| \leq \varepsilon$;

see [14, Prop. 2.c.4].

From (b) it is clear that every strictly singular operator belongs to $\mathscr{H}(X)$. On the other hand, if S is not strictly singular and is bounded from below on some infinite-dimensional subspace Z, then we have for the quotient map q: $X \to X/Z$ that $S \tilde{+} q$ is bounded from below. Since q obviously satisfies (b), this shows that $S \notin \mathscr{H}(X)$. ☐

Let us list some elementary properties of the operation $\tilde{-}$ that follow directly from the definition.

Proposition 2.8. *Suppose that $\mathscr{M}_1, \mathscr{M}_2 \subset \mathscr{OP}(X)$ contain the zero operator.*

(a) *$\mathscr{M}_2 \tilde{-} \mathscr{M}_1$ is an order ideal or $\sim$closed whenever $\mathscr{M}_2$ is.*

(b) *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are order ideals, then $\mathscr{M}_2 \tilde{-} \mathscr{M}_1$ is homogeneous whenever $\mathscr{M}_2$ is.*

Of particular relevance are subsets of $\mathscr{OP}(X)$ that are semigroups with respect to the operation $\tilde{+}$.

Proposition 2.9. *Suppose that $\mathscr{M}_1, \mathscr{M}_2 \subset \mathscr{OP}(X)$ contain the zero operator.*

(a) *$\mathscr{M}_1$ is a subsemigroup of $\mathscr{OP}(X)$ if and only if $\mathscr{M}_1 \tilde{-} \mathscr{M}_1 \supset \mathscr{M}_1$, in which case $\mathscr{M}_1 \tilde{-} \mathscr{M}_1 = \mathscr{M}_1$.*

(b) *Let $\mathscr{M}_1$ be a subsemigroup of $\mathscr{OP}(X)$, and let $\mathscr{M}_1 \subset \mathscr{M}_2$. Then $\mathscr{M}_2 \tilde{-} (\mathscr{M}_2 \tilde{-} \mathscr{M}_1)$ is again a subsemigroup.*

(c) *$\mathscr{M}_2 \tilde{-} \mathscr{M}_2$ is always a subsemigroup of $\mathscr{OP}(X)$.*

Proof. (a) is clear from the definition.

For (b) we note first that

$$\mathscr{M}_2 \tilde{-} (\mathscr{M}_2 \tilde{-} (\mathscr{M}_2 - \mathscr{M}_1)) = \mathscr{M}_2 - \mathscr{M}_1. \tag{2.1}$$

Indeed, by definition of $\tilde{-}$ we have

$$\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1) \supset \mathscr{M}_1, \tag{2.2}$$

whence

$$\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1)) \subset \mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1.$$

On the other hand, an application of (2.2) with $\mathscr{M}_1$ replaced by $\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1$ gives "$\supset$" in (2.1). Now, by elementary arithmetic involving $\tilde{+}$ and $\tilde{-}$ we have, writing $\mathscr{D} = \mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1$ for short,

$$\begin{aligned}
(\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}) \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}) &= \mathscr{M}_2 \,\tilde{-}\, (\mathscr{D} \,\tilde{+}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}))\\
&= \mathscr{M}_2 \,\tilde{-}\, ((\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}) \,\tilde{+}\, \mathscr{D})\\
&= (\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{D})) \,\tilde{-}\, \mathscr{D}\\
&= \mathscr{D} \,\tilde{-}\, \mathscr{D} \qquad \text{(by (2.1))}\\
&= (\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1) \,\tilde{-}\, \mathscr{D}\\
&= \mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_1 \,\tilde{+}\, \mathscr{D}).
\end{aligned}$$

Because $\mathscr{M}_1$ is a semigroup, one can easily deduce that $\mathscr{M}_1 \tilde{+} \mathscr{D} \subset \mathscr{D}$; indeed,

$$\begin{aligned}
\mathscr{M}_1 \,\tilde{+}\, \mathscr{D} &= (\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1) \,\tilde{+}\, \mathscr{M}_1\\
&= (\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_1 \,\tilde{+}\, \mathscr{M}_1)) \,\tilde{+}\, \mathscr{M}_1\\
&= ((\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1) \,\tilde{-}\, \mathscr{M}_1) \,\tilde{+}\, \mathscr{M}_1\\
&\subset \mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1.
\end{aligned}$$

Therefore

$$(\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}) \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{D}) \supset \mathscr{M}_2 \,\tilde{-}\, \mathscr{D},$$

completing the proof that $\mathscr{M}_2 \,\tilde{-}\, (\mathscr{M}_2 \,\tilde{-}\, \mathscr{M}_1)$ is a semigroup.

Finally, (c) is the special case $\mathscr{M}_1 = \{0\}$ of (b). □

The following definition is important for our abstract semigroup approach.

Definition 2.10. Let $\mathscr{M} \subset \mathscr{OP}(X)$, and let $\mathscr{M}_1 \subset \mathscr{M}$ be a subsemigroup of $\mathscr{OP}(X)$. $\mathscr{M}_1$ is called a *maximal subsemigroup* of $\mathscr{M}$ if every subsemigroup $\mathscr{M}_2 \subset \mathscr{M}$ which includes $\mathscr{M}_1$ coincides with $\mathscr{M}_1$. We call the intersection of all maximal subsemigroups of $\mathscr{M}$ the *central part* of $\mathscr{M}$ and denote it by $\mathrm{cp}(\mathscr{M})$.

Here is a characterisation of the central part of $\mathscr{M}$.

Theorem 2.11. *Let $\mathscr{M} \subset \mathscr{OP}(X)$ have the following properties: $0 \in \mathscr{M}$ and every element of $\mathscr{M}$ is contained in a subsemigroup of $\mathscr{M}$ (this happens for example if $\mathscr{M}$ is homogeneous). Then $\mathrm{cp}(\mathscr{M}) = \mathscr{M} \,\tilde{-}\, \mathscr{M}$.*

Proof. Let $\mathscr{M}_1$ be a maximal subsemigroup of $\mathscr{M}$. Put $\mathscr{M}_2 = \mathscr{M} \,\tilde{-}\, \mathscr{M}$. We have proved above in Proposition 2.9(c) that $\mathscr{M}_2$ is a subsemigroup, so $\mathscr{M}_2 \,\tilde{+}\, \mathscr{M}_1$ is a subsemigroup, too. By definition of $\mathscr{M}_2$ we have $\mathscr{M}_2 \,\tilde{+}\, \mathscr{M}_1 \subset \mathscr{M}$. So the maximality of $\mathscr{M}_1$ implies that $\mathscr{M}_1 \supset \mathscr{M}_2$. This proves the inclusion $\mathrm{cp}(\mathscr{M}) \supset \mathscr{M} \,\tilde{-}\, \mathscr{M}$.

Let us now prove the inverse inclusion. Let $T \in \operatorname{cp}(\mathscr{M}) \setminus (\mathscr{M} \tilde{-} \mathscr{M})$. Then there is some $T_1 \in \mathscr{M}$ such that $T_1 \tilde{+} T$ does not belong to $\mathscr{M}$. Consider the maximal subsemigroup $\mathscr{M}_3$ of $\mathscr{M}$ which contains T_1. Then $\mathscr{M}_3$ cannot contain T, so $\operatorname{cp}(\mathscr{M})$ cannot contain T either. $\square$

For every operator T and $\varepsilon > 0$ we define the *tube*

$$U_{T,\varepsilon} = \{x \in X \colon \|Tx\| < \varepsilon\}.$$

Let $\mathscr{M} \subset \mathscr{OP}(X)$. Put

$$\mathscr{M}_{\sim} = \{U_{T,\varepsilon} \cap S(X) \colon T \in \mathscr{M},\ \varepsilon > 0\}$$

Then $(\mathscr{M}_{\sim})^{\sim} = \mathscr{UB}(X) \tilde{-} \mathscr{M}$.

Proposition 2.12. *Let $\mathscr{M} \subset \mathscr{OP}(X)$ and let $\mathscr{N}$ be a collection of subsets in X. Then $\mathscr{N}^{\sim} \tilde{-} \mathscr{M} = \mathscr{N}_1{}^{\sim}$, where $\mathscr{N}_1$ consists of all intersections of the form $U_{T,\varepsilon} \cap A$, $T \in \mathscr{M}$, $A \in \mathscr{N}$, $\varepsilon > 0$. In particular, if $\mathscr{N}^{\sim} \tilde{-} \mathscr{M}$ is non-empty, then all the intersections $U_{T,\varepsilon} \cap A$ are non-empty and $\mathscr{N}^{\sim} \supset \mathscr{M}$.*

Proof. Let $T_1 \in \mathscr{N}^{\sim} \tilde{-} \mathscr{M}$. Then for every $T \in \mathscr{M}$ we have $T_1 \tilde{+} T \in \mathscr{N}^{\sim}$. This means that for every $A \in \mathscr{N}$ and $\varepsilon > 0$ there is an element $x \in A$ such that $\|(T_1 \tilde{+} T)x\| < \varepsilon$. This in turn implies that $x \in A \cap U_{T,\varepsilon}$ and $\|T_1 x\| < \varepsilon$. So $T_1 \in \mathscr{N}_1{}^{\sim}$.

Now let $T_1 \in \mathscr{N}_1{}^{\sim}$. Then for every $T \in \mathscr{M}$, every $A \in \mathscr{N}$ and $\varepsilon > 0$ there is an element $x \in A \cap U_{T,\varepsilon/2}$ such that $\|T_1 x\| < \varepsilon/2$. But by the definition of tubes, $\|Tx\| < \varepsilon/2$. So $\|(T_1 \tilde{+} T)x\| < \varepsilon$ and $T_1 \in \mathscr{N}^{\sim} \tilde{-} \mathscr{M}$. $\square$

3. Narrow operators

In this section we define the class of narrow operators on a Banach space with the Daugavet property. But first we need to introduce a closely related class of operators.

Definition 3.1. An operator $T \in \mathscr{OP}(X)$ is said to be a *strong Daugavet operator* if for every two elements $x, y \in S(X)$ and for every $\varepsilon > 0$ there is an element $z \in (y + U_{T,\varepsilon}) \cap S(X)$ such that $\|z + x\| > 2 - \varepsilon$. We denote the class of all strong Daugavet operators on X by $\mathscr{SD}(X)$.

It follows from Lemma 1.1 that a finite-rank operator on a space with the Daugavet property is a strong Daugavet operator, and conversely, if every rank-1 operator is strongly Daugavet, then X has the Daugavet property.

There is an obvious connection between strong Daugavet operators and the Daugavet equation.

Lemma 3.2. *If $T \colon X \to X$ is a strong Daugavet operator, then T satisfies the Daugavet equation* (1.1).

Proof. We assume without loss of generality that $\|T\| = 1$. Given $\varepsilon > 0$ pick $y \in S(X)$ such that $\|Ty\| \geq 1 - \varepsilon$. If $x = Ty/\|Ty\|$ and z is chosen according to Definition 3.1, then

$$2 - \varepsilon < \|z + x\| \leq \|z + Ty\| + \varepsilon \leq \|z + Tz\| + 2\varepsilon,$$

hence

$$\|z + Tz\| \geq 2 - 3\varepsilon,$$

which proves the lemma. □

We now relate the strong Daugavet property to a collection of subsets of X.

Definition 3.3. For every ordered pair of elements (x, y) of $S(X)$ and every $\varepsilon > 0$ let us define a set $D(x, y, \varepsilon)$ by

$$z \in D(x, y, \varepsilon) \iff \|z + x + y\| > 2 - \varepsilon \ \& \ \|z + y\| < 1 + \varepsilon.$$

By $\mathscr{D}(X)$ we denote the collection of all sets $D(x, y, \varepsilon)$, where $x, y \in S(X)$ and $\varepsilon > 0$.

Proposition 3.4. $\mathscr{SD}(X) = \mathscr{D}(X)^{\sim}$.

Proof. $T \in \mathscr{D}(X)^{\sim}$ if and only if for every pair $x, y \in S(X)$ and $\varepsilon > 0$ there is an element $z \in D(x, y, \varepsilon)$ such that $\|Tz\| < \varepsilon$. This in turn is equivalent to the following condition: for every pair $x, y \in S(X)$ and $\varepsilon > 0$ there is an element v such that $\|v\| < 1 + \varepsilon$, $\|x + v\| > 2 - \varepsilon$ and v belongs to the tube $y + U_{T,\varepsilon}$ (just put $v = z + y$). Evidently, the last equation coincides with the strong Daugavet property of the operator T. □

Let us consider an example.

Theorem 3.5. *For a compact Hausdorff space K, the class $\mathscr{SD}(C(K))$ of strong Daugavet operators coincides with the class of C-narrow operators.*

Proof. The fact that every C-narrow operator is a strong Daugavet operator has been proved in a slightly different form in [12, Th. 3.2]. Consider the converse implication. Let $T \in \mathscr{SD}(C(K))$. Fix a closed subset $F \subset K$ and $0 < \varepsilon < 1/4$. According to the definition it is sufficient to prove that there is a function $f \in S(C(K))$ for which the restriction to F is less than 2ε and $\|Tf\| < 2\varepsilon$ (cf. Example 2.6). Let us fix a neighbourhood U of F and an open set $V \subset K$, $V \cap U = \emptyset$. Select inductively functions $x_n, y_n \in S(C(K))$ and $f_n, g_n \in C(K)$ as follows. All the y_n are supported on U, and the x_n are non-negative functions supported on V. Given x_n and y_n pick $f_n \in D(x_n, y_n, \varepsilon)$ with $\|Tf_n\| < \varepsilon$, and let $g_n = f_1 + f_2 + \dots + f_n$. Then choose $y_{n+1} \in S(C(K))$ subject to the above support condition such that $\sup_{t \in F} |g_n(t)|\, y_{n+1}$ coincides on F with g_n, and let x_{n+1} be a non-negative continuous function supported on the subset of V where g_n attains its supremum on V up to ε, i.e., on the set $\{t \in V\colon g_n(t) > \sup_{s \in V} g_n(s) - \varepsilon\}$, etc. (There is no initial restriction in the choice of y_1 and x_1 apart from the support and positivity conditions.)

We first claim that

$$\|g_n\|_F := \sup_{t \in F} |g_n(t)| \leq 3 + n\varepsilon.$$

This is certainly true for $n = 1$ since $\|f_1 + y_1\| < 1 + \varepsilon$. Now induction yields for $t \in F$

$$\begin{aligned} |g_{n+1}(t)| &= |g_n(t) + f_{n+1}(t)| \\ &= \big| \|g_n\|_F \, y_{n+1}(t) + f_{n+1}(t) \big| \\ &= \big| y_{n+1}(t) + f_{n+1}(t) + (\|g_n\|_F - 1) y_{n+1}(t) \big| \\ &\le |y_{n+1}(t) + f_{n+1}(t)| + \|g_n\|_F - 1 \\ &\le 1 + \varepsilon + 2 + n\varepsilon = 3 + (n+1)\varepsilon. \end{aligned}$$

(We have tacitly assumed that $\|g_n\|_F \ge 1$ since the induction step is clear otherwise, because $\|f_{n+1}\| \le 2 + \varepsilon$.)

Next, we have that

$$\sup_{t \in V} g_n(t) > n(1 - 2\varepsilon).$$

Indeed, the functions x_1 and y_1 are disjointly supported; hence by the definition of $D(x_1, y_1, \varepsilon)$ there is a point in the support of x_1 at which $f_1 = g_1$ is bigger than $1 - \varepsilon$. Thus, the above inequality holds for $n = 1$. To perform the induction step we use the same argument to find a point t_0 in the support of x_{n+1} at which f_{n+1} exceeds $1 - \varepsilon$. At this point t_0 the function g_n attains its supremum on V up to ε. So

$$\begin{aligned} \sup_{t \in V} g_{n+1}(t) \ge g_{n+1}(t_0) &= g_n(t_0) + f_{n+1}(t_0) > \sup_{t \in V} g_n(t) + 1 - 2\varepsilon \\ &> n(1 - 2\varepsilon) + 1 - 2\varepsilon = (n+1)(1 - 2\varepsilon). \end{aligned}$$

Therefore $\|g_{n+1}\| > (n+1)(1 - 2\varepsilon)$, and on the other hand we have $\|Tg_n\| \le \sum_{k=1}^{n} \|Tf_n\| < n\varepsilon$. So for n big enough the function $f = g_n / \|g_n\|$ will satisfy the desired conditions. □

Actually, a somewhat smaller class of operators turns out to be crucial.

Definition 3.6. Let X be a space with the Daugavet property. Define the class of *narrow operators* by $\mathscr{NAR}(X) = \mathscr{SD}(X) \tilde{-} X^*$.

In other words, an operator T is said to be a narrow operator if, for every $x^* \in X^*$, $T \tilde{+} x^*$ is a strong Daugavet operator.

Incidentally, it follows from the defining property of a narrow operator that a Banach space on which at least one narrow operator is defined must automatically have the Daugavet property.

Proposition 3.4 and Proposition 2.8 imply that $\mathscr{NAR}(X)$ is a $\sim$closed homogeneous order ideal, and hence so is $\mathrm{cp}(\mathscr{NAR}(X))$.

We now show that we won't get anything new on $C(K)$ if K is perfect.

Theorem 3.7. *For a perfect compact Hausdorff space K, the classes of C-narrow operators and of narrow operators coincide on $C(K)$.*

Proof. Since a narrow operator is a strong Daugavet operator and a strong Daugavet operator is C-narrow (Theorem 3.5), it is left to prove that a C-narrow operator T on $C(K)$ is narrow if K is perfect. Let $x^* \in C(K)^*$ be a functional, represented by a regular Borel measure μ; we have to show that $T \tilde{+} x^*$ is a strong Daugavet operator.

Thus, let $f, g \in S(C(K))$ and $\varepsilon > 0$. Let $\varepsilon' = \varepsilon/(4 + \|T\|)$, and consider the open set $U = \{t\colon |f(t)| > 1-\varepsilon'\}$. Pick an open non-empty subset $V \subset U$ with the property that $f-g$ is almost constant on V in that for some number $c \in [-2, 2]$

$$|f(t) - g(t) - c| \le \varepsilon' \text{ for } t \in V,$$

and $|\mu|(V) \le \varepsilon'$; the latter is possible since K has no isolated points. Since T is C-narrow, there is some $\varphi \in S(C(K))$ vanishing off V such that $\|T\varphi\| \le \varepsilon'$; in fact, φ can (and will) be chosen positive [12, Lemma 1.4]. Let $h = \varphi f + (1-\varphi)g$. Then $\|h\| \le 1$, and

$$\|f + h\| \ge \sup_{t\in V} |f(t) + h(t)| \ge 2 - 2\varepsilon' \ge 2 - \varepsilon;$$

furthermore

$$\begin{aligned}
|x^*(g) - x^*(h)| &= |x^*(\varphi(f-g))| \le |\mu(V)|\,\|\varphi\|\,\|f-g\| \le 2\varepsilon' \\
\|T(g) - T(h)\| &= \|T(\varphi(f-g))\| \le \|T\|\,\|\varphi(f-g-c)\| + \|T\varphi\|\,|c| \\
&\le \|T\|\varepsilon' + 2\varepsilon'
\end{aligned}$$

so that

$$\|(T \tilde{+} x^*)(g-h)\| \le \varepsilon,$$

which proves that $T \tilde{+} x^*$ is a strong Daugavet operator. □

We shall show below (Section 6) that in general, on a space with the Daugavet property narrow and strong Daugavet operators are not the same.

We don't know if in general $\mathscr{NAR}(X)$ is a subsemigroup of $\mathscr{OP}(X)$ as it will be shown to be the case for $X = C(K)$ (Theorem 4.8), but we will show that its central part $\mathrm{cp}(\mathscr{NAR}(X))$ is always large. It contains, in particular, all strong Radon-Nikodým operators and all operators which do not fix copies of ℓ_1. Hence all the operators which are majorized by linear combinations of strong Radon-Nikodým operators and operators not fixing copies of ℓ_1, as well as $\sim$-limits of sequences of such operators belong to $\mathrm{cp}(\mathscr{NAR}(X))$.

We now formulate a number of lemmas. Eventually, Proposition 3.11 will present a geometric description of what distinguishes a narrow operator from a strong Daugavet operator.

Lemma 3.8. *Let $T \in \mathscr{NAR}(X)$. Then for every $x, y \in S(X)$, $\varepsilon > 0$ and every slice $S = S(x^*, \alpha)$ of the unit ball of X containing y there is an element $v \in S$ such that $\|x + v\| > 2 - \varepsilon$ and $\|T(y - v)\| < \varepsilon$.*

Proof. Fix some $0 < \delta < \varepsilon$ and find an element y_1 in norm-interior of S such that $\|y - y_1\| < \delta$. By Proposition 2.12, for every $0 < \delta_1 < \varepsilon$ there is an element $u \in U_{x^*,\delta_1} \cap D(x, y_1/\|y_1\|, \delta_1)$ such that $\|Tu\| < \delta_1$. If δ_1 is small enough, then $v := (y_1 + \|y_1\|u)/\|(y_1 + \|y_1\|u)\| \in S$. So, if in turn δ is small enough, then v satisfies our requirements. □

Lemma 3.9. *For every $\tau > 0$ and every pair of positive numbers a, b there is a $\delta > 0$ such that if $v, x \in S(X)$ and $\|x+v\| > 2-\delta$, then $\|ax+bv\| > a+b-\tau$.*

Proof. Select $\delta < \tau/\max(a,b)$. There are two symmetric cases: $a \le b$ or $b \le a$. Consider for example the first of them. If we assume that our statement is not true, then we obtain

$$\begin{aligned} 2-\delta < \|x+v\| &= \|(1-a/b)x + 1/b(ax+bv)\| \\ &\le 1 - a/b + 1/b(a+b-\tau) = 2 - \tau/b, \end{aligned}$$

a contradiction. □

Lemma 3.10. *Let $T \in \mathscr{NAR}(X)$.*

(a) *Let $S_1, \dots, S_n$ be a finite collection of slices and $U \subset B(X)$ be a convex combination of these slices, i.e., there are $\lambda_k \ge 0$, $k = 1, \dots, n$, $\sum_{k=1}^n \lambda_k = 1$, such that $\lambda_1 S_1 + \dots + \lambda_n S_n = U$. Then for every $\varepsilon > 0$, every $x_1 \in S(X)$ and every $w \in U$ there exists an element $u \in U$ such that $\|u + x_1\| > 2 - \varepsilon$ and $\|T(w-u)\| < \varepsilon$.*

(b) *The same conclusion is true if U is a relatively weakly open set.*

Proof. (a) First of all let us fix elements $y_k \in S_k$ such that $\lambda_1 y_1 + \dots + \lambda_n y_n = w$. Applying repeatedly Lemma 3.8 with sufficiently small ε_j to S_j, $y_j \in S_j$ and

$$x_j = \left(x_1 + \sum_{k=1}^{j-1} \lambda_k v_k\right) \Bigg/ \left\| x_1 + \sum_{k=1}^{j-1} \lambda_k v_k \right\|,$$

we may select elements $v_k \in S_k$ with $\|T(y_k - v_k)\| < \varepsilon$, $k = 1, \dots, n$, in such a way that for every $j = 1, \dots, n$

$$\left\| x_1 + \sum_{k=1}^{j} \lambda_k v_k \right\| > 1 + \sum_{k=1}^{j} \lambda_k (1-\varepsilon)$$

(to get the last inequality, we need to apply the previous lemma at each step). The element $u = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ will be as required.

(b) This follows from (a) since given $u \in U$ there is a convex combination V of slices such that $u \in V \subset U$; see [8, Lemma II.1] or [21]. □

Proposition 3.11. *An operator T on a Banach space X with the Daugavet property is narrow if and only if for every $x, y \in S(X)$, $\varepsilon > 0$ and every slice S of the unit ball of X containing y there is an element $v \in S$ such that $\|x+v\| > 2-\varepsilon$ and $\|T(y-v)\| < \varepsilon$.*

Proof. It only remains to show that the above condition is sufficient for T to be narrow. We first note that an operator satisfying that condition will also satisfy the conclusion of Lemma 3.10; see the proof of that lemma. Now, if $x_0^* \in X^*$, $x, y \in S(X)$ and $\varepsilon > 0$, consider the relatively weakly open set

$$U := \{z \in B(X) \colon |x_0^*(z-y)| < \varepsilon/2\}.$$

By Lemma 3.10 there exists some $w \in U$ such that $\|w + x\| > 2 - \varepsilon/2$ and $\|T(w-y)\| < \varepsilon/2$; note that $y \in U$. By definition this means that $T \tilde{+} x_0^*$ is a strong Daugavet operator; i.e., T is narrow. □

Let T be a strong Radon-Nikodým operator on a space X with the Daugavet property; this means that the closure of $T(B(X))$ is a set with the Radon-Nikodým property, cf. [4] for this notion. We shall show that such an operator is narrow. For $\varepsilon > 0$, consider the subset $A(T,\varepsilon)$ of $B(X)$ defined by $y \in A(T,\varepsilon)$ if there exists a convex combination U of slices of the unit ball such that $y \in U$ and $U \subset y + U_{T,\varepsilon}$.

Lemma 3.12. *The set $A(T,\varepsilon)$ introduced above is a convex dense subset of $B(X)$.*

Proof. The convexity is evident. To prove the density we need to show, by the Hahn-Banach theorem, that for every $x^* \in S(X^*)$ and every $0 < \delta < \varepsilon$ there is an element $y \in A(T,\varepsilon)$ such that $x^*(y) > 1 - \delta$ (in other words, $y \in S = S(x^*,\delta)$). Let us fix an element $x \in B(X)$ with $x^*(x) > 1 - \delta/2$ and consider the operator $T_1 = x^* \tilde{+} T$. Consider further $\overline{T_1(B(X))}$ and a $\delta/2$-neighbourhood W of $T_1 x$ in $\overline{T_1(B(X))}$. By the Radon-Nikodým property of the set $\overline{T_1(B(X))}$ there is a convex combination W_1 of slices of $\overline{T_1(B(X))}$ in W. The preimages in $B(X)$ of these slices of $\overline{T_1(B(X))}$ are slices in $B(X)$. The corresponding convex combination U of these slices in $B(X)$ lies in the preimage of W in $B(X)$, so this convex combination is contained in $(x + U_{T_1,\delta/2}) \cap B(X)$. Fix an element $y \in U$. By our construction $y \in U \subset (x + U_{T_1,\delta/2}) \cap B(X) \subset S$. On the other hand,

$$U \subset x + U_{T_1,\delta/2} \subset y + U_{T_1,\delta} \subset y + U_{T,\delta} \subset y + U_{T,\varepsilon},$$

so $y \in A(T,\varepsilon)$. □

The following result is a generalisation of [13, Th. 2.3]. It can be understood as a transfer theorem: in Definition 3.6 one can pass from one-dimensional operators to a much wider class of operators. Let us denote the class of strong Radon-Nikodým operators on X by $\mathscr{SRN}(X)$.

Theorem 3.13. *Let X be a space with the Daugavet property, T be narrow and T_1 be a strong Radon-Nikodým operator on X. Then $T \tilde{+} T_1$ is narrow; that is, we have $\mathscr{NAR}(X) \tilde{+} \mathscr{SRN}(X) = \mathscr{NAR}(X)$. In particular every strong Radon-Nikodým operator T_1 on X is a narrow operator.*

Proof. Let us fix $\varepsilon > 0$, $x, y \in S(X)$ and $y_1 \in A(T_1,\varepsilon)$ such that $\|y - y_1\| < \varepsilon$. According to the definition of $A(T_1,\varepsilon)$ there exists a convex combination U of slices of the unit ball such that $y_1 \in U$ and $U \subset y_1 + U_{T_1,\varepsilon}$. By Lemma 3.10 there is an element $z \in U$ such that $\|z + x\| > 2 - \varepsilon$ and $\|T(y_1 - z)\| < \varepsilon$. But the inclusion $z \in y_1 + U_{T_1,\varepsilon}$ means that $\|T_1(y_1 - z)\| < \varepsilon$. So

$$\|(T \tilde{+} T_1)(y - z)\| < \varepsilon\|T \tilde{+} T_1\| + \|(T \tilde{+} T_1)(y_1 - z)\| < \varepsilon\|T \tilde{+} T_1\| + 2\varepsilon.$$

Because ε is arbitrarily small, the last inequality shows that $T \tilde{+} T_1$ satisfies the definition of a strong Daugavet operator.

Now let $x^* \in X^*$ and consider $T_2 = T_1 \tilde{+} x^*$. This is a strong Radon-Nikodým operator, too. So $(T \tilde{+} T_1) \tilde{+} x^* = T \tilde{+} T_2$ is a strong Daugavet operator by what we have just proved; by definition, this says that $T \tilde{+} T_1$ is narrow. ☐

Corollary 3.14. *Let X be a Banach space with the Daugavet property.*

(a) $\mathscr{NAR}(X) \tilde{+} X^* = \mathscr{NAR}(X)$.

(b) $\mathrm{cp}(\mathscr{NAR}(X)) = \mathscr{SD}(X) \tilde{-} \mathscr{NAR}(X)$.

(c) $\mathscr{SRN}(X) \subset \mathrm{cp}(\mathscr{NAR}(X))$.

Proof. (a) follows from the previous theorem, because every finite-rank operator is a strong Radon-Nikodým operator.

For (b) use Theorem 2.11 and note that

$$\begin{aligned} \mathscr{SD}(X) \tilde{-} \mathscr{NAR}(X) &= \mathscr{SD}(X) \tilde{-} (\mathscr{NAR}(X) \tilde{+} X^*) \\ &= (\mathscr{SD}(X) \tilde{-} X^*) \tilde{-} \mathscr{NAR}(X) \\ &= \mathscr{NAR}(X) \tilde{-} \mathscr{NAR}(X). \end{aligned}$$

(c) is a restatement of Theorem 3.13. ☐

4. Operators which do not fix copies of ℓ_1

It is proved in [21] that an operator $T\colon X \to X$ on a space with the Daugavet property which does not fix a copy of ℓ_1 satisfies the Daugavet equation. Recall that $T \in \mathscr{OP}(X)$ does not fix a copy of ℓ_1 if there is no subspace $E \subset X$ isomorphic to ℓ_1 on which the restriction $T\colon E \to T(E)$ is an isomorphism. By Rosenthal's ℓ_1-theorem, this is equivalent to saying that for every bounded sequence $(x_n) \subset X$, the sequence of images (Tx_n) admits a weak Cauchy subsequence. We shall investigate the class of operators not fixing a copy of ℓ_1 in the present context.

We will use the following theorem, due to H.P. Rosenthal [18]:

Theorem 4.1. *Let X be a separable Banach space without ℓ_1-subspaces. If $A \subset X$ is bounded and $x^{**} \in X^{**}$ is a weak* limit point of A, then there is a sequence in A which converges to x^{**} in the weak* topology of X^{**}.*

In fact, we shall need a generalization of this result and first provide a lemma.

Lemma 4.2. *Let X be a Banach space without subspaces isomorphic to ℓ_1, and let $\{x_{n,m}\}_{n,m\in\mathbb{N}} \subset X$ be a bounded double sequence. Let $x^{**} \in X^{**}$ be a $\sigma(X^{**},X^*)$-limit point of every column $\{x_{n,m}\}_{n\in\mathbb{N}}$ of $\{x_{n,m}\}_{n,m\in\mathbb{N}}$. Then there are strictly increasing sequences $(n(k))$, $(m(k))$ of indices such that $x_{n(k),m(k)} \to x^{**}$ in $\sigma(X^{**},X^*)$.*

Proof. Consider an auxiliary space $Y = X \times \mathbb{R}$ and an auxiliary matrix $\{y_{n,m}\}_{n,m\in\mathbb{N}} \subset Y$, $y_{n,m} = (x_{n,m}, 1/n + 1/m)$. Since Y contains no copies of ℓ_1 either and since $(x^{**},0)$ is a $\sigma(Y^{**},Y^*)$-limit point of $\{y_{n,m}\}_{n,m\in\mathbb{N}}$, there is, according to Theorem 4.1, a sequence of the form $(y_{n(k),m(k)})$ which converges to $(x^{**},0)$ in $\sigma(Y^{**},Y^*)$. This means in particular that $x_{n(k),m(k)} \to x^{**}$ in $\sigma(X^{**},X^*)$ and $1/n + 1/m \to 0$. So $(n(k))$ and $(m(k))$

both tend to ∞, which, after passing to a subsequence, provides the desired sequence. $\square$

The next result is a direct generalisation of Theorem 4.1.

Theorem 4.3. *Let X be a separable Banach space without ℓ_1-subspaces, $(\Gamma, \preceq)$ be a directed set, and let $F\colon \Gamma \to X$ be a bounded function. Then for every $\sigma(X^{**}, X^*)$-limit point x^{**} of the function F there is a strictly increasing sequence $\gamma(1) \preceq \gamma(2) \preceq \dots$ in Γ such that $\big(F(\gamma(n))\big)$ converges to x^{**} in $\sigma(X^{**}, X^*)$.*

Proof. Using inductively Theorem 4.1 we can select a doubly indexed sequence $\{\gamma_{n,m}\}_{n,m\in\mathbb{N}}$ in Γ with the following properties:

1. for every $m \in \mathbb{N}$, $x^{**} \in X^{**}$ is a $\sigma(X^{**}, X^*)$-limit point of every column $\{F(\gamma_{n,m})\}_{n\in\mathbb{N}}$;
2. for every $m, n, k, l \in \mathbb{N}$, if $\max\{k, l\} < m$, then $\gamma_{k,l} \preceq \gamma_{n,m}$.

Applying Lemma 4.2 and passing to a subsequence if necessary, we obtain strictly increasing sequences $(n(k))$, $(m(k))$ such that $\max_{k<j}\{n(k), m(k)\} < m(j)$ and $\big(F(\gamma_{n(k),m(k)}\big)$ converges to x^{**} in $\sigma(X^{**}, X^*)$. To finish the proof put $\gamma(k) = \gamma_{n(k),m(k)}$. $\square$

We wish to prove that an operator not fixing a copy of ℓ_1 is narrow (Theorem 4.13 below). To cover the case of non-separable spaces as well we first show that the Daugavet property is separably determined. The next lemma prepares this result.

Lemma 4.4. *Let X be a Banach space with the Daugavet property. Then for any $\varepsilon > 0$ and $x, y \in S(X)$, there exists a finite-dimensional subspace $Y = Y(x, y, \varepsilon)$ of X with $x, y \in Y$ such that for every slice $S(x^*, \varepsilon/2)$ containing y there is some $y_1 \in S(Y) \cap S(x^*, \varepsilon)$ such that $\|y_1 + x\| > 2 - \varepsilon$.*

Proof. Assume there exist $\varepsilon > 0$ and $x, y \in S(X)$ such that for every finite-dimensional subspace $Y \subset X$ there is a slice $S(x_Y^*, \varepsilon/2)$ containing y with $\|y_1 + x\| \le 2 - \varepsilon$ for all $y_1 \in S(Y) \cap S(x_Y^*, \varepsilon)$. Take a weak* cluster point x^* of the net (x_Y^*) and let $x_0^* = x^*/\|x^*\|$. We have $x^*(y) \ge 1 - \varepsilon/2$ since $y \in S(x_Y^*, \varepsilon/2)$ and therefore $\|x^*\| \ge 1 - \varepsilon/2$. Now if $y_1 \in S(x_0^*, \varepsilon/2)$, then $x^*(y_1) \ge \|x^*\|(1 - \varepsilon/2) > 1 - \varepsilon$ and therefore $x_{Y_1}^*(y_1) > 1 - \varepsilon$ for some Y_1 that contains y_1. So by assumption $\|y_1 + x\| \le 2 - \varepsilon$, which contradicts Lemma 1.1 when applied to the slice $S(x_0^*, \varepsilon/2)$. $\square$

Theorem 4.5. *A Banach space X has the Daugavet property if and only if for every separable subspace $Y \subset X$ there is a separable subspace $Z \subset X$ which contains Y and has the Daugavet property.*

Proof. Suppose X has the Daugavet property. Let (v_n) be a dense sequence in Y. We select a sequence $V_1 \subset V_2 \subset \dots$ of finite-dimensional subspaces of X by the following inductive procedure. Put $V_1 = \operatorname{lin} v_1$. Suppose V_n has already been constructed. Fix a 2^{-n}-net (x_k^n, y_k^n), $k = 1, \dots, N_n$, in

$S(V_n) \times S(V_n)$ provided with the sum norm, select by Lemma 4.4 finite-dimensional subspaces $Y_k = Y(x_k^n, y_k^n, \varepsilon)$, $k = 1, \dots, N_n$, for $\varepsilon = 2^{-n}$ and define $V_{n+1} = \mathrm{lin}(\{v_{n+1}\} \cup Y_1 \cup \ldots \cup Y_{N_n})$.

If Z is defined to be the closure of the union of all the V_n, then $Y \subset Z$ and Z has the Daugavet property by Lemma 1.1.

Conversely, let $x \in S(X)$, $\varepsilon > 0$ and let $S \subset B(X)$ be a slice. Fix a point $z \in S$. If Z is a separable subspace with the Daugavet property containing x and z, then by Lemma 1.1 there exists some $y \in S \cap Z$ such that $\|y + x\| > 2 - \varepsilon$. Again by Lemma 1.1 this shows that X has the Daugavet property. □

We shall need the operator version of this theorem, which is based on the following lemma. The proofs of Lemma 4.6 and Theorem 4.7 are virtually the same as those of Lemma 4.4 and Theorem 4.5 (one uses Proposition 3.11).

Lemma 4.6. *Let X be a Banach space with the Daugavet property and let T be a narrow operator on X. Then for any $\varepsilon > 0$ and $x, y \in S(X)$, there exists a finite-dimensional subspace $Y = Y(x, y, \varepsilon)$ of X with $x, y \in Y$ such that for every slice $S(x^*, \varepsilon/2)$ containing y there is some $y_1 \in S(Y) \cap S(x^*, \varepsilon)$ with $\|Ty_1 - Ty\| < \varepsilon$ such that $\|y_1 + x\| > 2 - \varepsilon$.*

Theorem 4.7. *An operator T on a Banach space X is narrow if and only if for every separable subspace Y of X there is a separable subspace $Z \subset X$ containing Y such that the restriction of T to Z is narrow.*

This theorem leads to an important structural result on narrow operators on $C(K)$.

Theorem 4.8. *If K is a perfect compact Hausdorff space, then $\mathscr{NAR}(C(K))$ is a subsemigroup of $\mathscr{OP}(C(K))$.*

Proof. First, let K be a perfect compact metric space. It follows from Theorem 3.7 and [12, Th. 1.8] that the set of all narrow operators on $C(K)$ is stable under the operation $\tilde{+}$, i.e., it is a semigroup. (In fact, [12] only deals with $K = [0, 1]$, but the arguments work as well for a metric K.)

We shall now reduce the general case to the metric one. Let now K be a perfect compact Hausdorff space, and let T_1 and T_2 be two narrow operators on $C(K)$; we shall verify that $T_1 \tilde{+} T_2$ is narrow, using Theorem 4.7 above.

Thus, let Y be a separable subspace of $C(K)$. We shall first argue that there is a separable space Z_1 containing Y such that $T_1|_{Z_1}$ and $T_2|_{Z_1}$ are strong Daugavet operators. Let A be a countable dense subset of $S(Y)$. For every pair (x, y) in $A \times A$ and every $\varepsilon = 1/k$ there is some z_1 (resp. z_2) according to the definition of the strong Daugavet property of T_1 (resp. T_2). The countable collection of these z's and Y span a closed separable subspace X_1. Repeating this procedure starting from X_1 yields some closed separable subspace X_2, etc. The closed linear span Z_1 of $X_1, X_2, X_3, \dots$ then has the desired property.

Now by [22, Lemma 2.4] there is a separable space $Z_2 \supset Z_1$ isometric to some space $C(M_2)$ for a perfect compact metric space M_2. By the same token as above, we can extend Z_2 to a separable space Z_3 so that T_1 and T_2 are strong Daugavet operators on Z_3, and we can extend Z_3 to a separable space Z_4 isometric to some space $C(M_4)$ for a perfect compact metric space M_4, etc. Let Z be the closed linear span of $Z_1, Z_2, Z_3, \ldots$. Then $T_1|_Z$ and $T_2|_Z$ are strong Daugavet operators, and Z is isometric to some space $C(M)$ for a perfect compact metric space M. By what we already know, $(T_1 \tilde{+} T_2)|_Z$ is a narrow operator on $Z \cong C(M)$; recall that the classes of narrow and strong Daugavet operators coincide on $C(M)$.

Finally, Theorem 4.7 implies that $T_1 \tilde{+} T_2$ is narrow on $C(K)$, which proves the theorem. □

Next we introduce a topology related to an order ideal of operators.

Definition 4.9. Let $\mathscr{M} \subset \mathscr{OP}(X)$ be an order ideal of operators, closed under the operation $\tilde{+}$. Then the system of tubes $U_{T,\varepsilon}$, $T \in \mathscr{M}$, $\varepsilon > 0$, defines a base of neighbourhoods of 0 for some locally convex topology on X. We denote this topology by $\sigma(X, \mathscr{M})$.

If $\mathscr{M} = \mathscr{F}(X)$, the class of all finite-rank operators, then $\sigma(X, \mathscr{M})$ coincides with the weak topology; if $\mathscr{M} = \mathscr{OP}(X)$, then $\sigma(X, \mathscr{M})$ coincides with the norm topology. For classes which are in between one gets topologies which are between the weak and the norm topology. If $\mathscr{N}$ is a collection of subsets in X such that $\mathscr{N}^{\sim}$ is closed under the operation $\tilde{+}$, then $\sigma(X, \mathscr{N}^{\sim})$ is the strongest locally convex topology on X continuous with respect to the norm, in which the zero vector belongs to the closure of every element of $\mathscr{N}$.

Definition 4.10. A locally convex topology τ on X is said to be a *Daugavet topology* if for every two elements $x, y \in S(X)$, for every $\varepsilon > 0$ and every τ-neighbourhood U of y there is an element $z \in U \cap S(X)$ such that $\|z + x\| > 2 - \varepsilon$.

Of course, $\sigma(X, \mathscr{M})$ is a Daugavet topology if and only if every operator $T \in \mathscr{M}$ is a strong Daugavet operator.

Lemma 4.11. *Let X be a Banach space with the Daugavet property, T a narrow operator, $A = \{a_1, \ldots, a_n\} \subset S(X)$, $\varepsilon > 0$ and $y \in S(X)$. Then for every $\sigma(X, \mathrm{cp}(\mathscr{NAR}(X)))$-neighbourhood W of y there is an element $w \in W \cap S(X)$ such that $\|T(w - y)\| < \varepsilon$ and $\|w + a\| > 2 - \varepsilon$ for every $a \in A$.*

Proof. We shall argue by induction on n. First of all consider $n = 1$. Every $\sigma(X, \mathrm{cp}(\mathscr{NAR}(X)))$-neighbourhood of y can be represented as $W = y + U_{R,\delta}$, where $R \in \mathrm{cp}(\mathscr{NAR}(X))$. Since $T_1 = R \tilde{+} T$ is a strong Daugavet operator by definition of the central part, there is an element $w \in S(X)$ such that $\|w + a_1\| > 2 - \varepsilon$ and $\|T_1(w - y)\| < \min(\delta, \varepsilon)$. The last inequality means, in particular, that $\|T(w - y)\| < \varepsilon$ and $w \in W$.

Now suppose our assertion is true for n, let us prove it for $n + 1$. Let $A = \{a_1, \ldots, a_n, a_{n+1}\} \subset S(X)$, and let us assume that an element $w_1 \in$

$W \cap S(X)$ such that $\|T(w_1 - y)\| < \varepsilon/2$ and $\|w_1 + a_k\| > 2-\varepsilon$, $k = 1,\dots,n$, has already been selected. Then there is a weak neighbourhood U of w_1 such that the inequalities $\|u + a_k\| > 2 - \varepsilon$, $k = 1,\dots,n$, hold for every $u \in U$. The intersection $U \cap W$ is a $\sigma(X, \mathrm{cp}(\mathscr{NAR}(X)))$-neighbourhood of w_1, so according to our inductive assumption for $n = 1$, there is an element $w \in S(X) \cap U \cap W$ such that $\|w + a_{n+1}\| > 2 - \varepsilon$ and $\|T(w - w_1)\| < \varepsilon/2$. This element w satisfies all the requirements. □

Using an ε-net of the unit ball of the finite-dimensional subspace Z below one can easily deduce the following corollary.

Proposition 4.12. *Let X be a Banach space with the Daugavet property, T be a narrow operator and $Z \subset X$ be a finite-dimensional subspace. Then for every $\varepsilon > 0$, every $y \in S(X)$ and every $\sigma(X, \mathrm{cp}(\mathscr{NAR}(X)))$-neighbourhood W of y there is an element $w \in W \cap S(X)$ such that $\|T(w - y)\| < \varepsilon$ and $\|z + w\| > (1-\varepsilon)(\|z\| + \|w\|)$ for every $z \in Z$.*

Theorem 4.13. *Let X be a Banach space with the Daugavet property and let T be an operator on X which does not fix a copy of ℓ_1. Then $T \in \mathrm{cp}(\mathscr{NAR}(X))$, so in particular T is a narrow operator.*

Proof. Lemma 1(xii) of [6] implies that every operator which does not fix a copy of ℓ_1 can be factored through a space without ℓ_1-subspaces. So every operator which does not fix a copy of ℓ_1 can be majorized by an operator which maps into a space without ℓ_1-subspaces. Since the class of narrow operators is an order ideal, it is enough to prove our theorem for $T\colon X \to Y$, where Y has no ℓ_1-subspaces. Also, by Theorem 4.7 we may assume that X and Y are separable.

Let us fix a narrow operator R, $\varepsilon > 0$ and $x, y \in S(X)$. Let us introduce a directed set $(\Gamma, \preceq)$ as follows: the elements of Γ are finite sequences in $S(X)$ of the form $\gamma = (x_1,\dots,x_n)$, $n \in \mathbb{N}$, with $x_1 = x$. The (strict) ordering is defined by

$$(x_1,\dots,x_n) \prec (y_1,\dots,y_m) \iff n < m \ \&\ \{x_1,\dots,x_n\} \subset \{y_1,\dots,y_{m-1}\}$$

and of course $\gamma_1 \preceq \gamma_2$ if $\gamma_1 \prec \gamma_2$ or $\gamma_1 = \gamma_2$. Now define a bounded function $F\colon \Gamma \to Y \times \mathbb{R} \times \mathbb{R}$ by

$$F(\gamma) = (Tx_n, \alpha(\gamma), \|R(y - x_n)\|),$$

where

$$\alpha(\gamma) = \sup\bigl\{a > 0\colon \|z + x_n\| > a(\|z\| + \|x_n\|)\ \forall z \in \mathrm{lin}\{x_1, x_2, \dots, x_{n-1}\}\bigr\}.$$

Due to Proposition 4.12, for every weak neighbourhood U of y in $B(X)$, every $\varepsilon > 0$ and every finite collection $\{v_1,\dots,v_n\} \subset X$ there is some $v_{n+1} \in U$ for which $\alpha\bigl((v_1,\dots,v_{n+1})\bigr) > 1 - \varepsilon$ and $\|R(y - v_{n+1})\| < \varepsilon$. This means that $(Ty, 1, 0)$ is a weak limit point of the function F. So, by Theorem 4.3 there is a strictly $\prec$-increasing sequence $(\gamma_j) = \bigl((x_1,\dots,x_{n(j)})\bigr)$ for which $(Tx_{n(j)})$ tends weakly to Ty, $(\|R(y - x_{n(j)})\|)$ tends to 0 and $(\alpha(\gamma_j))$ tends to 1. Passing to a subsequence we can select points $x_{n(j)}$

in such a way that the sequence $\{x, x_{n(1)}, x_{n(2)}, \dots\}$ is ε-equivalent to the canonical basis of ℓ_1. According to Mazur's theorem, there is a sequence $z_n \in \operatorname{conv}\{x_{n(j)}\}_{j>n}$ such that $\|Ty - Tz_n\| \to 0$. Evidently $\|z_n + x\| > 2 - \varepsilon$ and $\|(R \tilde{+} T)(y - z_n)\| \to 0$, which means that $R \tilde{+} T \in \mathscr{SD}(X)$ and thus proves the theorem by Corollary 3.14(b). $\square$

There are other applications of Theorem 4.3 which are not related to the Daugavet property. As an example let us prove the following theorem which was earlier established by E. Behrends [2] under the more restrictive condition of separability of X^*.

Theorem 4.14. *Let X be a Banach space without ℓ_1-subspaces and $A_n \subset X$ be bounded subsets with $0 \in \overline{\operatorname{conv}}\, A_n$ for each $n \in \mathbb{N}$. Then there exists a sequence (a_n) in X with $a_n \in A_n$ for every n such that $0 \in \overline{\operatorname{conv}}\{a_1, a_2, \dots\}$.*

Proof. In each A_n there is a separable subset whose closed convex hull contains 0. So, passing to the linear span of these separable subsets we may assume that X is separable. Introduce a directed set $(\Gamma, \preceq)$ as follows: the elements of Γ are of the form

$$\gamma = \bigl(n, m, \{a_k\}_{k=n}^m, \{\lambda_k\}_{k=n}^m\bigr),$$

where $n, m \in \mathbb{N}$, $n < m$, $a_k \in A_k$, $\lambda_k > 0$, $\sum_{k=n}^m \lambda_k = 1$. Define $\preceq$ as follows: let $\gamma_1 = (n_1, m_1, \{a_k\}_{k=n_1}^{m_1}, \{\lambda_k\}_{k=n_1}^{m_1})$, $\gamma_2 = (n_2, m_2, \{b_k\}_{k=n_2}^{m_2}, \{\mu_k\}_{k=n_2}^{m_2})$; then $\gamma_1 \preceq \gamma_2$ if $m_1 < n_2$. Define $F\colon \Gamma \to X$ by the formula $F(\gamma) = \sum_{k=n}^m \lambda_k a_k$. Now, 0 is a weak limit point of F; see the proof of [2, Th. 4.3]. So, by Theorem 4.3 there is a sequence of elements

$$\gamma_j = (n_j, m_j, \{a_k\}_{k=n_j}^{m_j}, \{\lambda_k\}_{k=n_j}^{m_j})$$

such that $n_1 < m_1 < n_2 < m_2 < n_3 < \dots$ and $\sum_{k=n_j}^{m_j} \lambda_k a_k$ tends weakly to zero. To finish the proof one just needs to apply Mazur's theorem. $\square$

5. Rich subspaces

In [17] a subspace Y of L_1 is called rich if the quotient map $q\colon L_1 \to L_1/Y$ is L_1-narrow, and likewise a subspace Y of $C(K)$ is called rich in [12] if the quotient map $q\colon C(K) \to C(K)/Y$ is C-narrow. We are now in a position to discuss rich subspaces in general.

Definition 5.1. Let X be a Banach space with the Daugavet property. A subspace Y is said to be *almost rich* if the quotient map $q\colon X \to X/Y$ is a strong Daugavet operator. A subspace Y is said to be *rich* if the quotient map $q\colon X \to X/Y$ is a narrow operator.

By Theorem 3.7 the new definition comprises the old one for subspaces of $C(K)$.

The necessity to distinguish rich and almost rich subspaces will become apparent later when we show that the following theorem does not extend to almost rich subspaces; see Theorem 6.4.

Theorem 5.2. *A rich subspace Y of a Banach space X with the Daugavet property has the Daugavet property itself. Moreover, (Y, X) is a Daugavet pair.*

Proof. Consider elements $x \in S(X)$, $y \in S(Y)$, a slice $S = S(x^*, \varepsilon)$ and $y \in S$. According to our assumption the quotient map q: $X \to X/Y$ is a narrow operator. So there is an element $u \in S$ such that $\|u + x\| > 2 - \varepsilon$ and $\|q(y - u)\| = \|q(u)\| < \varepsilon$. The last condition means that the distance from u to Y is smaller than ε, so there is an element $v \in Y$ with $\|v - u\| < \varepsilon$. The norm of v is close to 1, viz. $1 - 2\varepsilon < \|v\| < 1 + \varepsilon$. Put $w = v/\|v\|$. For this w we have $\|w - u\| < 3\varepsilon$, so $w \in S(x^*, 4\varepsilon)$ and $\|w + x\| > 2 - 4\varepsilon$. □

This theorem leads to new hereditary properties for the Daugavet property.

Proposition 5.3. *Suppose Y is a subspace of a Banach space X with the Daugavet property.*

(a) *If the quotient space X/Y has the Radon-Nikodým property, then Y is rich.*

(b) *If the quotient space X/Y contains no copy of ℓ_1, then Y is rich in X.*

(c) *If $(X/Y)^*$ has the Radon-Nikodým property, then Y is rich.*

In either case Y has the Daugavet property itself.

Proof. (a) follows from Theorem 3.13, (b) from Theorem 4.13, and (c) follows from (b). □

That Y has the Daugavet property under assumption (a) has been proved earlier in [21].

Remark 5.4. If the quotient map q: $X \to X/Y$ belongs to $\mathrm{cp}(\mathscr{NAR}(X))$, then the restriction to Y of every narrow operator on X is a narrow operator itself. If Y is a rich subspace of a space X having the Daugavet property, then the restriction to Y of every operator $T \in \mathrm{cp}(\mathscr{NAR}(X))$ is a narrow operator.

Definition 5.5. We say that a subspace Y of a space X with the Daugavet property is *wealthy* if Y and every subspace of X containing Y have the Daugavet property.

It is plain that if Y is an (almost) rich subspace of a space X with the Daugavet property, then every bigger subspace is (almost) rich, too. Thus, if Y is rich, then it is wealthy. We now investigate the converse implication.

Lemma 5.6. *The following conditions for a subspace Y of a Banach space X with the Daugavet property are equivalent:*

(i) *Y is wealthy.*

(ii) *Every finite-codimensional subspace of Y is wealthy.*

(iii) *For every pair $x, y \in S(X)$, the linear span of Y, x and y has the Daugavet property.*

(iv) *For every $x, y \in S(X)$, for every $\varepsilon > 0$ and for every slice S of $S(X)$ which contains y there is an element $v \in \operatorname{lin}(\{x, y\} \cup Y) \cap S$ such that $\|x + v\| > 2 - \varepsilon$.*

Proof. Due to Proposition 5.3 every finite-codimensional subspace of a space with the Daugavet property has the Daugavet property itself (see also [13, Th. 2.14]); this is the reason for the equivalence of (i) and (ii). The implication (i) $\Rightarrow$ (iii) follows immediately from the definition of a wealthy subspace; (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are consequences of Lemma 1.1. $\square$

Let us say that a pair of elements $x, y \in S(X)$ is *ε-fine* if there is a slice S of $S(X)$ which contains y and the diameter of $S \cap \operatorname{lin}\{x, y\}$ is less than ε.

Lemma 5.7. *Let Y be a wealthy subspace of a Banach space X with the Daugavet property and let a pair $x, y \in S(X)$ be ε-fine. Then Y intersects $D(x, y, 2\varepsilon)$.*

Proof. First of all let us fix a slice $S = S(x^*, \varepsilon_1)$ from the definition of an ε-fine pair and fix a $\delta > 0$ such that the set $W = \{w \in \operatorname{lin}\{x, y\}\colon \|w\| < 1 + \delta, x^*(w) > 1 - \varepsilon_1\}$ still has diameter less than ε. Now let us find a finite-codimensional subspace $E \subset Y$ such that

1. $x^* = 0$ on E,
2. if $e \in E$ and $w \in \operatorname{lin}\{x, y\}$, then $\|w\| < (1 + \delta)\|e + w\|$;

the last condition can be satisfied by a variant of the Mazur argument leading to the basic sequence selection principle; see [10, Lemma 6.3.1]. According to our assumptions $\operatorname{lin}(\{x, y\} \cup E)$ has the Daugavet property. So there is an element $v \in \operatorname{lin}(\{x, y\} \cup E) \cap S$ such that $\|x + v\| > 2 - \varepsilon$. Let us represent v in the form $v = e + w$, where $e \in E$, $w \in \operatorname{lin}\{x, y\}$. By choice of E this means that $\|w\| < 1 + \delta$ and $x^*(w) = x^*(v) > 1 - \varepsilon_1$. Thus, $w \in W$ and $\|y - w\| < \varepsilon$. Finally we have that the element e belongs to $E \cap D(x, y, 2\varepsilon)$, which concludes the proof. $\square$

Let us recall the following result [13], which can also be deduced from our Proposition 4.12:

Lemma 5.8. *Let X be a Banach space with the Daugavet property and $Z \subset X$ be a finite-dimensional subspace. Then, for every $\varepsilon > 0$ in every slice of the unit sphere of X there is an element x such that*

$$\|z + x\| > (1 - \varepsilon)(\|z\| + \|x\|) \qquad \forall z \in Z.$$

We now present two easy lemmas.

Lemma 5.9. *A subspace Y of a Banach space with the Daugavet property which is almost rich together with all of its 1-codimensional subspaces is rich.*

Proof. Let $q\colon X \to X/Y$ be the quotient map and let $x^* \in S(X^*)$; further let $Y_1 = Y \cap \ker x^*$ and let $q_1\colon X \to X/Y_1$ be the corresponding quotient map. Then $Y_1 = Y$ or Y_1 is 1-codimensional in Y. Now, in either case we have $\|q(x)\|+|x^*(x)| \le 2\|q_1(x)\|$ for all $x \in X$. Since q_1 is a strong Daugavet operator by assumption, so is $q \tilde{+} x^*$, and q is narrow. ☐

Lemma 5.10. *A subspace Y of a Banach space X with the Daugavet property is almost rich if and only if Y intersects all the elements of $\mathscr{D}(X)$.*

Proof. If Y intersects all the elements of $\mathscr{D}(X)$, then the quotient map q: $X \to X/Y$ is unbounded from below on every element of $\mathscr{D}(X)$. So the quotient map belongs to $\mathscr{D}(X)^{\sim}$ which coincides with the class of strong Daugavet operators by Proposition 3.4.

Now consider the converse statement. If Y is almost rich, then for every $\varepsilon > 0$ the map q is unbounded from below on every set of the form $D(x,y,\varepsilon/2)$. This means that there is an element $z \in Y$ for which $\operatorname{dist}(z, D(x,y,\varepsilon/2)) < \varepsilon/2$. In this case z belongs to $D(x,y,\varepsilon)$, so the intersection of this set with Y is non-empty. ☐

The following is the key result for establishing that wealthy subspaces are rich.

Lemma 5.11. *Every wealthy subspace Y of a Banach space X having the Daugavet property is almost rich.*

Proof. According to Lemma 5.10 we need to prove that for every positive $\varepsilon < 1/10$ and every pair $x, y \in S(X)$ the subspace Y intersects $D(x,y,\varepsilon)$. To do this, according to Lemma 5.7, it is enough to show that for every $\varepsilon > 0$ and every pair $x, y \in S(X)$ there is an ε-fine pair $x_1, y_1 \in S(X)$ which approximates (x,y) well; i.e., $\|x - x_1\| + \|y - y_1\| < \varepsilon$. Let us fix a positive $\delta < \varepsilon^2/8$ and select an element $z \in S(X)$ in such a way that for every $w \in \operatorname{lin}\{x,y\}$ and for every $t > 0$

$$\|w + tz\| \ge (1-\delta)(\|w\| + |t|)$$

(we use Lemma 5.8). Put $x_1 = x + \varepsilon z$, $y_1 = y$. To show that (x_1, y) is an ε-fine pair it is sufficient to demonstrate that, for every $v \in \operatorname{lin}\{x_1, y\}$ with $\|v\| \ge \varepsilon$, $\max\{\|y+v\|, \|y-v\|\} > 1$. To do this let us argue ad absurdum. Take some $v = ay + b(x + \varepsilon z)$ with $\|v\| \ge \varepsilon$ and assume that $\max\{\|y+v\|, \|y-v\|\} = 1$. Then

$$\begin{aligned} 1 &= \max\{\|y + ay + b(x+\varepsilon z)\|, \|y - ay - b(x + \varepsilon z)\|\} \\ &\ge (1-\delta)(\max\{\|y + ay + bx + \|, \|y - ay - bx\|\} + |b|\varepsilon) \\ &\ge (1-\delta)(1 + |b|\varepsilon). \end{aligned}$$

So $|b| \le \varepsilon/4$. But in this case $|a| > \varepsilon/2$ and

$$\max\{\|y+v\|, \|y-v\|\} > \max\{\|y+ay\|, \|y-ay\|\} - \varepsilon/3 > 1 + \varepsilon/6,$$

which provides a contradiction. ☐

Theorem 5.12. *The following properties of a subspace Y of a Daugavet space X are equivalent:*

(i) *Y is wealthy.*

(ii) *Y is rich.*

(iii) *Every finite-codimensional subspace of Y is rich.*

Proof. It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), see the remark following Definition 5.5. Now suppose (i). Every 1-codimensional subspace of Y is wealthy by Lemma 5.6 and is hence almost rich by Lemma 5.11. An appeal to Lemma 5.9 completes the proof. $\square$

6. Operators on L_1

In this section we shall study strong Daugavet and narrow operators on L_1. We first introduce a technical definition.

Let (Ω, Σ, μ) be an atomless probability space. A function $f \in L_1 = L_1(\mu)$ is said to be a *balanced ε-peak* on $A \in \Sigma$ if $f \geq -1$, $\operatorname{supp} f \subset A$, $\int_\Omega f\, d\mu = 0$ and $\mu\{t\colon f(t) = -1\} > \mu(A) - \varepsilon$. The collection of all balanced ε-peaks on A will be denoted by $P(A, \varepsilon)$.

Theorem 6.1. $\mathscr{NAR}(L_1) = \{P(A,\varepsilon)\colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$.

Proof. Let $T \in \mathscr{NAR}(L_1)$, $\delta, \varepsilon > 0$, and $A \in \Sigma$. Consider a slice in L_1 of the form

$$S = \Big\{ f \in B(L_1)\colon \int_A f\, d\mu > 1 - \delta \Big\}.$$

Applying Lemma 3.8 to this slice, the elements $x = -\chi_A/\mu(A)$, $y = \chi_A/\mu(A)$ and δ we get a function $v \in S$ such that

$$\|v - \chi_A/\mu(A)\| > 2 - \delta, \quad \|T(v - \chi_A/\mu(A))\| < \delta. \tag{6.1}$$

Denote by B the set $\{t \in A\colon v(t) > 0\}$. The condition $v \in S$ implies that $\|v - \chi_B v\| < \delta$, so

$$\|v\chi_B - \chi_A/\mu(A)\| > 2 - 2\delta.$$

Next, introduce $C = \{t \in A\colon v(t) > 1/\mu(A)\}$. By the last inequality

$$\|v\chi_C - \chi_A/\mu(A)\| > 2 - 2\delta, \quad \|v - \chi_C v\| < 3\delta$$

and

$$\mu(C) < \delta\mu(A); \tag{6.2}$$

to see this observe that

$$\begin{aligned} 2 - 2\delta < \Big\| \chi_B v - \frac{\chi_A}{\mu(A)} \Big\| &\leq \int_C \Big(\chi_B v - \frac{1}{\mu(A)} \Big)\, d\mu + \frac{1}{\mu(A)}(\mu(A) - \mu(C)) \\ &\leq 2 - 2\frac{\mu(C)}{\mu(A)}. \end{aligned}$$

Put $f = (\mu(A)/\beta)\chi_C v - \chi_A$ with $\beta = \int_C v\, d\mu$ so that $\int_\Omega f\, d\mu = 0$. Since $\int_A v\, d\mu > 1 - \delta$ we have from $\|v - \chi_C v\| < 3\delta$ that $\beta \geq 1 - 4\delta$. By (6.1) we conclude that

$$\|Tf\| = \mu(A) \Big\| T\Big(\frac{\chi_C v}{\beta} - \frac{\chi_A}{\mu(A)} \Big) \Big\| \leq \mu(A) \Big(\|T\| \, \Big\| \frac{\chi_C v}{\beta} - v \Big\| + \delta \Big)$$

and

$$\Big\|\frac{\chi_C v}{\beta} - v\Big\| \le \Big\|\frac{\chi_C v - v}{\beta}\Big\| + \Big\|\frac{v}{\beta} - v\Big\| \le \frac{3\delta}{\beta} + \Big(\frac{1}{\beta} - 1\Big) \le \frac{7\delta}{1-4\delta},$$

and if δ is small enough, by (6.2) $f \in P(A,\varepsilon)$. This proves the inclusion $\mathscr{NAR}(L_1) \subset \{P(A,\varepsilon)\colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$.

To prove the opposite inclusion we use Proposition 3.11. Let us fix $T \in \{P(A,\varepsilon)\colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$. Let $x, y \in S(L_1)$, $y^* \in S(L_\infty)$ and $\varepsilon > 0$ be such that $\langle y^*, y\rangle > 1 - \varepsilon$. Without loss of generality we may assume that there is a partition $A_1, \dots, A_n$ of Ω such that the restrictions of x, y and y^* on A_k are constants, say a_k, b_k and c_k respectively. By our assumption T is unbounded from below on each of the $P(A_k,\delta)$ for every $\delta > 0$, $k = 1, \dots, n$. Let us fix functions $f_k \in P(A_k,\delta)$ such that $\|Tf_k\| < \delta$, $k = 1, \dots, n$, and put

$$v = \sum_{k=1}^{n} b_k(\chi_{A_k} + f_k).$$

By definition of balanced δ-peaks $\langle y^*, v\rangle > 1-\varepsilon$, $\|v\| = 1$, and $\|T(y-v)\|$ and $\mu(\operatorname{supp} v)$ become arbitrarily small when δ is small enough. Thus δ can be chosen so that v fulfills the conditions $\|T(y-v)\| < \varepsilon$ and $\|x+v\| > 2-\varepsilon$. $\square$

The characterisation of narrow operators on L_1 proved above looks similar to the definition of L_1-narrow operators. It is easy to prove that every L_1-narrow operator is narrow. We don't know whether the classes of narrow operators and L_1-narrow operators on L_1 coincide.

The aim of the remainder of this section is to construct an example of a strong Daugavet operator on L_1 which is not narrow. In fact, we shall define a subspace $Y \subset L_1[0,1]$ so that the quotient map $q\colon L_1 \to L_1/Y$ is a strong Daugavet operator, but Y fails the Daugavet property. By Theorem 5.2, q cannot be narrow. Likewise, Y is almost rich, but not rich.

Let $I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n \in \mathbb{N}_0$ and $k = 1, 2, \dots, 2^n$. Fix $N \in \mathbb{N}$. We define

$$\begin{aligned}
g_{0,1} &= (2^N - 1)\chi_{I_{N,1}} - \chi_{I_{0,1}\setminus I_{N,1}} \\
g_{1,1} &= (2^{N^2-N} - 1)\chi_{I_{N^2,1}} - \chi_{I_{N,1}\setminus I_{N^2,1}} \\
g_{1,k} &= g_{1,1}(t - \tfrac{k-1}{2^n}), \qquad k = 2, \dots, 2^N, \\
&\ \ \vdots \\
g_{n,1} &= (2^{N^{n+1}-N^n} - 1)\chi_{I_{N^{n+1},1}} - \chi_{I_{N^n,1}\setminus I_{N^{n+1},1}} \\
g_{n,k} &= g_{n,1}(t - \tfrac{k-1}{2^{N^n}}), \qquad k = 2, \dots, 2^{N^n}.
\end{aligned}$$

Denote by P_n the "peak set" of the n'th generation, i.e.,

$$P_n = \Big\{t \in [0,1]\colon \sum_{k=1}^{2^{N^n}} g_{n,k}(t) > 0\Big\},$$

and $P = \bigcup_n P_n$. Clearly $|P_n| = 2^{N^n}/2^{N^{n+1}} = \left(1/2^{N-1}\right)^{N^n}$ and $|P| \leq 1/(2^N-1)$. Notice also that $\int_0^1 g_{n,k}(t)\,dt = 0$ for all n and k.

First we formulate a lemma. All the norms appearing below are L_1-norms.

Lemma 6.2. *Let*

$$g = \sum_{n=0}^{M} \sum_{k=1}^{2^{N^n}} a_{n,k} g_{n,k}.$$

Then

$$\|g\chi_{[0,1]\setminus P}\| \leq 3\|g\chi_P\|.$$

Proof. Denote

$$g'' = \sum_{\operatorname{supp} g_{n,k} \subset P} a_{n,k} g_{n,k}, \quad g' = g - g''.$$

Since g' and g coincide off P, we clearly have

$$\|g'\chi_{[0,1]\setminus P}\| = \|g\chi_{[0,1]\setminus P}\|. \tag{6.3}$$

We also have that

$$\|g'\chi_P\| \leq \|g\chi_P\|. \tag{6.4}$$

Indeed, we can write P as a countable union of disjoint (half-open) intervals; denote by I any one of these. Then g' is constant on I, and $\int_0^1 g''(t)\,dt = 0$. Hence

$$\|g'\chi_I\| = \left|\int_0^1 g'(t)\chi_I(t)\,dt\right| = \left|\int_0^1 (g'(t)\chi_I(t) + g''(t)\chi_I(t))\,dt\right| \leq \|g\chi_I\|.$$

Summing up over all I gives the result.

Next, we claim that

$$\|g'\chi_{[0,1]\setminus P}\| \leq 3\|g'\chi_P\|. \tag{6.5}$$

To see this, we label the intervals I from the previous paragraph as follows. For every $l \in \mathbb{N}$ write $B_0 = P_0$ and $B_l = P_l \setminus \bigcup_{i=1}^{l-1} P_i$. Each B_l can be written as $\bigcup_{d \in D_l} I_{N^{l+1},d}$ where D_l is some subset of $\{1, \ldots, 2^{N^{l+1}}\}$ with cardinality $< 2^{N^l}$. Let us write $g' = \sum_{n=0}^{M} \sum_{k=1}^{2^{N^n}} b_{n,k} g_{n,k}$. We then have the estimates

$$\int_0^1 |g'(t)\chi_{B_0}(t)|\,dt = |b_{0,1}|\frac{2^N-1}{2^N}$$

and

$$\begin{aligned}\int_0^1 |g'(t)\chi_{B_l}(t)|\,dt &= \sum_{d\in D_l}\int_{I_{N^{l+1},d}}\Big|-b_{0,1}-\sum_{n=1}^{l-1}\sum_{k=1}^{2^{N^n}} b_{n,k}\chi_{\operatorname{supp} g_{n,k}} \\ &\qquad + b_{l,(d-1)/(2^{N-1})^{N^l}+1}\big(2^{N^{l+1}-N^l}-1\big)\Big|\,dt \\ &\geq \sum_{k=1}^{2^{N^l}}\Big(\frac{1}{2^{N^l}}-\frac{1}{2^{N^{l+1}}}\Big)|b_{l,k}| \\ &\qquad -\frac{1}{(2^{N-1})^{N^l}}|b_{0,1}|-\frac{1}{(2^{N-1})^{N^l}}\sum_{n=1}^{l-1}\sum_{k=1}^{2^{N^l}}|b_{n,k}|.\end{aligned}$$

Summing up over all l gives us

$$\begin{aligned}\int_P |g'(t)|\,dt &\geq |b_{0,1}|\Big(\frac{2^N-1}{2^N}-\sum_{m=1}^{\infty}\frac{1}{(2^{N-1})^{N^m}}\Big) \\ &\qquad +\sum_{l=1}^{\infty}\Big(\frac{1}{2^{N^l}}-\frac{1}{2^{N^{l+1}}}-\sum_{m=l+1}^{\infty}\frac{1}{(2^{N-1})^{N^m}}\Big)\sum_{k=1}^{2^{N^l}}|b_{l,k}| \\ &\geq \frac12|b_{0,1}|+\frac12\sum_{l=1}^{\infty}\frac{1}{2^{N^l}}\sum_{k=1}^{2^{N^l}}|b_{l,k}|.\end{aligned}$$

On the other hand, by the triangle inequality

$$\int_0^1 |g'(t)|\,dt \leq 2\Big(|b_{0,1}|+\sum_{l=1}^{\infty}\frac{1}{2^{N^l}}\sum_{k=1}^{2^{N^l}}|b_{l,k}|\Big),$$

hence the claim follows.

The lemma now results from (6.3)–(6.5) $\square$

Theorem 6.3. *Let $Y_N \subset L_1[0,1]$ be the closed subspace generated by the system $\{g_{n,k}\}$ and the constants. Then the quotient map $q_N\colon L_1 \to L_1/Y_N$ is a strong Daugavet operator for all N, but Y_N fails the Daugavet property if $N \geq 4$.*

Proof. Let us fix $x, y \in S(L_1)$ and $\varepsilon > 0$. Without loss of generality we may assume that $x = \sum_{k=1}^{2^{N^n}} a_{n,k}\chi_{I_{n,k}}$ for a big enough n to be chosen later.

Put $h = \sum_{k=1}^{2^{N^n}} a_{n,k}g_{n,k}$. Then

$$x + h = \sum_{k=1}^{2^{N^n}} 2^{N^{n+1}-N^n}\chi_{N^{n+1},d_{n,k}}a_{n,k}$$

with $d_{n,k} = 1 + (k-1)(2^{N-1})^{N^n}$. So

$$\|x+h\| = \sum_{k=1}^{2^{N^n}} \frac{|a_{n,k}|}{2^{N^n}} = \|x\| = 1,$$

and $\mathrm{supp}(x+h) \subset P_n$. Since $|P_n| \to 0$ we can pick n big enough to satisfy $\|x+h+y\| > 2-\varepsilon$. This shows that q_N is a strong Daugavet operator.

To show that Y_N fails the Daugavet property if $N \geq 4$, take $g^* = \chi_{[0,1]\setminus P} \in Y_N^*$ and $\varepsilon = 2|P|$. Since $\mathbf{1} \in S(Y_N)$, we get

$$\|g^*\| \geq g^*(\mathbf{1}) = 1 - \varepsilon/2 > 1 - \varepsilon.$$

Thus, $S(g^*, \varepsilon) \cap B(Y_N) \neq \emptyset$. We show that there is no f in this slice such that $\|f - \mathbf{1}\| > 2 - \varepsilon$.

Suppose, on the contrary, that there is such an f. Without loss of generality we can assume that

$$f = a_0\mathbf{1} + g$$

where g is as in Lemma 6.2.

It follows from our conditions that

$$\|f\chi_P\| = \int_P |f(t)|\,dt = \|f\| - g^*(|f|) \leq 1 - g^*(f) < \varepsilon. \tag{6.6}$$

Hence,

$$1 \geq \int_0^1 f(t)\,dt = \int_P f(t)\,dt + g^*(f) > 1 - 2\varepsilon,$$

and since $\int_0^1 f(t)\,dt = a_0$, we get

$$1 - 2\varepsilon < a_0 \leq 1. \tag{6.7}$$

By (6.6) and (6.7),

$$\|g\chi_P\| \leq \varepsilon + |P| < 2\varepsilon, \tag{6.8}$$

thus (6.7) and (6.8) yield

$$\|g\chi_{[0,1]\setminus P}\| \geq \|g\| - 2\varepsilon = \|f - a_0\mathbf{1}\| - 2\varepsilon \geq \|f - \mathbf{1}\| - 4\varepsilon > 2 - 5\varepsilon.$$

But now Lemma 6.2 and (6.8) imply

$$2 - 5\varepsilon < \|g\chi_{[0,1]\setminus P}\| \leq 3\|g\chi_P\| < 6\varepsilon,$$

which yields $\varepsilon > 2/11$, i.e., $|P| > 1/11$, which is false for $N \geq 4$. □

Theorems 6.3 and 5.2 immediately yield the following result.

Theorem 6.4. *There is an almost rich subspace of $L_1[0,1]$ which fails the Daugavet property and hence fails to be rich. Thus, on $L_1[0,1]$ the class of strong Daugavet operators does not coincide with the class of narrow operators.*

7. Questions

We finish this paper with some questions which have remained open. We intend to deal with these problems in a future publication.

1. Does the class of narrow operators on a Banach space X form a subsemigroup of $\mathscr{OP}(X)$? [Added in the final version: We have recently constructed a counterexample on the space $X = C([0,1], L_1[0,1])$.]
2. Is every narrow operator on L_1 also L_1-narrow?
3. Is the sum of two L_1-narrow operators from L_1 to L_1 again L_1-narrow? This question is clearly related to the previous ones; we remark that the proof in [17, p. 69] which purportedly shows this to be true appears to have a gap.
4. If X has the Daugavet property, does X have a subspace isomorphic to ℓ_2?
5. If T is an operator on a space X with the Daugavet property which does not fix a copy of ℓ_2, is T then narrow? We remark that the answer is affirmative in the case $X = C[0,1]$ by our Theorem 4.13 and a result due to Bourgain [3].

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