Wavelet characterizations for anisotropic Besov spaces: case $0 < p < 1$

Gustavo Garrigós* Reinhard Hochmuth Anita Tabacco

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1 Introduction

In previous works [8, 9, 7] the authors have studied independently the characterizion of anisotropic Besov spaces $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ by means of wavelet decompositions. The interest of this question comes from applications to non-linear approximation and numerical resolution to semi-elliptic linear and non-linear PDE's (see [7, 9, 10]). In our previous works, some limitations appeared that either restricted our results to values of $p, q \geq 1$ [7], or forced us to use "non-homogeneous" MRA's (i.e., MRA's not necessarily scaling via a dilation matrix M) [8, 9]. The goal of the present note is to overcome these limitations and give a complete answer for the whole range $0 < p, q < \infty$, within the MRA setting introduced in [7]. (For another approach we refer the reader to [1].)

To be more precise, we consider multilevel decompositions of the form

$$f = P_0 f + \sum_{j=0}^{\infty} Q_j f, \quad f \in L^p(\mathbb{R}^n),\ 0 < p < \infty.$$

The operators P_j and $Q_j := P_{j+1} - P_j$ denote the projectors associated with an "admissible" multiresolution analysis, $\{V_j\}_{j\in\mathbb{Z}}$, which is assumed to have a sufficiently smooth compactly supported scaling function. In order to analyze spaces in the family $B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n)$, $\alpha > 0$, where $\boldsymbol{a} = (a_1, \dots, a_n) > 0$ is a fixed anisotropy, we consider only MRA's whose scales change according to the dilation matrix:

$$M = \operatorname{diag}\,(\lambda^{\frac{1}{a_1}}, \dots, \lambda^{\frac{1}{a_n}}), \quad \text{for some } \lambda > 1 \tag{1.1}$$

(so they give the right homogeneity: $|\det M|^{\frac{1}{p}}\, |f(M\,\cdot)|_{B^{\alpha\boldsymbol{a}}_{p,q}} = \lambda^{\alpha} |f|_{B^{\alpha\boldsymbol{a}}_{p,q}}$). We then can prove the following theorem which is the main result of this note.

Theorem 1.2 *Let $0 < p, q < \infty$ and $\alpha > 0$. Then, for an MRA as above with a sufficiently smooth scaling function we have:*

$$B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}^n) \ :\ \sum_{j=0}^{\infty} \lambda^{\alpha j q} \|Q_j f\|_p^q < \infty \Big\}.$$

Moreover, we have the equivalence of norms:

$$\|f\|_{B^{\alpha\boldsymbol{a}}_{p,q}} \sim \|P_0 f\|_p + \Big[\sum_{j=0}^{\infty} \lambda^{\alpha j q} \|Q_j f\|_p^q \Big]^{\frac{1}{q}}. \tag{1.3}$$

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The same theorem was proven in [7], for $1 \le p, q < \infty$, however the arguments used there do not extend to the range $0 < p, q < 1$. On one hand, certain interpolation results for anisotropic Besov spaces seem not to appear in the existing literature. On the other hand, the proofs rely on the boundedness of the projectors P_j in $L^p(\mathbb{R}^n)$, which no longer holds if $p < 1$. This kind of difficulties already appear in the isotropic case, being succesfully handled for the first time in the celebrated paper [6].

In the work of the third author [8, 9], the same question was taken up, but this time for Besov spaces in the unit square $B^{\boldsymbol{\alpha}}_{p,q}([0,1]^2)$. The difficulties of the quasi-Banach situation were bridged with appropriate modifications of the ideas in [6]. There is, however, one main difference in the approach introduced in [9]. Namely, the family of spaces $\{V^\sharp_{\mathbf{j}_n}\}_{n\in\mathbb{Z}}$ considered in [9] does not form strictly speaking a multiresolution analysis (i.e., dilated according to a matrix M). It is instead an appropriately chosen collection of subspaces from the tensor product of two 1-dimensional MRA's: $\left\{V^\sharp_{\mathbf{j}} := V^1_{j_1} \otimes V^2_{j_2} \mid \mathbf{j} = (j_1, j_2) \in \mathbb{Z}^2\right\}$. This also forces the scaling functions to be tensor products of 1-dimensional ones.

In this paper, we intend to combine techniques from our previous works to obtain results valid for very general multiresolution analysis, with minimal smoothness assumptions. This requires an appropriate description of the projector P_j when $0 < p < 1$ (as an almost-best approximation to the space $V^{(p)}_j$) similar to the one introduced in [2] for the 1-dimensional case. Further, we shall deduce interpolation properties and an embedding theorem for anisotropic Besov spaces as in [9]. As in the previous papers [7, 9], the main tools will be two inequalities of Bernstein and Jackson type for $\{V_j\}_{j\in\mathbb{Z}}$. From them and appropriate modifications of the method of DeVore-Popov [6, 2] we shall obtain our final results.

2 The general framework

2.1 Besov spaces

Let us fix $0 < p, q < \infty$ and $\boldsymbol{a} = (a_1, \dots, a_n)$. We assume $a_j > 0$ and $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = n$. Given $\alpha > 0$ we write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) = \alpha\boldsymbol{a}$.

Let $k_1, \dots, k_n$ be positive integers so that $k_j > \alpha_j$, $j = 1, \dots, n$, and let $\{\boldsymbol{e}_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$. Then, we say that a function $f \in L^p(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$ whenever

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}} := \sum_{j=1}^n \left[\int_0^\infty \left(t^{-\alpha_j}\|\Delta^{k_j}_{t\boldsymbol{e}_j} f\|_p\right)^q \frac{dt}{t}\right]^{\frac{1}{q}} < \infty. \tag{2.1}$$

As usual, we denote by $\Delta^k_h f$ the k^{th}-iteration of the increment operator: $\Delta_h f(x) = f(x+h) - f(x)$. Standard computations involving the modulus of continuity show that the quasi-norms $\|f\|_{B^{\boldsymbol{\alpha}}_{p,q}} = \|f\|_p + |f|_{B^{\boldsymbol{\alpha}}_{p,q}}$ are equivalent for any choice of the integers k_j as above [5, 9]. One more equivalent quasi-norm, which will be useful below, is obtained by replacing (2.1) with

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}} = \sum_{|\boldsymbol{\ell}|=L} \left[\int_{(0,\infty)^n} \left(|h|_{\boldsymbol{a}}^{-\alpha}\|\Delta^{\boldsymbol{\ell}}_h f\|_p\right)^q \frac{dh}{|h|^n_{\boldsymbol{a}}}\right]^{1/q}, \tag{2.2}$$

for any integer $L > \max\{\alpha_1, \dots, \alpha_n\}$. Here we recall the standard notation

$$\Delta^{\boldsymbol{\ell}}_h f(x) = (\Delta^{\ell_1}_{h_1\boldsymbol{e}_1} \circ \ldots \circ \Delta^{\ell_n}_{h_n\boldsymbol{e}_n}) f(x), \quad \boldsymbol{\ell} = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$$

$$|h|_{\boldsymbol{a}} = \sum_{j=1}^n |h_j|^{a_j}, \quad h = (h_1, \dots, h_n) \in \mathbb{R}^n.$$

A final equivalent quasi-seminorm is a discrete version of (2.2)

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}} = \sum_{|\boldsymbol{\ell}|=L} \left[\sum_{j\in\mathbb{Z}} \left(\lambda^{j\alpha}\omega^{\boldsymbol{\ell}}_p(f; \lambda^{-j})\right)^q\right]^{1/q}, \tag{2.3}$$

where $\omega^{\boldsymbol{\ell}}_p(f;t) = \sup_{|h|_{\boldsymbol{a}} \le t} \|\Delta^{\boldsymbol{\ell}}_h f\|_p, \quad t > 0.$

2.2 Multiresolution analyses in $L^p(\mathbb{R})$, $0 < p < \infty$

In this section we follow the same setting as in [7]. Let M be a dilation matrix as in (1.1) which we shall assume to have integer entries. We point out that such assumption forces a restriction in the multi-indices: $\boldsymbol{a} \in \mathbb{R}_+ \log \mathbb{Z}_+^n$ (see §3.3 in [7]). Note also that, in our notation $m := |\det M| = \lambda^n$.

Let $\varphi, \tilde{\varphi}$ be a pair of biorthogonal M-scaling functions, which throughout the paper we shall assume continuous and compactly supported. The multiresolution analysis generated by φ is defined as usual:

$$V_j^{(p)} := \overline{\operatorname{span}}_{L^p}\{\varphi_{j,k}^{(p)} \equiv |\det M|^{j/p}\varphi(M^j \cdot -k) : k \in \mathbb{Z}^n\}, \qquad j \in \mathbb{Z}.$$

Note that $V_j^{(p)} = T^j V_0^{(p)}$, where T is the isometry in $L^p(\mathbb{R}^n)$ given by $Tf(x) = |\det M|^{1/p} f(Mx)$. A key property in what follows is the p-stability of the generator of $V_j^{(p)}$.

Proposition 2.4 *In the conditions above, $\{\varphi_k := \varphi_{0,k}\}_{k\in\mathbb{Z}^n}$ is a p-stable basis of $V_0^{(p)}$. That is, every $f \in V_0^{(p)}$ can be written uniquely as $f = \sum_{k\in\mathbb{Z}^n} c_k\varphi_k$ and*

$$\|\sum_{k\in\mathbb{Z}^n} c_k\varphi_k\|_{L^p(\mathbb{R}^n)} \sim \Big(\sum_{k\in\mathbb{Z}^n} |c_k|^p\Big)^{\frac{1}{p}}.$$

Proof: The proof for $1 \le p < \infty$ is standard (see the Appendix in [7]). We assume therefore $0 < p < 1$. Then, for every sequence $\{c_k\}_{k\in\mathbb{Z}^n} \in \ell^p(\mathbb{Z}^n)$, we have

$$\int_{\mathbb{R}^n} |\sum_{k\in\mathbb{Z}^n} c_k\varphi(x-k)|^p\,dx \le \sum_{k\in\mathbb{Z}^n} |c_k|^p\,\|\varphi\|_{L^p}^p.$$

Thus, the series $\sum_{k\in\mathbb{Z}^n} c_k\varphi_k$ converges in L^p and defines a function in $V_0^{(p)}$. Conversely, let $f = \sum_{k\in\mathbb{Z}^n} c_k\varphi_k \in V_0^{(p)}$, for a finite sequence of scalars $\{c_k\}$. Then, the biorthogonality gives $c_k = \langle f, \tilde{\varphi}_k\rangle$, and therefore

$$|c_k| \le \int_{\mathbb{R}^n} |f|\,|\tilde{\varphi}_k| \le \|\tilde{\varphi}\|_\infty \int_{\widetilde{J}_k} |f|,$$

where $\widetilde{J}_k = \operatorname{Supp}\tilde{\varphi}_k = \widetilde{J}_0 + k$, $k \in \mathbb{Z}^n$, is a compact set. Now, since

$$V_0^{(p)}|_{\widetilde{J}_k} := \Big\{f|_{\widetilde{J}_k} \;:\; f \in V_0^{(p)}\Big\} = \operatorname{span}\Big\{\varphi(\cdot-\ell)|_{\widetilde{J}_k} \;:\; \ell \in \mathbb{Z}^n\Big\}$$

is a finite dimensional space, we must have

$$\int_{\widetilde{J}_k} |f| \le c\,\Big(\int_{\widetilde{J}_k} |f|^p\Big)^{\frac{1}{p}}, \quad \forall f \in V_0^{(p)},$$

with a constant $c > 0$ independent of f and k. Thus,

$$\sum_{k\in\mathbb{Z}^n} |c_k|^p \le c \sum_{k\in\mathbb{Z}^n} \int_{\widetilde{J}_k} |f|^p \lesssim \int_{\mathbb{R}^n} |f|^p,$$

at least for $f \in \operatorname{span}\{\varphi_k\}$. A density argument establishes the proposition. □

Let us now construct the projectors onto $V_j^{(p)}$. When $p \ge 1$, there is a standard definition:

$$P_j = P_j^{(p)} : L^p(\mathbb{R}^n) \longrightarrow V_j^{(p)}, \qquad f \longmapsto P_j f = \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}_{j,k}^{(p')}\rangle \varphi_{j,k}^{(p)}.$$

These are bounded linear operators satisfying:

(1) $\|f - P_j f\|_p \le C\, E_j^{(p)}(f) := C \inf_{g \in V_j^{(p)}} \|f - g\|_p$;

(2) $\|f - P_j f\|_p \to 0$, as $j \to +\infty$.

(3) $\|P_j f\|_p \to 0$, as $j \to -\infty$ (only when $1 < p < \infty$).

This leads to a *multilevel decomposition* of every $f \in L^p(\mathbb{R}^n)$:

$$f = P_0 f + \sum_{j=0}^{\infty} Q_j f \quad \Big(= \sum_{j \in \mathbb{Z}} Q_j f, \quad \text{if } p > 1 \Big)$$

where $Q_j = Q_j^{(p)} := P_{j+1} - P_j$, and the convergence is in the L^p-norm. As it is shown in [7], there exist compactly supported functions $\psi_1, \tilde{\psi}_1, \ldots, \psi_{m-1}, \tilde{\psi}_{m-1}$, so that $\{\psi_{a;j,k}^{(p)}\}_{a,k}$ forms a p-stable basis of $W_j^{(p)} = \operatorname{Im} Q_j$, with dual basis $\{\tilde{\psi}_{a;j,k}^{(p')}\}_{a,k}$. From these properties one can develop a wavelet analysis for functions $f \in L^p(\mathbb{R}^n)$, that then is carried out to the spaces $B_{p,q}^{\boldsymbol{\alpha}}(\mathbb{R}^n)$ [7].

The situation when $0 < p < 1$ is somewhat different, but still can be pursued provided we find projectors P_j onto $V_j^{(p)}$ satisfying at least conditions (1) and (2) above (see the abstract framework in [2]). One first constructs the space of "piecewise $V_0^{(p)}$-functions":

$$\mathcal{V}_0^{(p)} = \Big\{ g \in L^p(\mathbb{R}^n) \quad | \quad \forall \ell \in \mathbb{Z}^n, \ \exists f^\ell \in V_0^{(p)} \ : \ g|_{I_\ell} = f^\ell|_{I_\ell} \Big\},$$

where we denote $I_\ell = [0,1]^n + \ell$, $\ell \in \mathbb{Z}^n$. Note that $\mathcal{V}_0^{(p)}$ is *a priori* much larger than $V_0^{(p)}$. Now, we define a mapping assigning to every $f \in L^p(\mathbb{R}^n)$ a function $R_0 f \in \mathcal{V}_0^{(p)}$ so that

$$\|f - R_0 f\|_{L^p(I_\ell)} \le 2 \inf_{g \in V_0^{(p)}} \|f - g\|_{L^p(I_\ell)}, \quad \forall \ell \in \mathbb{Z}^n. \tag{2.5}$$

That is, in each cube I_ℓ, $R_0 f$ is an *almost-best approximation of f in $V_0^{(p)}$*. Note that $R_0 f = f$ if $f \in V_0^{(p)}$, but R_0 is not necessarily continuous, nor even linear. Still, from (2.5) it easily follows that R_0 is bounded:

$$\|R_0 f\|_{L^p(\mathbb{R}^n)}^p \le \|f\|_{L^p(\mathbb{R}^n)}^p + \sum_{\ell \in \mathbb{Z}^n} \|f - R_0 f\|_{L^p(I_\ell)}^p \le 3\, \|f\|_{L^p(\mathbb{R}^n)}^p. \tag{2.6}$$

We can now define a projector $P_0 = P_0^{(p)}$ onto $V_0^{(p)}$, composing R_0 with the standard orthogonal projector onto $V_0^{(2)}$:

$$f \in L^p(\mathbb{R}^n) \longmapsto P_0 f := \sum_{k \in \mathbb{Z}^n} \langle R_0 f, \tilde{\varphi}_k \rangle \varphi_k.$$

Indeed, the same argument we gave in the proof of Proposition 2.4, together with (2.6), shows that $P_0 f \in V_0^{(p)}$, and moreover

$$\|P_0 f\|_p \lesssim \|f\|_p, \quad f \in L^p(\mathbb{R}^n). \tag{2.7}$$

Note, further, that again $P_0 f = f$ if $f \in V_0^{(p)}$. Finally, an elementary computation using (2.5), (2.7) and the quasi-triangular inequality gives

$$\|f - P_0 f\|_p \lesssim \inf_{g \in V_0^{(p)}} \|f - g\|_p, \quad \forall f \in L^p(\mathbb{R}^n)$$

(see (6.33) in [2]). Therefore, letting $P_j := P_j^{(p)} = T^j P_0 T^{-j}$, $j \in \mathbb{Z}$, we have defined a family of projectors onto $V_j^{(p)}$ satisfying condition (1) above. Condition (2) will be a consequence of a Jackson-type inequality that we shall show in the next section.

3 Bernstein and Jackson inequalities

Here we show the key inequalities in this paper, from which we shall derive our results below. These inequalities will hold for any of our "admissible" MRA's, provided their corresponding scaling functions are smooth enough. The simpler one to state is the following:

PROPOSITION 3.1 (Bernstein-type inequality) *If $\varphi \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)$, then*

$$|f|_{B^{\boldsymbol{\alpha}}_{p,q}} \lesssim \lambda^{\alpha j} \|f\|_{L^p}, \quad \forall f \in V_j^{(p)}. \tag{3.2}$$

PROOF: Let us choose an integer $L > \max\{\alpha_1, \ldots, \alpha_n\}$. By scaling, it suffices to show (3.2) for $j = 0$. We only include the proof for $0 < p < 1$ (the other case is similar, see [7]). If $f = \sum_{k \in \mathbb{Z}^n} c_k \varphi_k \in V_0^{(p)}$, then for every $|h| \leq 1$ we have

$$\begin{aligned} \|\Delta_h^L f\|_p^p &= \sum_\gamma \int_{I+\gamma} |\Delta_h^L f(x)|^p \, dx \\ &\leq \sum_\gamma \sum_{m \in K_*} |c_{\gamma+m}|^p \int_I |\Delta_h^L \varphi_m(x)|^p \, dx = C \, \|\Delta_h^L \varphi\|_p^p \, \|f\|_p^p , \end{aligned}$$

where K_* denotes a finite set of indices in $\mathbb{Z}^n$ (independent of f or h). Thus,

$$\begin{aligned} |f|_{B^{\boldsymbol{\alpha}}_{p,q}} &\leq \sum_{i=1}^n \Big\{ \Big[\int_0^1 \big(t^{-\alpha_i} \|\Delta^L_{t\mathbf{e}_i} f\|_p\big)^q \frac{dt}{t} \Big]^{1/q} + \Big[\int_1^\infty \big(t^{-\alpha_i} \|\Delta^L_{t\mathbf{e}_i} f\|_p\big)^q \frac{dt}{t} \Big]^{1/q} \Big\} \\ &\lesssim \|f\|_p \sum_{i=1}^n \Big\{ \Big[\int_0^1 \big(t^{-\alpha_i} \|\Delta^L_{t\mathbf{e}_i} \varphi\|_p\big)^q \frac{dt}{t} \Big]^{1/q} + \Big[\int_1^\infty \frac{dt}{t^{1+\alpha_i q}} \Big]^{1/q} \Big\} \lesssim \|f\|_p . \end{aligned}$$

This completes the proof of (3.2). □

We assume now that, given an integer $L > \max\{\alpha_1, \ldots, \alpha_n\}$, the space $V_0^{(p)}$ contains locally the polynomials of degree $\leq L-1$. This is the case, for instance, when $\varphi \in H^{L\boldsymbol{\alpha}}(\mathbb{R}^n)$ (see [4]). In general, it is not difficult to see that, just under the assumption $\varphi, \tilde{\varphi} \in C_c(\mathbb{R}^n)$, $V_0^{(p)}$ always contains locally the constants (see, e.g, the 1-dimensional case in §2.7 of [3]).

PROPOSITION 3.3 (Jackson-type inequality) *If $V_0^{(p)}$ contains locally the polynomials of degree $\leq L-1$, then*

$$\|f - P_j f\|_p \lesssim \lambda^{-\alpha j} \, |f|_{B^{\boldsymbol{\alpha}}_{p,q}}, \quad \forall f \in B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n). \tag{3.4}$$

PROOF: Again, a scaling argument reduces matters to $j = 0$. We shall also assume $0 < p < 1$ (otherwise see [7]). Following the classical steps, we shall first need two lemmata:

LEMMA 3.5 (Local boundedness) *In the conditions above, there exists a finite set $K_\# \subset \mathbb{Z}^n$ such that, for every $\gamma \in \mathbb{Z}^n$, we have*

$$\|P_0 f\|_{L^p(I_\gamma)} \lesssim \|f\|_{L^p(J_\gamma)}, \quad \forall f \in L^p(\mathbb{R}^n), \tag{3.6}$$

where $J_\gamma = \cup_{m \in K_\#} (I_\gamma + m)$.

LEMMA 3.7 (Whitney-type inequality) *Let $f \in L^p(\mathbb{R}^n)$ and $J = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$. Then, for every $L \geq 1$ there exists a polynomial Q in $\mathbb{R}^n$ of degree $\leq L-1$ such that*

$$\|f - Q\|^p_{L^p(J)} \leq C \sum_{|\boldsymbol{\ell}| = L} \frac{1}{|J|} \int_0^{b_1 - a_1} \cdots \int_0^{b_n - a_n} \|\Delta_h^{\boldsymbol{\ell}} f\|^p_{L^p(J)} \, dh,$$

where $C = C(n, p, L) > 0$.

The second lemma is a classical result which can be found, e.g., in [11]. For convenience, we have formulated it in terms of the *averaged modulus of continuity* (rather than its equivalent version with $\omega_p^{\boldsymbol{\ell}}(f;|J|)$). The first lemma follows easily from the definition of P_0 and the arguments used in Proposition 2.4. We leave the details to the reader.

We turn now to the proof of (3.4) for $j=0$. Let us fix $\gamma \in \mathbb{Z}^n$ and a polynomial Q of degree $\leq L-1$. Note that from our assumptions we have $Q \in \mathcal{V}_0^{(p)}$. Then, using Lemma 3.5 and (2.5) we obtain:

$$\begin{aligned} \|f-P_0f\|_{L^p(I_\gamma)} &\lesssim (\|f-Q\|_{L^p(I_\gamma)} + \|Q-R_0f\|_{L^p(J_\gamma)}) \\ &\lesssim (\|f-Q\|_{L^p(J_\gamma)} + \|f-R_0f\|_{L^p(J_\gamma)}) \lesssim \|f-Q\|_{L^p(J_\gamma)}. \end{aligned}$$

Now, with an appropriate choice of $Q=Q^\gamma$, we can invoke Lemma 3.7 to obtain

$$\begin{aligned} \|f-P_0f\|^p_{L^p(\mathbb{R}^n)} &= \sum_{\gamma\in\mathbb{Z}^n} \|f-P_0f\|^p_{L^p(I_\gamma)} \lesssim \sum_{|\boldsymbol{\ell}|=L} \frac{1}{c^n}\int_{[0,c]^n} \|\Delta_h^{\boldsymbol{\ell}} f\|^p_{L^p(\mathbb{R}^n)}\,dh \\ &\lesssim \sum_{|\boldsymbol{\ell}|=L}\left[\frac{1}{(c')^n}\int_{[0,c']^n} \|\Delta_h^{\boldsymbol{\ell}} f\|^q_{L^p(\mathbb{R}^n)}\,dh\right]^{\frac{p}{q}}, \end{aligned}$$

where in the last equality we have used the equivalence of the p^{th} and q^{th} averaged moduli of continuity (for which we still have a constant $c'=c'(L,p,q)>0$). Thus,

$$\|f-P_0f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{|\boldsymbol{\ell}|=L}\left[\frac{1}{(c')^n}\int_{[0,c']^n} \frac{|h|_{\boldsymbol{a}}^{n+\alpha q}}{|h|_{\boldsymbol{a}}^{n+\alpha q}}\|\Delta_h^{\boldsymbol{\ell}} f\|^q_{L^p(\mathbb{R}^n)}dh\right]^{1/q} \lesssim |f|_{B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)},$$

using in the last step the definition of seminorm in (2.2). □

Remark 3.8 Sharper results can be obtained if we consider anisotropic polynomials instead of isotropic ones. Lemma 3.7 can be generalized using anisotropic moduli or averaged partial differences as it was done in [9].

Corollary 3.9 *In the conditions of §2.2, the projectors $P_j = P_j^{(p)}$ satisfy*

$$\lim_{j\to\infty}\|f-P_jf\|_{L^p(\mathbb{R}^n)} = 0, \quad \forall f\in L^p(\mathbb{R}^n).$$

4 Main results

We have now all the tools to prove the main results in this paper. Throughout this section we assume that the scaling function $\varphi \in B^{\alpha_0\boldsymbol{a}}_{p,q}(\mathbb{R}^n)$ and $V_0^{(p)}$ contains locally the polynomials of degree $\leq L-1$, for a fixed integer $L\geq 1$.

Theorem 4.1 *Let $0<p,q<\infty$. In the assumptions above, for every $0<\alpha<\alpha^*$ where $\alpha^* := \min\{\alpha_0, \frac{L}{a_1},\dots,\frac{L}{a_n}\}$ we have:*

$$B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n) = \Big\{f\in L^p(\mathbb{R}^n)\,:\,\sum_{j=0}^\infty \lambda^{\alpha j q}\|Q_jf\|_p^q<\infty\Big\} = \Big\{f\in L^p(\mathbb{R}^n)\,:\,\sum_{j=0}^\infty \lambda^{\alpha j q}E_j^{(p)}(f)^q<\infty\Big\}.$$

In this case, we have the equivalence of quasi-seminorms:

$$|f|_{B^{\alpha\boldsymbol{a}}_{p,q}} \sim \Big[\sum_{j\in\mathbb{Z}}\big(\lambda^{\alpha j}\|Q_jf\|_p\big)^q\Big]^{\frac{1}{q}} \sim \Big[\sum_{j\in\mathbb{Z}}\big(\lambda^{\alpha j}E_j^{(p)}(f)\big)^q\Big]^{\frac{1}{q}}.$$

Proof: Throughout the proof we let $\mu = \min\{1, p\}$. We shall only show the equivalence involving the $E_j^{(p)}(f)$'s, the other following easily from

$$\|Q_j f\|_p^\mu \le E_j^{(p)}(f)^\mu + E_{j+1}^{(p)}(f)^\mu \le 2 \sum_{\ell \ge j} \|Q_\ell f\|_p^\mu, \quad j \in \mathbb{Z},$$

and inequalities of Hardy type. We first remark that

$$E_j^{(p)}(f) \le \|f - P_j f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{|\boldsymbol{\ell}| = L} \omega_p^{\boldsymbol{\ell}}(f, c\lambda^{-j}), \quad j \in \mathbb{Z},$$

as we have seen in the proof of Jackson's inequality. Therefore, using the definition of seminorm in (2.3), we conclude

$$\Big[\sum_{j \in \mathbb{Z}} \big(\lambda^{\alpha j} E_j^{(p)}(f)\big)^q\Big]^{\frac{1}{q}} \lesssim |f|_{B_{p,q}^{\alpha \boldsymbol{a}}},$$

For the converse inequality, it suffices to show that for any $f \in L^p(\mathbb{R}^n)$ and $J \in \mathbb{N}$

$$\omega_p^{\boldsymbol{\ell}}(f, \lambda^{-J}) \lesssim \lambda^{-J\alpha_0} \Big[\sum_{j=-\infty}^{J} \big(\lambda^{\alpha_0 j} E_j^{(p)}(f)\big)^\mu\Big]^{\frac{1}{\mu}}, \quad |\boldsymbol{\ell}| = L. \tag{4.2}$$

Indeed, again an argument involving Hardy's inequality will give (for $0 < \alpha < \alpha_0$):

$$\Big[\sum_{j \in \mathbb{Z}} \big(\lambda^{\alpha j} \omega_p^{\boldsymbol{\ell}}(f, c\lambda^{-J})\big)^q\Big]^{\frac{1}{q}} \lesssim \Big[\sum_{j \in \mathbb{Z}} \big(\lambda^{\alpha j} E_j^{(p)}(f)\big)^q\Big]^{\frac{1}{q}}, \tag{4.3}$$

from which we shall conclude the theorem.

To show (4.2) note that we can write, for any positive integer J:

$$\Delta_h^{\boldsymbol{\ell}} f = \Delta_h^{\boldsymbol{\ell}}(f - P_J f) + \sum_{j=-\infty}^{J-1} \Delta_h^{\boldsymbol{\ell}}(Q_j f)$$

(since by Bernstein's inequality $\lim_{j \to -\infty} |P_j f|_{B_{p,q}^{\boldsymbol{\alpha}}} = 0$). Further, the same arguments as in the proof of Bernstein's inequality give, for $|h|_{\boldsymbol{a}} \le \lambda^{-J}$,

$$\begin{aligned} \|\Delta_h^{\boldsymbol{\ell}} f\|_p^\mu \quad &\le \quad E_J^{(p)}(f)^\mu + \sum_{j=-\infty}^{J-1} \|\Delta_h^{\boldsymbol{\ell}}(Q_j f)\|_p^\mu \lesssim E_J^{(p)}(f)^\mu + \sum_{j=-\infty}^{J-1} \|Q_j f\|_p^\mu \, \|\Delta_h^{\boldsymbol{\ell}} \varphi_{j+1,0}\|_p^\mu \\ &\lesssim \quad E_J^{(p)}(f)^\mu + \sum_{j=-\infty}^{J} E_j^{(p)}(f)^\mu \, \omega_p^{\boldsymbol{\ell}}(\varphi, \lambda^{-(J-j)})^\mu. \end{aligned}$$

Now, since $\varphi \in B_{p,q}^{\alpha_0 \boldsymbol{a}} \subset B_{p,\infty}^{\alpha_0 \boldsymbol{a}}$, we have

$$\omega_p^{\boldsymbol{\ell}}(\varphi, \lambda^{-(J-j)}) \lesssim \lambda^{-(J-j)\alpha_0} |\varphi|_{B_{p,q}^{\alpha_0 \boldsymbol{a}}}, \quad j \in \mathbb{Z}.$$

Substituting in the sum above and taking the sup over $|h|_{\boldsymbol{a}} \le \lambda^{-J}$, we obtain (4.2). □

Corollary 4.4 *In the conditions of the theorem, if $f \in B_{p,q}^{\alpha \boldsymbol{a}}(\mathbb{R}^n)$ then we can write $f = P_0 f + \sum_{j=0}^{\infty} Q_j f$ with convergence in the norm of $B_{p,q}^{\alpha \boldsymbol{a}}$.*

PROOF: By Corollary 3.9, it suffices to verify that

$$\lim_{j\to\infty} |f - P_j f|_{B^{\alpha\boldsymbol{a}}_{p,q}} = 0, \quad f \in B^{\alpha\boldsymbol{a}}_{p,q}(\mathbb{R}^n).$$

Now, using the elementary fact $E^{(p)}_\ell(f - P_j f) \lesssim E^{(p)}_{\max(j,\ell)}(f)$, it follows from the characterization of Theorem 4.1

$$|f - P_j f|^q_{B^{\alpha\boldsymbol{a}}_{p,q}} \lesssim \sum_{\ell\in\mathbb{Z}} (\lambda^{\ell\alpha} E^{(p)}_\ell(f - P_j f))^q \lesssim \sum_{\ell\geq j} (\lambda^{\ell\alpha} E^{(p)}_\ell(f))^q \to 0, \quad \text{as } j\to\infty.$$

□

COROLLARY 4.5 *Let $0 < p, q < \infty$, $\alpha > 0$ and $\boldsymbol{\alpha} = \alpha\boldsymbol{a}$. Then, for the real interpolation method we have the identity:*

$$\Big(L^p(\mathbb{R}^n), B^{\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n)\Big)_{\theta,q} = B^{\theta\boldsymbol{\alpha}}_{p,q}(\mathbb{R}^n), \qquad \theta \in (0,1).$$

Moreover, if $\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, we also have

$$\Big(B^{\alpha_0\boldsymbol{a}}_{p,q_0}, B^{\alpha_1\boldsymbol{a}}_{p,q_1}\Big)_{\theta,q} = B^{\alpha_\theta\boldsymbol{a}}_{p,q} \quad \text{and} \quad \Big(B^{\alpha_0\boldsymbol{a}}_{p_0,q_0}, B^{\alpha_1\boldsymbol{a}}_{p_1,q_1}\Big)_{\theta,q_\theta} = B^{\alpha_\theta\boldsymbol{a}}_{p_\theta,q_\theta}.$$

PROOF: The first result follows from the Bernstein and Jackson inequalities, and the abstract framework in [2]. The last two results are a consequence of the previous characterization theorem, which gives the same real interpolation properties for $B^{\boldsymbol{\alpha}}_{p,q}$ and the sequence spaces $\ell^\alpha_q(L^p)$ (see Theorem 6.1 in [6] and §4.1.4 in [9]).

□

COROLLARY 4.6 *Let $0 < p, q < \infty$, $\alpha > 0$ and*

$$\frac{1}{\tau} = \frac{\alpha}{n} + \frac{1}{p}. \tag{4.7}$$

Then, $B^{\alpha\boldsymbol{a}}_{\tau,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ and

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{B^{\alpha\boldsymbol{a}}_{\tau,p}(\mathbb{R}^n)}, \quad \forall f \in B^{\alpha\boldsymbol{a}}_{\tau,p}(\mathbb{R}^n). \tag{4.8}$$

PROOF: The proof relies on the following elementary fact:

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \lambda^{j\alpha} \|f\|_{L^\tau(\mathbb{R}^n)}, \quad \forall f \in V^{(\tau)}_j. \tag{4.9}$$

Indeed, letting $\mu = \min\{1,p\}$ and using Theorem 4.1 we obtain

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &\lesssim \Big[\|P^{(\tau)}_0 f\|^\mu_p + \sum_{j=0}^\infty \|Q^{(\tau)}_j f\|^\mu_p\Big]^{\frac{1}{\mu}} \\ &\lesssim \Big[\|P^{(\tau)}_0 f\|^\mu_\tau + \sum_{j=0}^\infty (\lambda^{j\alpha}\|Q^{(\tau)}_j f\|_\tau)^\mu\Big]^{\frac{1}{\mu}} \lesssim \|f\|_{B^{\alpha\boldsymbol{a}}_{\tau,\mu}}. \end{aligned}$$

If $\mu = p \leq 1$ this gives the desired result. If $p > 1$, one chooses $1 \leq p_0 < p < p_1 < \infty$ so that

$$\|f\|_{p_i} \lesssim \|f\|_{B^{\alpha_i\boldsymbol{a}}_{\tau,1}}, \quad i = 0,1,$$

where α_i, p_i, τ are related as in (4.7). Using real interpolation one obtains (4.8).

To show (4.9) it suffices to consider $j = 0$ and $f = \sum_k c_k\varphi_k$, for a finite sequence of scalars $\{c_k\}_k$. But using the equivalence of (quasi)-norms in a finite dimensional space we have

$$\|f\|^p_{L^p(\mathbb{R}^n)} \leq \sum_{\gamma\in\mathbb{Z}^n} \int_{I_\gamma} |\sum_k c_k\varphi_k|^p\,dx \lesssim \sum_{\gamma\in\mathbb{Z}^n} \left(\int_{I_\gamma} |\sum_k c_k\varphi_k|^\tau\,dx\right)^{\frac{p}{\tau}} \lesssim \|f\|^p_{L^\tau(\mathbb{R}^n)},$$

where in the last equality we have used the inclusion $\ell^1 \hookrightarrow \ell^{\frac{p}{\tau}}$ (since $\tau < p$).

$\square$

In our final theorem we give a wavelet characterization for the spaces $B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)$, when $0<\tau<1$ and α is in an appropriate range. Note that, a priori, there does not exist a τ-stable basis of wavelet type for the spaces $W^{(\tau)}_j := \operatorname{Im} Q^{(\tau)}_j$. In fact, the multilevel decomposition $f = P^{(\tau)}_0 f + \sum_{j=0}^{\infty} Q^{(\tau)}_j f$ of a function $f \in L^\tau(\mathbb{R}^n)$ may not even be a direct sum, limiting the applicability of this technique. In general, when dealing with Besov spaces one can use the extra embedding $B^{\alpha\boldsymbol{a}}_{\tau,\tau} \hookrightarrow L^p$ in Corollary 4.6, which at least for $n(\frac{1}{\tau}-1) < \alpha < \frac{n}{\tau}$ gives $p \geq 1$. This allows us to write every $f \in B^{\alpha\boldsymbol{a}}_{\tau,\tau}$ as:

$$f = \sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\varphi}_k\rangle \varphi_k + \sum_{a=1}^{m-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle \psi^{(p)}_{a;j,k} = \sum_{a=1}^{m-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle \psi^{(p)}_{a;j,k}.$$

Next theorem characterizes, among the functions with such representation, those belonging to $B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)$. As usual, we shall assume that the scaling function $\varphi \in B^{\alpha_0\boldsymbol{a}}_{\tau,\tau}$, and V_0 contains locally the polynomials of degree $\leq L-1$.

THEOREM 4.10 *Let $0<\tau<1$, $n(\frac{1}{\tau}-1) < \alpha < \min\{\frac{n}{\tau},\alpha^*\}$ and $p = p(\alpha,\tau)$ given by (4.7). Then,*

$$B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n) = \Big\{ f \in L^p(\mathbb{R}^n) \ : \ \|P^{(p)}_0 f\|^\tau_\tau + \sum_{j=0}^{\infty} \lambda^{j\tau\alpha} \|Q^{(p)}_j f\|^\tau_p < \infty \Big\}.$$

In this case, we have the equivalence of quasi-seminorms:

$$|f|_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}} \quad \sim \quad \Big[\sum_{j\in\mathbb{Z}}\sum_{a=1}^{m-1}\sum_{k\in\mathbb{Z}^n} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau\Big]^{\frac{1}{\tau}}.$$

PROOF: Suppose that $f \in B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)$. We first claim that $P^{(p)}_0 f \in V^{(\tau)}_0$. Indeed, writing $P^{(p)}_0 f = \sum_k \langle f, \tilde{\varphi}_k\rangle \varphi_k$, and using the embedding $B^{\alpha\boldsymbol{a}}_{\tau,\tau}(J) \hookrightarrow L^p(J)$ for J compact (see [9]), we have

$$|\langle f, \tilde{\varphi}_k\rangle| \leq \Big[\int_{\mathrm{Supp}\tilde{\varphi}_k} |f|^p \, dx\Big]^{\frac{1}{p}} \|\tilde{\varphi}\|_{p'} \ \lesssim \ \|f\|_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathrm{Supp}\tilde{\varphi}_k)}.$$

Therefore,

$$\sum_{k\in\mathbb{Z}^n} |\langle f, \tilde{\varphi}_k\rangle|^\tau \ \lesssim \ \sum_{k\in\mathbb{Z}^n} \|f\|^\tau_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathrm{Supp}\tilde{\varphi}_k)} \ \lesssim \ \|f\|^\tau_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)},$$

which, by the τ-stability of $\{\varphi_k\}_k$, gives $P^{(p)}_0 f \in V^{(\tau)}_0$ and

$$\|P^{(p)}_0 f\|_\tau \sim \Big[\sum_{k\in\mathbb{Z}^n} |\langle f, \tilde{\varphi}_k\rangle|^\tau\Big]^{\frac{1}{\tau}} \ \lesssim \ \|f\|_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}}.$$

Next, we show that

$$\Big[\sum_{j\in\mathbb{Z}}\sum_{a=1}^{m-1}\sum_{k\in\mathbb{Z}^n} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau\Big]^{\frac{1}{\tau}} \ \lesssim \ |f|_{B^{\alpha\boldsymbol{a}}_{\tau,\tau}}, \quad \forall f \in B^{\alpha\boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n). \tag{4.11}$$

Observe that

$$\sum_{j\in\mathbb{Z}} \lambda^{\alpha j\tau} \|Q^{(p)}_j f\|^\tau_\tau \leq \sum_{j,a,k} \lambda^{\alpha j\tau} \, |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau \, \|\psi^{(p)}_{a;j,k}\|^\tau_\tau = \sum_{j,a,k} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau \, \|\psi_a\|^\tau_\tau,$$

so (4.11) will give the direct inclusion of the theorem.

By density (e.g., Corollary 4.4), it is enough to verify (4.11) for $f \in B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)$, where we assume $\alpha < \alpha_1 < \min\{\frac{n}{\tau}, \alpha^*\}$. In fact, we claim that (4.11) follows from

$$\Big[\sum_{a,k} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau\Big]^{\frac{1}{\tau}} \lesssim \lambda^{-j(\alpha_1-\alpha)} |f|_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}}, \quad j \in \mathbb{Z}. \tag{4.12}$$

Indeed, since $\langle P^{(\tau)}_j f, \tilde{\psi}^{(p')}_{a;j,k}\rangle = 0$, (4.12) together with Theorem 4.1, will give as well

$$\sum_{a,k} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau \lesssim \lambda^{-j\tau(\alpha_1-\alpha)} |f - P^{(\tau)}_j f|^\tau_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}} \lesssim \lambda^{-j\tau(\alpha_1-\alpha)} \sum_{\ell=j}^{\infty} \lambda^{\ell\tau\alpha_1} E^{(\tau)}_\ell(f)^\tau.$$

Thus,

$$\sum_{a,j,k} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau \lesssim \sum_{\ell\in\mathbb{Z}} \lambda^{\ell\tau\alpha_1} E^{(\tau)}_\ell(f)^\tau \Big(\sum_{j=-\infty}^{\ell} \lambda^{-j\tau(\alpha_1-\alpha)}\Big) \lesssim \sum_{\ell\in\mathbb{Z}} \lambda^{\ell\tau\alpha} E^{(\tau)}_\ell(f)^\tau.$$

Let us now show (4.12), which by invariance we just need to verify for $j = 0$. We let $p_1 = p(\alpha_1, \tau)$ as in (4.7). Then, writing $Q_0 f = \sum_{a,k} \langle f, \tilde{\psi}_{a;k}\rangle \psi_{a;k}$, we see that for every polynomial Q of degree $\le L-1$ we must have

$$\begin{aligned} |\langle f, \tilde{\psi}_{a;k}\rangle| &= \left| \int_{\mathrm{Supp}\tilde{\psi}_{a;k}} (f - Q)\, \overline{\tilde{\psi}_{a;k}}\, dx \right| \le \|\tilde{\psi}_a\|_{p_1'} \|f - Q\|_{L^{p_1}(\mathrm{Supp}\tilde{\psi}_{a;k})} \\ &\lesssim \|f - Q\|_{L^\tau(\mathrm{Supp}\tilde{\psi}_{a;k})} + |f - Q|_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}(\mathrm{Supp}\tilde{\psi}_{a;k})}. \end{aligned}$$

Now, using $|f - Q|_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}} = |f|_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}}$, and Whitney's inequality we obtain:

$$|\langle f, \tilde{\psi}_{a;k}\rangle| \lesssim |f|_{B^{\alpha_1 \boldsymbol{a}}_{\tau,\tau}(\mathrm{Supp}\tilde{\psi}_{a;k})}.$$

Raising to the power τ and summing in $k \in \mathbb{Z}^n$, we end up with (4.12).

Let us now show the inverse inclusion of the theorem. First of all note that, by the assumptions on f, we have

$$\|f\|^\tau_\tau \le \|P^{(p)}_0 f\|^\tau_\tau + \sum_{j=0}^{\infty} \|Q^{(p)}_j f\|^\tau_p < \infty.$$

Next we show that, whenever $g \in V^{(p)}_j$,

$$|g|_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}} \lesssim \lambda^{j\alpha} \|g\|_\tau. \tag{4.13}$$

From this we can easily conclude the proof since

$$|f|^\tau_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}} \le |P^{(p)}_0 f|^\tau_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}} + \sum_{j=0}^{\infty} |Q^{(p)}_j f|^\tau_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}} \le \|P^{(p)}_0 f\|^\tau_\tau + \sum_{j=0}^{\infty} \lambda^{j\tau\alpha} \|Q^{(p)}_j f\|^\tau_\tau.$$

Note that, as a posteriori consequence of (4.13) and (4.11), we obtain

$$\lambda^{j\alpha} \|Q^{(p)}_j f\|_\tau \sim \Big[\sum_{a,k} |\langle f, \tilde{\psi}^{(p')}_{a;j,k}\rangle|^\tau\Big]^{\frac{1}{\tau}}.$$

To show (4.13) we may assume $j = 0$. We then follow the same steps as for Bernstein's inequality, except for the minor refinement we indicate below:

$$|g|^\tau_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}(\mathbb{R}^n)} \le \sum_{\gamma\in\mathbb{Z}^n} |g|^\tau_{B^{\alpha \boldsymbol{a}}_{\tau,\tau}(I_\gamma)} \lesssim \sum_{\gamma\in\mathbb{Z}^n} \sum_{m\in K_*} |c_{m+\gamma}|^\tau \lesssim \sum_{\gamma\in\mathbb{Z}^n} \int_{\mathrm{Supp}\tilde{\varphi}_\gamma} |g|^\tau\, dx \lesssim \|g\|^\tau_{L^\tau(\mathbb{R}^n)}.$$

The proof of the theorem is now complete. □

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