Quantitative homogenization of global attractors for reaction-diffusion systems with rapidly oscillating terms

–Dedicated to Alain Bensoussan on the occasion of his sixtieth birthday–

Bernold Fiedler*, Mark I. Vishik**

* Freie Universität Berlin
Arnimallee 2–6
14195 Berlin, Germany

** Russian Academy of Sciences
Institute for Problems of Information Transmission
Bolshoy Karetny, per.19
101 447 Moscow GSP-4, Russia

Abstract

For rapidly spatially oscillating nonlinearities f and inhomogeneities g we compare solutions u^ε of reaction-diffusion systems

$$\partial_t u^\varepsilon = a\Delta u^\varepsilon - f(\varepsilon, x, x/\varepsilon, u) + g(\varepsilon, x, x/\varepsilon)$$

with solutions u^0 of the formally homogenized, spatially averaged system

$$\partial_t u^0 = a\Delta u^0 - f^0(x, u^0) + g^0(x, u^0).$$

We consider a smooth bounded domain $x \in \Omega \subseteq \mathbb{R}^n$, $n \leq 3$, with Dirichlet boundary conditions. We also impose sufficient regularity and dissipation conditions, such that solutions exist globally in time and, in fact, converge to their compact global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$, respectively, in $L_2(\Omega)$.

Based on ε-independent a priori estimates we prove

$$\|u^\varepsilon(\cdot, t) - u^0(\cdot, t)\|_{L_2(\Omega)} \leq C\varepsilon e^{\rho t},$$

uniformly for all $t \geq 0$ and $0 < \varepsilon \leq \varepsilon_0$. Here the solutions u^ε and u^0 start at the same initial condition $u = u_0(x) \in H^1(\Omega)$ for $t = 0$, and $C = C(\|u_0\|_{H^1})$.

Based on an ε-independent H^2-bound on the global attractors $\mathcal{A}^\varepsilon$ as well as an exponential attraction rate ν of the homogenized attractor $\mathcal{A}^0$ in $L_2(\Omega)$, we also prove fractional order upper semicontinuity of the global attractors for $\varepsilon \searrow 0$,

$$dist_{L_2(\Omega)}(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C\varepsilon^{\gamma'}$$

for $\gamma' = (1 + \rho/\nu)^{-1}$. This result requires the homogenized nonlinearity $f^0(x, w)$ to be near a potential vector function $f^1(x, w) = \nabla_w F(x, w)$ with scalar potential F.

Both quantitative homogenization estimates are proved only for quasiperiodic dependence of f, g on the spatially rapidly oscillating variable x/ε. Moreover, the finitely many frequencies describing this quasiperiodic dependence are assumed to satisfy Diophantine conditions, as are familiar from small divisor problems in Kolmogorov-Arnold-Moser theory. Alternatively, the results hold if f, g admit a sufficiently regular divergence representation which describes their explicit spatial dependence. All results apply to, and are illustrated for, the case of FitzHugh-Nagumo systems with spatially rapidly oscillating quasiperiodic coefficients and inhomogeneities.

In the companion paper [FV00], based on analytic semigroup methods, similar results are obtained for the quantitative homogenization of solutions and invariant manifolds. Examples include homogenization of the Navier-Stokes system under periodic boundary conditions and for spatially rapidly oscillating quasiperiodic forces.

1 Introduction

Reaction-diffusion systems with rapid spatial oscillations in their nonlinearity f and in the inhomogeneous term g take the form

$$\partial_t u = a\Delta u - f(\varepsilon, x, x/\varepsilon, u) + g(\varepsilon, x, x/\varepsilon). \tag{1.1}$$

Here $u \in \mathbb{R}^N$, and x is in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \leq 3$. To be specific, we consider Dirichlet boundary conditions $u = 0$ for $x \in \partial\Omega$. The constant diffusion matrix a is symmetric and strictly positive definite. For $\varepsilon > 0$ tending to zero *rapid spatial oscillations* arise in f and g, caused by the dependency on $z = x/\varepsilon$.

It is our goal, in the present paper, to compare the global behavior of solutions $u = u^\varepsilon(x,t)$ of system (1.1), for $\varepsilon \searrow 0$, with solutions $u = u^0(x,t)$ of the formally *homogenized system*

$$\partial_t u = a\Delta u - f^0(x, u) + g^0(x) \tag{1.2}$$

on $x \in \Omega$, again with Dirichlet boundary conditions. The formally homogenized nonlinearity f^0 and inhomogeneity g^0 can be defined, for example, by spatial averaging

$$\begin{aligned} f^0(x,w) &:= \lim_{\varepsilon\searrow 0} |\varepsilon^{-1}\Omega|^{-1} \int\limits_{\varepsilon^{-1}\Omega} f(\varepsilon, x, z, w)dz \\ g^0(x) &:= \lim_{\varepsilon\searrow 0} |\varepsilon^{-1}\Omega|^{-1} \int\limits_{\varepsilon^{-1}\Omega} g(\varepsilon, x, z)dz \end{aligned} \tag{1.3}$$

over the scaled domain $\varepsilon^{-1}\Omega \subseteq \mathbb{R}^n$. The limits exist under suitable assumptions on f and g.

Our comparison between solutions u^ε and homogenized solutions u^0 will be quantitative, over a global time horizon, and with respect to the L_2-norm $|\cdot|_H = \|\cdot\|_{L_2}$ on the Hilbert space $H = L_2(\Omega)$. Specifically consider solutions u^ε, u^0 of (1.1), (1.2) respectively, with the same initial condition

$$u^\varepsilon(x,t) = u^0(x,t) = u_0(x), \qquad \text{for } t = 0. \tag{1.4}$$

We assume $u_0 \in H_0^1$ possesses square integrable first derivatives, $\|u_0\|_{H^1} < \infty$, and satisfies Dirichlet boundary conditions. In theorems 3.1, 3.2 below we will then derive an estimate, uniformly for all $t \geq 0$, of the form

$$|u^\varepsilon(\cdot,t) - u^0(\cdot,t)|_H \leq C(\|u_0\|_{H^1}) \cdot \varepsilon e^{\varrho t}, \tag{1.5}$$

for some $\varrho > 0$ and suitable constants $C = C(\|u_0\|_{H^1})$. In addition to more or less standard assumptions involving regularity, growth and dissipativeness conditions, this result requires f, g to depend quasiperiodically on the rapid space variable $z = x/\varepsilon$. Moreover the spatial frequencies involved in this quasiperiodicity will be required to satisfy a Diophantine condition in the sense of [Cas57]. For complete details and an explicit condition including some discussion see section 3, as well as the independent companion paper [FV00].

Going beyond estimate (1.5), which only compares solutions starting from identical initial conditions, we also aim at a quantitative homogenization estimate for the *global attractors* $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ of (1.1) and (1.2), respectively. Under dissipation conditions the global attractor can be characterized, equivalently, as the maximal compact invariant set, or the minimal set attracting every bounded set, or the set of (initial conditions of) bounded global solutions for $t \in \mathbb{R}$.

Here both invariance and global solution refer to times $t \in \mathbb{R}$, both positive and negative, rather than just forward solutions. See for example [BV92], [Hal88], [Lad91], [Rau00], [Tem88], [Vis92] for background and history of this notion.

Our main result, theorem 4.2 below, establishes a quantitative homogenization estimate

$$dist(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \sup_{u_0 \in \mathcal{A}^\varepsilon} dist(u_0, \mathcal{A}^0) \leq C\varepsilon^{\gamma'} \tag{1.6}$$

of fractional order γ', for some constants $C, \gamma' > 0$. The distance function *dist* is measured by the L_2-norm $\|\cdot\|_H$. The exponent γ' is in fact given by

$$\gamma' = (1 + \varrho/\nu)^{-1} \tag{1.7}$$

where ϱ is the exponential separation rate of the quantitative homogenization estimate (1.5) above, and $\nu > 0$ measures the in fact exponential rate of attraction of the global attractor $\mathcal{A}^0$ of the homogenized system (1.2),

$$dist(u^0(t), \mathcal{A}^0) \leq Ce^{-\nu t}, \tag{1.8}$$

for all $t \geq 0$, where ν is a positive constant, and $C = C(\|u^0(0)\|_H)$ is a monotone function. To achieve such an exponential rate of attraction, we will require the homogenized system (1.2) to be near a gradient-like, variational setting. More precisely $f^0(x,w) = f_1^0(x,w) + \eta f_2^0(x,w)$, where $f_1^0(x,w) = \nabla_w F_1^0(x,w) \in \mathbb{R}^N$ derives from a scalar potential F_1^0, the vector function $f_2^0(x,w)$ is subordinated to $f_1^0(x,w)$, and $\eta > 0$ is sufficiently small. See section 2 for more details.

Rather than unravelling the complete technical assumptions in the general setting of reaction-diffusion systems (1.1), (1.2), we present an example. We choose the celebrated FitzHugh-Nagumo system

$$\begin{aligned} \partial_t u_1 &= a_1 \Delta u_1 - \Phi(u) - b_4 v + g_1 \\ \partial_t u_2 &= a_2 \Delta u_2 + b_5 u - b_6 v + g_2 \end{aligned} \tag{1.9}$$

which has been widely used as the prototype of excitable media models, including its origin in nerve impulse modelling and the modelling of semiconductor devices. See [Smo83], [Rot84], [VVV94] for references including the applied background.

We assume the diffusion coefficients a_1, a_2 to be constant. The coefficients b_j are assumed to be of the form

$$b_j = b_j(\varepsilon, x, x/\varepsilon), \tag{1.10}$$

uniformly bounded and with smooth dependence on all arguments. Moreover we assume quasiperiodicity in the argument $z = x/\varepsilon \in \mathbb{R}^n$ with frequency vectors $\alpha_\sigma \in \mathbb{R}^{l_\sigma}$, $\sigma = 1, \cdots, n$. This means that

$$b_j(\varepsilon, x, z) = \Gamma_j(\varepsilon, x, z_1\alpha_1, \cdots, z_n\alpha_n), \tag{1.11}$$

where Γ_j is smooth in each of its $1+n+l_1+\cdots+l_n$ components and, except in ε and x, also 2π-periodic. For the frequency vectors α_σ we assume the celebrated Diophantine condition

$$|k_\sigma \cdot \alpha_\sigma| \geq c|k_\sigma|^{-(l_\sigma - 1 + \delta)} \tag{1.12}$$

for $\sigma = 1, \cdots, n$, some constants $c, \delta > 0$ and all integer vectors $k_\sigma \in \mathbb{Z}^{l_\sigma} \setminus \{0\}$. See also the related condition (1.17) below. The quasiperiodicity assumptions for the smooth bounded inhomogeneities $g_j = g_j(\varepsilon, x, x/\varepsilon)$ are completely analogous and are omitted. The cubic nonlinearity $\Phi(u)$ is assumed to be of the form

$$\Phi(u) = b_3 u^3 + b_2 u^2 + b_1 u, \tag{1.13}$$

with coefficients b_j satisfying (1.10)–(1.12) above, as well. In terms of the representation of b_j by the 2π-periodic functions Γ_j from (1.11) it is particularly easy to define the homogenized version of the FitzHugh-Nagumo system (1.9). Indeed, the homogenized system takes the same form (1.9) with spatially averaged coefficients b_j^0 and inhomogeneities g_j. Specifically

$$b_j^0(x) := (2\pi)^{-l} \int\limits_{T^l} \Gamma_j(0, x, \omega_1, \cdots \omega_n) d\omega_1 \cdots d\omega_n \tag{1.14}$$

with $l = l_1 + \cdots + l_n$ and $T^l = \mathbb{R}^l / 2\pi \mathbb{Z}^l$.

The results of this paper, as applied to the homogenization of the FitzHugh-Nagumo system (1.9), can then be summarized as follows.

Theorem 1.1 *In the setting of (1.9)–(1.14) above, assume lower bounds*

$$\begin{array}{rcl} b_3 & \geq & b_3 > 0 \\ b_6 & \geq & b_6 > 0 \end{array} \tag{1.15}$$

for suitable positive constants b_3 and b_6, independently of $\varepsilon > 0$.

Then the global quantitative homogenization estimate (1.5) holds for the difference between solutions of the spatially rapidly oscillating FitzHugh-Nagumo system (1.9) and its homogenization.

Theorem 1.2 *In the setting of (1.10)–(1.15) above, assume that the averaged off-diagonal coefficients b_4^0 and b_5^0 are sufficiently small:*

$$|b_4^0| + |b_5^0| \le \eta_0. \tag{1.16}$$

Moreover assume that all equilibria of the decoupled homogenized system with $b_4^0 = b_5^0 = 0$ are hyperbolic.

Then the global attractors $\mathcal{A}^\varepsilon$ of the spatially rapidly oscillating FitzHugh-Nagumo system (1.9) and its homogenized counterpart $\mathcal{A}^0$ satisfy a fractional homogenization estimate (1.6).

The proofs of theorems 1.1, 1.2 are immediate corollaries to theorems 3.1, 3.2 and to theorem 4.2 below, respectively, and will be omitted. Instead, we recommend to read the present paper with the specific example of homogenization of the FitzHugh-Nagumo system in mind. Condition (1.16) above forces the homogenized FitzHugh-Nagumo system to be near a (decoupled) gradient system. For a more sophisticated Lyapunov function in the coupled case see [CS86].

The remainder of the paper is organized as follows. In section 2, we develop the standard a priori L_2-estimates for solutions of semilinear reaction-diffusion systems. In particular we prove an H^2-bound for the global attractors $\mathcal{A}^\varepsilon$ which is uniform in ε; see proposition 2.2. Although this material is reasonably standard we have to carefully avoid any differentiation with respect to x, which would immediately destroy ε-uniformity of our estimates.

In section 3, we give two proofs of the global quantitative homogenization estimate (1.5). In theorem 3.1 we prove the estimate under suitable regularity assumptions on a divergence representation of coefficients like b_j, of the form

$$b_j(\varepsilon, x, z) - b_j^0(x) = \sum_{\sigma=1}^{n} \partial_{z_\sigma} B_\sigma^j(\varepsilon, x, z). \tag{1.17}$$

In theorem 3.2 we then prove that the Diophantine conditions (1.12) are indeed strong enough to provide a divergence representation (1.17) as required in theorem 3.1.

Section 4 is devoted to a fractional order homogenization estimate for the distance of global attractors. In lemma 4.1 we isolate an abstract principle, reminiscent of an underlying argument in [HR89]. See also [Rau00]. We show how the two indgredients of

- a global quantitative homogenization estimate (1.5), and
- an exponential convergence rate (1.8) for the global attractor $\mathcal{A}^0$ of the homogenized system

together result in the desired fractional order distance estimate (1.6) for the L_2-distance $dist(\mathcal{A}^\varepsilon, \mathcal{A}^0)$. In theorem 4.2 we implement this abstract observation for our specific setting of reaction-diffusion systems.

The exponential attraction of $\mathcal{A}^0$, in our specific context, is a consequence of earlier works [BV92] and [GV97]. The necessary arguments, culminating in the exponential estimate (1.8), are deferred in section 5. In the companion paper [FV00], an alternative approach to spatial homogenization of rapid quasiperiodic oscillations is sketched in the framework of analytic semigroup theory. Emphasis there is on stronger estimates of the type

$$\|u^\varepsilon(\cdot,t) - u^0(\cdot,t)\|_{H^\sigma} \leq C\varepsilon^\gamma, \tag{1.18}$$

for suitable positive constants C, σ, γ, under Fourier-friendly periodic boundary conditions $x \in T^n$, and over a finite time horizon. The estimates are achieved in a triangle

$$0 < \gamma < \min(\sigma - n/2, 2 - \sigma) \tag{1.19}$$

and hold, likewise, for local stable and unstable manifolds of equilibria. We hope to reconcile the two approaches in future work.

The work [CV00] studies the case when $f(\varepsilon, x, x/\varepsilon, u)$ and $g(\varepsilon, x, x/\varepsilon)$ in (1.1) tend to $f^0(x,u)$ and $g^0(x)$ in some weak sense. There it was proved that the attractors $\mathcal{A}^\varepsilon$ tend to the attractor $\mathcal{A}^0$ of the averaged equation (1.2) in the corresponding functional space, but no quantitative estimate was given.

The more general case where the constant diffusion term $a\Delta u$ in (1.1) is replaced by $\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x, x/\varepsilon)\partial_{x_j} u)$ has also been widely investigated but mostly in finite cylinders $\Omega \times \{0 < t < T\}$, $T < +\infty$ and for periodic dependence on x/ε or by weak convergence methods. See for example the books [BLP78], [ZKO94]. Neither quantitative strong estimates nor convergence of global attractors or invariant manifolds have been obtained in this case, so far.

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2 Uniform a priori estimates

In this section we provide ε-*independent* estimates for solutions of the system

$$\partial_t u = a\Delta u - f(x, x/\varepsilon, u) + g(x, x/\varepsilon), \qquad u \in \mathbb{R}^N, \tag{2.1}$$

in a smooth bounded domain $\Omega \subseteq \mathbb{R}^n, n \le 3$, with Dirichlet boundary conditions and initial conditions

$$u(x, 0) = u_0(x) \tag{2.2}$$

at $t = 0$. For fixed $\varepsilon > 0$, existence and uniqueness of solutions $u(t, \cdot)$ in $L_\infty(\mathbb{R}^+, L_2(\Omega)), L_2(\mathbb{R}^+, H_0^1)$ etc., are standard, under our assumptions on a f, g specified below; see for example [BV92], [CV00], and [Tem88]. Note however that f and g are rapidly oscillating for $\varepsilon \searrow 0$ due to the terms x/ε. Although our ε-independent regularity estimates will proceed along the well-established lines of standard L_2-theory, the terms x/ε in f and g force us to carefully circumvent taking x-derivatives of (2.1). Keeping this imperative well in mind we frequently suppress ε-dependence as well as dependence on x and even on u, abbreviating (2.1) by

$$\partial_t u = a\Delta u - f + g, \tag{2.3}$$

where of course

$$\begin{aligned} f = f(u) &= f^\varepsilon(u) = f(x,u) := f(x, x/\varepsilon, u) \\ g &= g(x, x/\varepsilon). \end{aligned} \tag{2.4}$$

Below we only sketch the strategy to derive a priori estimates. Rigorous proofs are established using the Galerkin method, see for example [BV92], [CV00], [Tem88].

Our assumptions on a, f, g, uniformly with respect to (suppressed) ε, are the following. Let $a = a^* = (a_{ij})_{i,j=1,\cdots,N} \geq a_0 \cdot \mathrm{id} > 0$ be symmetric positive definite and independent of x, ε. Rescaling x, if necessary, we will assume $a_0 = 1$ without loss of generality. Denote $u = (u_1, \cdots, u_N)$, $f = (f_1, \cdots, f_N)$, $g = (g_1, \cdots, g_N)$. We assume that

$$\int_\Omega |g|^2 dx := |g|_H^2 \leq C_0 < +\infty \tag{2.5}$$

and in particular that $g(x, x/\varepsilon)$ is L_2-integrable. We also assume that the vector functions $f = f(x,z,w)$ and $f_w = \partial_w f(x,z,w)$ are continuous with respect to $x \in \Omega, z \in \mathbb{R}^n, w \in \mathbb{R}^N$ and satisfy the growth conditions given below. Supplementary requirements about these functions will be specified in later sections.

Uniformly for (suppressed) $0 < \varepsilon \leq \varepsilon_0$ the vector-function f is assumed to admit a splitting

$$f = f^1 + f^2 \tag{2.6}$$

where f^1 and f^2 satisfy the following conditions:

$$f^1(x,w) \cdot w \geq C \sum_{r=1}^N |w_r|^{p_r} - C_1 \tag{2.7}$$

$$\sum_r |f_r^1|^{q_r} \le C_2 \sum_{r=1}^N (|w_r|^{p_r} + 1). \tag{2.8}$$

Here $p_r \ge 2$ and $1/q_r + 1/p_r = 1$. For spatial dimensions $n = 3$ we also assume $p_r \le 4$ and

$$\sum_r |f_r^1|^2 \le C_3 (\sum_r |w_r|^6 + 1) \tag{2.9}$$

for all $w \in \mathbb{R}^N$. For $n \le 2$ the exponent 6 in (2.9) can be replaced by any σ, $2 \le \sigma < +\infty$, and $p_r \le \sigma$. Moreover $f^1(x, w)$ is assumed to possess a potential F with respect to w : there exists $F = F(x, x/\varepsilon, w)$ such that

$$f^1(x, x/\varepsilon, w) = \nabla_w F(x, x/\varepsilon, w) \tag{2.10}$$

and F satisfies the estimates

$$C_4 \sum_r |w_r|^{p_r} - C_5 \le F(x, z, w) \le C_6 \sum_r |w_r|^{p_r} + C_7. \tag{2.11}$$

We also suppose, that

$$\partial_w f(x, z, w) \zeta \cdot \zeta \ge -C_7\, \zeta \cdot \zeta \tag{2.12}$$

$$\sum_{r=1}^N |f_r^2(x, z, w)|^2 \le C_8 \left(\sum_{r=1}^N |w_r|^{p_r} + 1 \right) \tag{2.13}$$

for all $\zeta \in \mathbb{R}^n, w \in \mathbb{R}^N, z \in \mathbb{R}^n,$ and $x \in \Omega$.

Theorem 2.1 *Under the assumptions (2.5)–(2.13) the Cauchy poblem (2.1), (2.2) with $u_0 \in H := L_2(\Omega))^N$ possesses a unique weak solution $u(x, t)$. The norm $|u(t)| := |u(\cdot, t)|_H$ is uniformly bounded in t and satisfies the inequalities*

$$|u(t)|^2 \le e^{-\alpha t} |u(0)|^2 + C_9 \tag{2.14}$$

$$\begin{aligned} \beta_1 \int_t^{t+1} \|u\|_{H^1}^2 ds + \beta_2 \sum_{r=1}^N \int_t^{t+1} \|u(s)\|_{L_{p_r}}^{p_r} ds \quad &\le |u(t)|^2 + C_{10} \le \\ &\le e^{-\alpha t} |u(0)|^2 + C_{11} \end{aligned} \tag{2.15}$$

uniformly for all $t > 0$, with suitable positive constants α, β_j, C_j.

The proof is standard and can be found in [BV92], [CV00], [Tem88]. Let us recall, that to obtain (2.14) the equation (2.1) is multiplied by u and integrated over Ω. Then the Gronwall inequality is used. Subsequent integration in t yields (2.15).

Let us denote by $\{S(t) = S(t,\varepsilon),\ t \geq 0\}$ the semigroup corresponding to system (2.1) for fixed $\varepsilon,\ 0 < \varepsilon \leq \varepsilon_0$:

$$S(t)u_0 = u(t), \tag{2.16}$$

where $u(t) = u(x,t)$ is the solution of the Cauchy problem (2.1), (2.2).

For any $t \geq 0$ the semigroup $S(t) : H \to H$ is continuous. This fact follows from (2.12). Indeed, let $S(t)u_0^i = u^i(t), i = 1, 2$ be two solutions. Then (2.12) implies

$$\begin{aligned} &\partial_t |u^1 - u^2|^2 + (a\nabla(u^1 - u^2), \nabla(u^1 - u^2)) = \\ = \ &-(f(u^1) - f(u^2), u^1 - u^2) \leq C_7 |u^1 - u^2|^2. \end{aligned} \tag{2.17}$$

Gronwall's inequality then implies

$$|u^1(t) - u^2(t)|^2 \leq e^{C_7 t} |u_0^1 - u_0^2|^2$$

and the continuity of $S(t)$ in H is proved.

Proposition 2.2 *Let the assumptions of theorem 2.1 hold. Then the semigroups $\{S(t,\varepsilon), t \geq 0\}$ possess global attractors $\mathcal{A} = \mathcal{A}^\varepsilon$ in H which even are bounded in H^2,*

$$\|\mathcal{A}^\varepsilon\|_{H^2} \leq R < +\infty, \tag{2.18}$$

uniformly for $0 < \varepsilon \leq \varepsilon_0$.

Proof:

The proof follows a well-established path. We first prove a priori estimates for the solutions u in the norms $|\cdot|,\ \|\cdot\|,\ \|\cdot\|_{H^2}$ of $H = L_2,\ H^1 = H_0^1$ and H^2, uniformly with respect to ε and t. In addition we estimate $|\partial_t u|$ and

$\|\partial_t u\|$. These estimates will provide a uniformly absorbing set B_2, bounded in H^2 and hence compact in H. By the general theorem on the existence of compact global attractors, see [BV92], [Hal88], [Tem88], this is sufficient to prove existence and ε-uniform H^2-boundedness of the global attractors $\mathcal{A}^\varepsilon$, as was claimed in (2.18).

We shall prove first that for any $\varepsilon > 0$ there exists the attractor $\mathcal{A}^\varepsilon$ of $\{S(t,\varepsilon), t \geq 0\}$ in H. The continuity of $S(t)$ in H was established above. Let us prove that there exists a compact absorbing set bounded in H^1 for any fixed ε. For any fixed $\delta > 0$, the set

$$B_0 = \{u \mid u \in H, |u|^2 \leq (1+\delta)C_9\}, \tag{2.19}$$

is an absorbing set for $\{S(t)\}$ in H, due to (2.14). Therefore the global attractor $\mathcal{A}^\varepsilon$ exists in H. Now we prove that there exists an absorbing set B_1 of $S(t,\varepsilon)$, which is bounded in $H^1 = (H_0^1)^N$, uniformly with respect to ε, $0 < \varepsilon \leq \varepsilon_0$. To this end we need some standard a priori estimates for solutions $u(t)$ of the Cauchy problem (2.1), (2.2), which can be rigorously established, for instance, by the Galerkin method.

Multiplying both sides of (2.1) with $(t-\tau)\partial_t u(t)$ for $\tau \geq 0, t > \tau$, and integrating with respect to t and x we get

$$\begin{aligned}
&\int_\tau^{\tau+1} (\tau_1 - \tau)|\partial_t u(\tau_1)|^2 d\tau_1 + \frac{1}{2}\int_\tau^{\tau+1} \partial_{\tau_1}\left((\tau_1-\tau)a\nabla u(\tau_1), \nabla u(\tau_1)\right)\, d\tau_1 - \\
&-\frac{1}{2}\int_\tau^{\tau+1} (a\nabla u(\tau_1), \nabla u(\tau_1)) d\tau_1 + \int_\tau^{\tau+1} (\tau_1-\tau)\left(f_1(u(\tau_1)), \partial_t u(\tau_1)\right)\, d\tau_1 + \\
&+\int_\tau^{\tau+1} (\tau_1-\tau)\left(f_2(u(\tau_1)), \partial_t u(\tau_1)\right) d\tau_1 = \\
&= \int_\tau^{\tau+1} \left(g, (\tau_1-\tau)\partial_t u(\tau_1)\right) d\tau_1
\end{aligned} \tag{2.20}$$

Let us estimate the terms with f_1 and f_2 separately. From (2.10), (2.11) and

(2.15) we obtain the f_1-estimate

$$\int\limits_{\tau}^{\tau+1}(\tau_1-\tau)(f_1(u(\tau_1)),\partial_t u(\tau_1))\,d\tau_1 =$$

$$= \int\limits_{\tau}^{\tau+1}((\tau_1-\tau)\frac{\partial}{\partial\tau_1}F(u(\tau_1)),1)_H d\tau_1 =$$

$$= F(u(\tau+1),1) - \int\limits_{\tau}^{\tau+1}(F(u(\tau_1)),1)\,d\tau_1 \geq \tag{2.21}$$

$$\geq C_4\sum_{r=1}^{N}\|u_r(\tau+1)\|^{p_r}_{L_{p_r}(\Omega)} - C_5|\Omega| - C_6\sum_{i=1}^{N}\int\limits_{\tau}^{\tau+1}(\|u_r(\tau)\|^{p_r}_{L_{p_r}(\Omega)}d\tau - C_7|\Omega| \geq$$

$$\geq C_4\sum_{r=1}^{N}\|u_r(\tau+1)\|^{p_r}_{L_{p_r}(\Omega)} - C_{10}e^{-\alpha t}|u(0)|^2 - C_{11},$$

From (2.13) we deduce the f_2-estimate

$$\int\limits_{\tau}^{\tau+1}((\tau_1-\tau)f_2(x,x/\varepsilon,u(\tau_1)),\ \partial_t u(\tau_1))_H\,d\tau_1| \leq \tag{2.22}$$

$$\leq \frac{1}{2}C_8\sum_{r=1}^{N}\int\limits_{\tau}^{\tau+1}\|u(\tau)\|^{p_r}_{L_{p_r}}(\tau_1-\tau)d\tau + \frac{1}{2}\int\limits_{\tau}^{\tau+1}(\tau_1-\tau)|\partial_t u(\tau_1)|^2 d\sigma_1 + C_{12}$$

Combining (2.20)–(2.22) and (2.15) we estimate

$$\begin{aligned} &\tfrac{1}{2}\int\limits_{\tau}^{\tau+1}(\tau_1-\tau)|\partial_t u(\tau_1)|^2 d\tau_1 + \tfrac{1}{2}|\nabla u(\tau+1)|^2 + \tfrac{1}{2}C_4\sum_{r=1}^{N}\|u_r(\tau+1)\|^{p_r}_{L_{p_r}} \leq \\ \leq\ & C_{13}e^{-\alpha t}|u_0|^2 + C_{14}. \end{aligned} \tag{2.23}$$

Here we have used $a \geq \mathrm{id}$, by a fixed rescaling of x, Ω. Hence

$$\begin{aligned} &\int\limits_{\tau+\frac{1}{2}}^{\tau+1}|\partial_t u(\tau_1)|^2 d\tau_1 + |\nabla u(\tau+1)|^2 + \sum_{r=1}^{N}\|u_r(\tau+1)\|^{p_r}_{L_{p_r}} \leq \\ \leq\ & Ce^{-\alpha\tau}|u_0|^2 + C' \end{aligned} \tag{2.24}$$

where C, C' do not depend on $\tau \geq 0$.

At this point we would like to obtain higher regularity estimates, both in time t and space x. Since differentiation of (2.1) with respect to x is strictly forbidden, due to (suppressed) dependence of f, g on x/ε, we cautiously proceed by differentiation with respect to t, only. Let us differentiate equation (2.1) with respect to t and denote $\partial_t u = p$,

$$\partial_t p = a \triangle p - f'_u(u) p. \tag{2.25}$$

Multiplying this equation with $(t-\tau)p$ and integrating over Ω we get

$$\begin{aligned}
&\frac{1}{2}\partial_t((t-\tau)|p(\tau)|_H^2) - \frac{1}{2}|p(\tau)|^2 + (t-\tau)(a\nabla p(t), \nabla p(t)) = \\
=& -((t-\tau)f'_u(u(t))p(t), p(t)) \leq \\
\leq& C_1(t-\tau)(p(t), p(t)).
\end{aligned} \tag{2.26}$$

Here $C_1 = C_7$ from (2.12). Integrating in t and proceeding as in (2.20)–(2.22) we obtain

$$\begin{aligned}
&|p(\tau+1)|^2 + \int_\tau^{\tau+1} (\tau_1 - \tau)|\nabla p(\tau_1)|^2 \, d\tau_1 \leq \\
\leq \; & C_1 \int_\tau^{\tau+1} ((\tau_1 - \tau) + 1/2)|p(\tau_1)|^2 \, d\tau_1.
\end{aligned} \tag{2.27}$$

Using (2.24) this implies

$$\begin{aligned}
&|p(\tau+1)|^2 + \int_{\tau+1/2}^{\tau+1} |\nabla p(\tau_1)^2 d\tau_1 \leq \\
&\leq C_3 e^{-\alpha\tau}|u(0)|^2 + C_4
\end{aligned} \tag{2.28}$$

provided that $\tau \geq 1/2$. For $\tau \geq 1$ we therefore obtain

$$\int_\tau^{\tau+1} |\partial_t \nabla u(\tau_1)|_H^2 d\tau \leq C_5 e^{-\alpha\tau}|u(0)|^2 + C_6. \tag{2.29}$$

Consequently, $\nabla u(t)$ is continuous in t with values in H

$$\nabla u(t) \in C([\tau, \tau+1], H), \quad \tau \geq 1. \tag{2.30}$$

By (2.24) the H^1-ball

$$B_1 = \{u \mid u \in H^1 = (H_0^1)^N, \|u\|_{H^1} \leq R\} \tag{2.31}$$

is an absorbing set for the semigroups $\{S(t,\varepsilon), t \geq 0\}, 0 < \varepsilon \leq \varepsilon_0$, if we choose $R = (2C')^{1/2}$. We emphasize that B_1 does not depend on ε. Moreover, B_1 is bounded in H^1 and therefore is compact in H.

Using the growth conditions (2.9) and (2.13) for $f = f(u) = f^1(u) + f^2(u)$ and the Sobolev embedding theorem, we have for solutions $u = u(x,t)$ of (2.1)

$$|f(u(t))|_H^2 \leq \int_\Omega (|u(t)|^6 + 1)dt \leq C'(|\nabla u(t)|_H^6 + 1) \tag{2.32}$$

To estimate the term $|\nabla u|_H$ we invoke (2.24), with $t = \tau + 1 \geq 2$ and obtain

$$\|u(t)\|_{H^1} = \|\nabla u(t)\|_H \leq C_0 e^{-\alpha t}|u(0)|_H^2 + C_0' \tag{2.33}$$

Using (2.28) for $t \geq 2$

$$|p(t)|_H = |\partial_t u(t)|_H \leq C_6 e^{-\alpha t}|u(0)|^2 + C_7$$

We now rewrite equation (2.1) as

$$\Delta u = a^{-1}(\partial_t u + f(u) - g). \tag{2.34}$$

Because $g \in H$ we deduce from (2.28)–(2.34) that

$$|\Delta u(\cdot,t)|_H \leq C_8(|u(0)|_H^3 + 1)e^{-\alpha_1 t} + C_9 \tag{2.35}$$

holds for $t \geq 2$. Evidently our previous estimates can be adapted to replace $t \geq 2$ by $t \geq \delta > 0$, where δ is fixed arbitrary small. Summarizing, (2.35) implies that we can choose $R > 0$ such that the H^2-ball

$$\mathcal{B}_2 = \{u \mid u \in H^2 \cap H_0^1, \quad \|u\|_{H^2} \leq R\} \tag{2.36}$$

is a bounded absorbing set for the semigroups $S(t,\varepsilon)$, uniformly for all $0 < \varepsilon \leq \varepsilon_0$. We repeat that $\mathcal{B}_2$ does not depend on ε and is evidently compact in H. In conclusion, the semigroup $S(t,\varepsilon)$, which is continuous in H for any $t \geq 0$, possesses an absorbing set $\mathcal{B}_2$ which is compact in H. Therefore $S(t,\varepsilon)$ possesses a compact global attractor $\mathcal{A} = \mathcal{A}_\varepsilon$ in H, such that

$$\mathcal{A}^\varepsilon \subseteq \mathcal{B}_2, \qquad \|\mathcal{A}^\varepsilon\|_{H^2} \leq R \tag{2.37}$$

for all $0 < \varepsilon \leq \varepsilon_0$; see [BV92], [Tem88]. This proves proposition 2.2. $\bowtie$

In the next section we introduce the notion of the averaged equation associated to the spatially rapidly oscillating system (2.1): an autonomous equation of the form

$$\partial_t u = a\Delta u - f^0(x,u) + g^0(x), \qquad u|_{\partial\Omega} = 0. \tag{2.38}$$

Here a, f^0 and g^0 satisfy the same conditions as formulated at the beginning of this section, but do not depend on ε. Specifically, none of the variables is suppressed in (2.38). In particular all estimates deduced above hold true for solutions $u(x,t)$ of the Cauchy problem (2.38), (2.2). Therefore, the analogue of proposition 2.2 also holds for the semigroup $S^0(t)$ corresponding to (2.38),

$$S^0(t)u_0(x) = u(x,t), \tag{2.39}$$

where $u(x,t)$ is the solution of (2.38) with initial data

$$u|_{t=0} = u_0(x) \in H. \tag{2.40}$$

Later we shall use the following

Proposition 2.3 *Let $u_0(x) \in H^1$. Then the solution $u(x,t) = S^0(t)u_0(x)$ belongs to $C([0,T],H^1)$, for any $T > 0$.*

Proof:

In view of (2.30), where $\tau \geq 1$ can be replaced by $\tau \geq \delta > 0$ with δ arbitrary small, the continuity of $u \in C((0,T],H^1)$ is already proved. We now have to carefully monitor convergence of $u(\cdot,t) \in H^1$ for $t \searrow 0$. Estimates (2.14) and (2.15) evidently hold true for solutions $u(x,t)$ of the Cauchy problem (2.38), (2.40). Multiplying both sides of (2.1) with $\partial_t u(t)$, only, and integrating from $\tau = 0$ to $\tau = t$ this time, (2.20) becomes

$$\begin{aligned}
&\int_0^t |\partial_t u(\tau_1)|_H^2 d\tau_1 + \tfrac{1}{2}\int_0^t \partial_t(a\nabla u(\tau_1), \nabla u(\tau_1))d\tau_1 + \\
&+\int_0^t (f_1(u(\tau_1)), \partial_t u(\tau_1))d\tau_1 + \int_0^t (f_2(u(\tau_1)), \partial_t u(\tau_1))d\tau_1 = \\
&= \int_0^t (g, \partial_t u(\tau_1)d\tau_1.
\end{aligned} \tag{2.41}$$

Proceeding analogously to (2.21) and (2.22) we estimate

$$\frac{1}{2}|\nabla u(t)|_H^2 + C\sum_{r=1}^{N} \|u_r(t)\|_{L_{p_r}}^{p_r} \le \frac{1}{2}|\nabla u(0)|_H^2 + C_0 t + C_0'. \tag{2.42}$$

So for $0 \le t \le T$

$$|\nabla u(t)|_H^2 \le C_1(|\nabla u(0)|_H^2 + T) \tag{2.43}$$

Replacing f_j by f_j^0 in (2.9), (2.13) we get $|f^0(x, u(t))|_H^2 \le C_2(1 + |u(t)|^6)$. With (2.42) we estimate

$$\begin{aligned}
\int_0^t |f^0(x, u(\tau_1))|_H^2 d\tau_1 \quad &\le C_3 \int_0^t \|u(t)\|_{L_6}^6 dt + C_4 T \le \\
&\le C_5 \int_0^t |\nabla u(0)|_H^6 dt + C_4 T = \\
&= C_5 T|\nabla(u(0))|_H^6 + C_4 T,
\end{aligned} \tag{2.44}$$

uniformly for $0 < t \le T$. The function $u(x,t)$ is a solution of the equation

$$\begin{aligned}
\partial_t u - a\Delta u = -f^0(u(t)) + g^0 \quad &:= G(x,t), \qquad u|_{\partial\Omega} = 0 \\
u|_{t=0} \quad &= u_0(x) \in H_0^1.
\end{aligned} \tag{2.45}$$

In (2.44), we have established that

$$\begin{aligned}
G(x,t) \quad &\in \quad L_2([0,T], H) \\
\|G\|^2 \quad &= \quad \|G\|_{L_2([0,T],H)}^2 \quad < \infty.
\end{aligned} \tag{2.46}$$

Since $Au := -a\Delta u$ with Dirichlet boundary conditions is a selfadjoint positive operator, this implies that the solution $u(x,t)$ of (2.40), (2.41) is continuous, down to $t = 0$:

$$u(x,t) \in C([0,T], H_0^1) \tag{2.47}$$

This fact can, for instance, be proved using the Fourier method or the variations-of-constants formular. An elementary argument runs as follows. Let $\{e_j(x)\}$ be the eigenfunctions of A :

$$Ae_j(x) = \lambda_j e_j(x), \qquad e_j|_{\partial\Omega} = 0. \tag{2.48}$$

Then

$$u(x,t) = \sum_{j=1}^{\infty} C_j(t) e_j(x) \tag{2.49}$$

Decompose $G(x,t) = \sum_j G_j(t) e_j(x)$ and note that $\|G\|^2_{L_2(\Omega\times(0,T))} = \sum_j \int_0^T |G_j(\tau)|^2 d\tau < \infty$. For brevity let $u_0 = 0$. Then

$$\partial_t C_j(t) + \lambda_j C_j(t) = G_j(t), \qquad C_j(0) = 0$$

and therefore

$$C_j(t) = \int_0^t e^{-\lambda_j(t-\tau)} G_j(\tau) d\tau.$$

We have

$$\begin{aligned} \|u(\cdot,t)\|^2_{H^1} &= \sum_j \lambda_j |C_j(t)|^2 = \sum_j \lambda_j (\int_0^t e^{-\lambda_j(t-\tau)} G_j(\tau)d\tau)^2 \le \\ &\le \sum_j \lambda_j \int_0^t |G_j(\tau)|^2 d\tau \int_0^t e^{-2\lambda_j(t-\tau)} d\tau \le \\ &\le \tfrac{1}{2} \sum_j \int_0^t |G_j(\tau)|^2 d\tau. \end{aligned} \tag{2.50}$$

Previously we have proved that $u(x,t) \in C((0,T), H^1)$. So, it is sufficient to prove that

$$\|u(x,t)\|_{H^1} \to 0 \text{ as } t \to 0.$$

But (2.50) indeed implies

$$\|u(\cdot,t)\|^2_{H^1} \le \sum_j \int_0^t |G_j(\tau)|^2 d\tau = \int_0^t \int_\Omega |G(x,\tau)|^2 dx d\tau \to 0,$$

for $t \to 0$, because $G(x,t) \in L_2((0,T),H)$. This proves proposition 2.3. $\bowtie$

Note that estimates (2.41)–(2.43) hold true for any solution $u(x,t) = u^\varepsilon(x,t)$ of the Cauchy problem (2.1), (2.2) with initial data $u_0(x) \in H_0^1(\Omega) := H^1(\Omega)$. So we deduce from (2.43), with $T = 1$, that

$$|\nabla u(t)|_H^2 \le C_1(|\nabla u(0)|_H^2 + 1), \quad 0 \le t \le 1. \tag{2.51}$$

On the other hand the estimate (2.24) with $\tau + 1 = t$ implies

$$|\nabla u(t)|_H^2 \le C|u(0)|_H^2 + C', \quad t \ge 1. \tag{2.52}$$

Together, (2.51) and (2.52) imply

$$\|u(\cdot,t)\|_{H^1} = \|u^\varepsilon(\cdot,t)\|_{H^1} \le C_0(\|u_0\|_{H^1} + 1) \tag{2.53}$$

for any solution $u(x,t)$ of (2.1) and (2.2) with initial data $u_0 \in H^1$, uniformly for all $t \ge 0$ and all $0 \le \varepsilon \le \varepsilon_0$.

3 Rapid oscillations and the averaged equation: global estimate of the distance of solutions

Consider the Cauchy problem for the system (1.1),

$$\partial_t u = a\Delta u - f(x, x/\varepsilon, u) + g(x, x/\varepsilon), \tag{3.1}$$

with Dirichlet boundary conditions, $u/\partial\Omega = 0$, and initial data

$$u|_{t=0} = u_0(x) \in H_0^1. \tag{3.2}$$

For simplicity and brevity we suppose that the components $f_r(x,z,w)$, $r = 1, \cdots, N$, $x \in \Omega$, $z \in \mathbb{R}^n$, $w \in \mathbb{R}^N$, of $f(x,z,w)$ have the following structure

$$f_r(x,z,w) = \sum_{j=1}^{M} b_r^j(x,z) f_{jr}(w) \tag{3.3}$$

Strengthening assumptions (2.7)–(2.13) slightly, we require

$$\sum_{j=1}^{N} |\partial_w f_{jr}(x,z,w)|^2 \le C(|w|^4+1). \tag{3.4}$$

For all $r=1,\cdots,N,\ j=1,\cdots,M$ we suppose that $b_r^j(x,z)$ are bounded

$$|b_r^j(x,z)| \le C \tag{3.5}$$

and the average $b_r^{0j}(x)$ of $b_r^j(x,x/\varepsilon)$ exists in $L_{\infty,w*}(\Omega)$, for $\varepsilon \to 0$:

$$b_r^j(x,x/\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} b_r^{0j}(x). \tag{3.6}$$

This means that for any $\varphi(x) \in L_1(\Omega)$

$$\langle b_r^j(x,x/\varepsilon), \varphi(x)\rangle \underset{\varepsilon\to 0}{\longrightarrow} \langle b_r^{0j}(x), \varphi(x)\rangle$$

where $\langle\cdot,\cdot\rangle$ indicates duality.

We also assume that

$$g(x,x/\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} g^0(x) \tag{3.7}$$

in H_w. In other words

$$(g(x,x/\varepsilon), \varphi(x))_H \underset{\varepsilon\to 0}{\longrightarrow} (g^0(x), \varphi(x))_H$$

for any $\varphi \in H$. The equation

$$\partial_t u^0 = a\Delta u^0 - f^0(x,u^0) + g^0(x), \qquad u^0|_{\partial\Omega} = 0 \tag{3.8}$$

is called the *average* of (3.1), if

$$f_r^0(x,u^0) = \sum_{j=1}^{M} b_r^{0j}(x) f_{jr}(u^0). \tag{3.9}$$

We supply (3.8) with the same initial data (3.2) as for (3.1),

$$u^0|_{t=0} = u_0(x). \tag{3.10}$$

We now specify additional conditions which enable us to estimate the distance between the solutions $u(x,t)$ and $u^0(x,t)$ in the metric of $H = L_2(\Omega)$. Denote

$$\tilde{b}_r^j(x,z) = b_r^j(x,z) - b_r^{0j}(x),$$

for $x \in \Omega, z \in \mathrm{I\!R}^n$. We suppose that there exist functions $B_{r\sigma}^j(x,z)$ which are uniformly bounded for all $x \in \Omega, z \in \mathrm{I\!R}^n$,

$$|B_r^j\sigma(x,z)| \leq C_0 \tag{3.11}$$

and which represent $\tilde{b}_r^j$ such that

$$\tilde{b}_r^j(x,z) = \sum_{\sigma=1}^{n} \partial_{z_\sigma} B_{r\sigma}^j(x,z), \tag{3.12}$$

for all $r = 1,\cdots,N$ and $j = 1,\cdots,M$. With respect to the x-derivatives we assume an ε-independent L_3-bound

$$\|\partial_{x_\sigma}^1 B_{r\sigma}^j(\cdot,\cdot/\varepsilon)\|_{L_3(\Omega)} = \left(\int_\Omega \partial_{x_\sigma}^1 B_{r\sigma}^j(x,x/\varepsilon)|^3 dx\right)^{1/3} \leq C_0, \tag{3.13}$$

uniformly for all $r = 1,\cdots,N,\ \sigma = 1,\cdots,n,\ j = 1,\cdots,M.$

Here $\partial_{x_\sigma}^1$ indicate partial derivatives with respect to the first argument x of the function $B_{r\sigma}^j(x,z)$, only. We call (3.12) a *divergence representation* of $\tilde{b}_r^j$ by $B_{r\sigma}^j$. Analogously we denote $\tilde{g}(x,z) = g(x,z) - g^0(x)$ and require the existence of a function $J_\sigma(x,z)$ such that $\tilde{g}(x,z)$ admits a divergence representation

$$\tilde{g}_r(x,z) = \sum_{\sigma=1}^{n} \partial_{z_\sigma} J_{r\sigma}(x,z). \tag{3.14}$$

We assume bounds

$$\begin{aligned} |J_{r\sigma}(\cdot,\cdot/\varepsilon)|_H &\leq C_0, \\ \|\partial_{x_\sigma}^1 J_{r\sigma}(\cdot,\cdot/\varepsilon)\|_{L_{6/5}(\Omega)} &\leq C_0, \end{aligned} \tag{3.15}$$

for $r = 1,\cdots,N,\ \sigma = 1,\cdots,n$, and all $0 < \varepsilon \leq \varepsilon_0$.

Theorem 3.1 *Let $f(x,z,w)$ satisfy conditions (2.6)–(2.13), (3.3)–(3.6) and (3.11)–(3.13). Let $g(x,z)$ satisfy conditions (2.5), (3.7), (3.14) and (3.15). Then there exist a monotone function $C(s) > 0$ and a constant $\rho > 0$ such that the solutions $u^\varepsilon(x,t) = u(x,t)$ and $u^0(x,t)$ of the respective Cauchy problems (3.1), (3.2) and (3.8), (3.10) with equal initial data $u_0(x) \in H_0^1$ satisfy the distance estimate*

$$|u^\varepsilon(x,t) - u^0(x,t)|_H \leq C(\|u_0\|_{H^1})\varepsilon e^{\varrho t}, \tag{3.16}$$

uniformly for $0 \leq t < \infty$.

Proof:

Denote $v(x,t) = u^\varepsilon(x,t) - u^0(x,t)$. Subtracting (3.8) from (3.1) we get

$$\begin{aligned} &\partial_t v - a\Delta v + f^0(x,u^\varepsilon) - f^0(x,u^0) = \\ &= -(f(x,x/\varepsilon,u^\varepsilon) - f^0(x,u^\varepsilon)) + g(x,x/\varepsilon) - g^0(x). \end{aligned} \tag{3.17}$$

Multiplying both sides of (3.17) by v and integrating over Ω we get

$$\begin{aligned} &\tfrac{1}{2}\partial_t|v|^2 + (a\nabla v, \nabla v) \leq \\ \leq\ &(f^0(x,u^\varepsilon) - f^0(x,u^0), v) - (f(x,x/\varepsilon,u^\varepsilon) - f^0(x,u^\varepsilon), v) + \\ &+(g(x,x/\varepsilon) - g^0(x), v), \end{aligned} \tag{3.18}$$

where $|u| = |u|_H, H = (L_2(\Omega))^N$. To prove the theorem, we estimate the terms of the right hand side of (3.18), one by one.

Assumption (2.12) implies

$$\begin{aligned} &-(f^0(x,u^\varepsilon) - f^0(x,u^0), v) = \\ &= -(\int_0^1 \partial_u f^0(x,(1-s)u^0 + su^\varepsilon)ds \cdot v, v) \leq \\ &\leq C_1(v,v) \end{aligned} \tag{3.19}$$

for some positive constant C_1.

Applying (3.3) and (3.12) to the second term of the right hand side of (3.18) we get

$$
\begin{aligned}
&-(f^0(x,x/\varepsilon,u^\varepsilon)-f^0(x,u^\varepsilon),v)= \\
&=\sum_{r=1}^{N}\sum_{j=1}^{M}((b_r^j(x,x/\varepsilon)f_{jr}(u^\varepsilon)-b_r^{0j}(x)f_{jr}(u^\varepsilon),v_r)= \\
&=\sum_{r,j}(\tilde{b}_r^j(x,x/\varepsilon)f_{jr}(u^\varepsilon),v_r).
\end{aligned}
\tag{3.20}
$$

We now insert the divergence representation (3.12) of $\tilde{b}_r^j$ as follows. First we note, with obvious notation, that

$$
\partial_{z_\sigma}B(x,x/\varepsilon)=\epsilon\frac{d}{dx_\sigma}B(x,x/\varepsilon)-\epsilon\partial^1_{x_\sigma}B(x,x/\varepsilon). \tag{3.21}
$$

Inserting this into the right hand side of (3.20), we obtain

$$
\begin{aligned}
&&-(f(x,x/\varepsilon,u^\varepsilon)-f^0(x,u^\varepsilon),v)= \\
&=&-\sum_{r,j,\sigma}(\partial_{z_\sigma}B^j_{r\sigma}(x,x/\varepsilon),f_{jr}(u^\varepsilon)v_r)= \\
&=&+\varepsilon\sum_{r,j,\sigma}(B^j_{r\sigma}(x,x/\varepsilon),\partial_{x_\sigma}(f_{jr}(u^\varepsilon)v_r))- \\
&&-\varepsilon\sum_{r,j,\sigma}(\partial^1_{x_\sigma}B^j_{r\sigma}(x,x/\varepsilon),f_{jr}(u^\varepsilon)v_r).
\end{aligned}
\tag{3.22}
$$

Using boundedness (3.11) of the functions $B^j_{r\sigma}$ and the norms $\|\partial^1_{x_\sigma}B^j_{r\sigma}(\cdot,\cdot/\varepsilon)\|_{L_3}$, and integrating by parts, we deduce from (3.22) that

$$
\begin{aligned}
&&|(f(x,x/\varepsilon,u^\varepsilon)-f^0(x,u^\varepsilon),v)|_H\le \\
&\le&\varepsilon C_0\sum_{r,j,\sigma}|(\partial_{x_\sigma}(f_{jr}(u^\varepsilon)v_r),1)|+ \\
&&+\varepsilon\sum_{r,j,\sigma}\|\partial^1_{x_\sigma}B^j_{r\sigma}(\cdot,\cdot/\varepsilon)\|_{L_3}\|f_{jr}(u^\varepsilon)\|_{L_2}\|v\|_{L_6}\le \\
&\le&\varepsilon C_0\sum_{r,j,\sigma}(|(\partial_{x_\sigma}f_{jr}(u^\varepsilon),v_r)|+|(f_{jr}(u^\varepsilon),\partial_{x_\sigma}v_r)|)+ \\
&&+\varepsilon C_0\sum_{r,j,\sigma}\|f_{jr}(u^\varepsilon)\|_{L_2}\|v\|_{L_6}.
\end{aligned}
\tag{3.23}
$$

The first term on the right hand side of (3.23) is less than

$$
\begin{aligned}
&&\varepsilon C_0\sum_{r,j,\sigma}|f_{jr}(u^\varepsilon)|_H\cdot|\partial_{x_\sigma}v^r|_H+\varepsilon C_0\sum_{r,j,\sigma}|(\partial_u f_{jr}(u^\varepsilon)\partial_{x_\sigma}u^\varepsilon,v_r)|\le \\
&\le&\varepsilon C_0C(1+\|u^\varepsilon\|^3_{L_6})\|v\|_{H^1}+\varepsilon C_0C(\|u^\varepsilon\|_{L_6}+1)^2\|u^\varepsilon\|_{H^1}\|v\|_{L_6}\le \\
&\le&\varepsilon C_2(1+\|u_0\|^3_{H^1})\|v\|_{H^1}\le C_3\varepsilon^2\quad+\tfrac{1}{4}\|v\|^2_{H^1}
\end{aligned}
\tag{3.24}
$$

where $C_3 = C_3(\|u_0\|_1)$. We have used (2.2), (2.9), (2.13), (3.4) and (2.53) here. Analogously we deduce that the second and third term on the right hand side of (3.23) are estimated by

$$C_3\varepsilon^2 + \frac{1}{4}\|v\|^2_{H^1}.$$

This completes our estimate of the second term on the right hand side of (3.18),

$$|(f(x,x/\varepsilon,u^\varepsilon) - f^0(x,u^\varepsilon),v)| \leq 2C_3\varepsilon^2 + \frac{1}{2}\|v\|^2_{H^1}. \tag{3.25}$$

To estimate the third term in (3.18) we use the divergence representation (3.14) of $\tilde{g}(x,z) = g(x,x/\varepsilon) - g(x)$. From (3.15) we deduce, analogously to the above arguments,

$$\begin{aligned} &|(g(x,x/\varepsilon) - g^0(x),v)| = |(\tilde{g}(x,x/\varepsilon),v)| = \\ &= |\sum_{\sigma=1}^{n}(\partial_{z_\sigma}J_\sigma(x,x/\varepsilon),v)| \leq C_4\varepsilon^2 + \tfrac{1}{2}\|v\|^2_{H^1}. \end{aligned} \tag{3.26}$$

Adding the estimates (3.19), (3.25) and (3.26) of the three terms on the right hand side of (3.18) we get

$$\partial_t|v|^2 \leq 2C_1|v|^2 + C_5\varepsilon^2. \tag{3.27}$$

Here we have used that $a \geq \mathrm{id}$, and hence $(a\nabla v, \nabla v) \geq \|v\|^2_{H^1}$. Applying Gronwall's inequality to (3.27) proves (3.16), and the theorem. $\bowtie$

Let us now give some sufficient conditions which guarantee the existence of divergence representations (3.12), (3.14). Since the treatment of $\tilde{b}^j_r$ and $\tilde{g}_r$ will be completely analogous, we replace any of these expressions by the new symbol γ. So, let $\gamma(x,z) = \gamma(x_1,\cdots,x_n,z,\cdots,z_n)$ be a sufficiently smooth function, as specified below, which is quasiperiodic with respect to $z = (z_1,\cdots,z_n)$. This means that:

- there exists a function $\Gamma(x,\omega_{11},\cdots,\omega_{1l_1},\cdots,\omega_{n1},\cdots,\omega_{nl_n}) := \Gamma(x,\omega_1,\cdots,\omega_n)$ which is 2π-periodic with respect to each ω_{ij}. Here $\omega_1 = (\omega_{11},\cdots,\omega_{1l_1}) \in \mathrm{IR}^{l_1},\cdots,\omega_n = (\omega_{n_1},\cdots,\omega_{nl_n}) \in \mathrm{IR}^{l_n}$.

- there exist rationally independent frequencies $\alpha_{11}, \cdots, \alpha_{1l_1}, \ , \cdots, \alpha_{n1}, \cdots, \alpha_{nl_n}$ such that, in various notations

$$
\begin{aligned}
\gamma(x, x/\varepsilon) &= \gamma(x_1, \cdots, x_n, x_1/\varepsilon, \cdots, x_n/\varepsilon) := \\
&= \Gamma(x, \alpha_{11}x_1/\varepsilon, \cdots, \alpha_{1l_1}x_1/\varepsilon, \cdots, \alpha_{n1}x_n/\varepsilon, \cdots, \alpha_{nl_n}x_n/\varepsilon) = \\
&= \Gamma(x, \alpha_1 x_1/\varepsilon, \cdots, \alpha_n x_n/\varepsilon) = \\
&= \Gamma(n, \alpha \cdot x/\varepsilon).
\end{aligned}
\tag{3.28}
$$

Let $\tilde{\Gamma}(x, \omega) = \Gamma(x, \omega) - \Gamma_0(x)$, where $\Gamma^0(x)$ is the average of $\Gamma(x, \omega)$ with respect to ω :

$$\Gamma_0(x) = |T^l|^{-1} \int_{T^l} \Gamma(x, \omega_1, \cdots, \omega_n) d\omega_1 \cdots d\omega_n, \tag{3.29}$$

where $T^l = T^{l_1} \times \cdots \times T^{ln}$, and $T^{l_i} = \mathbb{R}^{l_i}/(\mathbb{Z} \cdot 2\pi)^{l_i}$ is the l_i-dimensional torus. Assume that the Fourier series

$$\Gamma(x, \omega) = \sum_k \Gamma_k(x) e^{ik\cdot\omega} \tag{3.30}$$

is rapidly convergent. We use the notation $k \in \mathbb{Z}^l, \ l = l_1 + \cdots + l_n, k \cdot \omega = k_1, \omega_1 + \cdots + k_n \omega_n$. For a precise notion of rapid convergence see (3.43) below. This defines our notion of quasiperiodicity.

Finally let

$$
\begin{aligned}
\tilde{\gamma}(x, z) &= \gamma(x, z) - \Gamma_0(x) = \\
&= \sum_{k \neq 0} \Gamma_k(x) \exp(i \sum_{j=1}^{n} k_j \cdot \alpha_j z_j)
\end{aligned}
\tag{3.31}
$$

where $k_j = (k_{j1}, \cdots, k_{jl_j}) \in \mathbb{Z}^{l_j}, \ \alpha_j \in \mathbb{R}^{l_j}$, and $z_j \in \mathbb{R}$.

For any such quasiperiodic function $\tilde{\gamma}(x, z)$, we now construct a corresponding divergence representation by functions $S_\sigma(x, z), \sigma = 1, \cdots, n$,

$$\tilde{\gamma}(x, z) = \sum_{\sigma=1}^{n} \partial_{z_\sigma} S_\sigma(x, z). \tag{3.32}$$

We shall find $S_\sigma(x,z)$ of the form

$$S_\sigma(x,z) = \sum_{k\in\mathbb{Z}^l\setminus\{0\}} \zeta_k^\sigma(x)\exp(i\sum_{j=1}^n k_j\cdot\alpha_j z_j) \tag{3.33}$$

From (3.31)–(3.33) we derive:

$$\begin{aligned}&\sum_{k\neq 0}\Gamma_k(x)\exp(i\sum_j k_j\cdot\alpha_j z_j) = \tilde\gamma(x,z) = \\ &= \sum_{k\neq 0}\sum_{\sigma=1}^n (k_\sigma,\alpha_\sigma)\zeta_k(x)\exp(i\sum_{j=1}^n k_j\cdot\alpha_j z_j).\end{aligned}$$

So (3.32) will hold if

$$\sum_{\sigma=1}^n (k_\sigma\cdot\alpha_\sigma)\zeta_k^\sigma(x) = -i\Gamma_k(x), \tag{3.34}$$

for all $k\in\mathbb{Z}^l\setminus\{0\}$. We now choose $0\le\beta_{k,\sigma}\le 1$ such that $\beta_{k,\sigma}=0$ for $k_\sigma\cdot\alpha_\sigma=0$, and $\sum_\sigma\beta_{k,\sigma}=1$. This is possible because $\sum_\sigma k_\sigma\cdot\alpha_\sigma\neq 0$ by rational independence of the frequencies $\alpha_{11},\cdots,\alpha_{nl_n}$. With these coefficients $\beta_{k,\sigma}$ at hand, a solution of (3.34) is given by

$$\zeta_k^\sigma(x) := \begin{cases} -i\Gamma_k(x)\beta_{k,\sigma}/(k_\sigma\cdot\alpha_\sigma) & \text{if } k_\sigma\cdot\alpha_\sigma\neq 0\\ 0 & \text{otherwise.}\end{cases} \tag{3.35}$$

This definition is the source of small denominators $k_\sigma\cdot\alpha_\sigma$, which will require a Diophantine condition.

So we can define

$$S_\sigma(x,z) = -i\sum_{\substack{k\neq 0\\ k_\sigma\cdot\alpha_\sigma\neq 0}}\beta_{k\sigma}\frac{\Gamma_k(x)}{k_\sigma\cdot\alpha_\sigma}\exp\left(i\sum_j k_j\cdot\alpha_j z\right) \tag{3.36}$$

for $\sigma=1,\cdots,n$. Let us repeat that the sum in (3.36) runs over $k\in\mathbb{Z}^l\setminus\{0\}$, but terms with $k_\sigma\cdot\alpha_\sigma=0$ are to be omitted.

We address the norm constraints (3.15) next. In terms of γ, S_σ, replacing $\tilde g_r, J_{r\sigma}$, the norm constraints (3.15) become

$$\begin{aligned}|S_\sigma(\cdot,\cdot/\varepsilon)|_H &\le C_0\\ \|\partial^1_{x_\sigma}S_\sigma(\cdot,\cdot/\varepsilon)\|_{L_q(\Omega)} &\le C_0\end{aligned} \tag{3.37}$$

with $q = 6/5$. To satisfy (3.37) we require the rapid decay of $|\Gamma_k(\cdot)|_H$, $\|\partial_x \Gamma_k(\cdot)\|_{L_q}$ and – most importantly – the Diophantine approximation condition. From (3.35) and (3.36)

$$\begin{aligned} |S_\sigma(\cdot,\cdot/\varepsilon)|_H &\leq \sum_{k_\sigma\cdot\alpha_\sigma\neq 0} \beta_{k\sigma} \tfrac{|\Gamma_k|_H}{|k_\sigma\cdot\alpha_\sigma|} \leq \\ &\leq \sum_{k\in\mathbb{Z}^l\setminus\{0\}} \tfrac{1}{c}|k_\sigma|^{l_\sigma-1+\delta}|\Gamma_k|_H. \end{aligned} \tag{3.38}$$

Here we have used the Diophantine condition

$$|k_\sigma\cdot\alpha_\sigma| \geq c|k_\sigma|^{-(l_\sigma-1+\delta)} \tag{3.39}$$

Let $\mathcal{D}_\sigma \subseteq \mathbb{R}^{l_\sigma}$ denote the set of frequencies $\alpha_\sigma \in \mathbb{R}^{l_\sigma}$ for which there exist small positive c, δ such that the Diophantine condition (3.39) holds, for all $k_\sigma \in \mathbb{Z}^{l\sigma}\setminus\{0\}$. Then it is known from Diophantine approximation theory, that $\mathcal{D}_\sigma$ is a set of full Lebesgue measure in $\mathbb{R}^{l_\sigma}$,

$$\mathrm{meas}_{\mathbb{R}^{l_\sigma}}(\mathbb{R}^{l_\sigma}\setminus\mathcal{D}_\sigma) = 0; \tag{3.40}$$

see for example [Cas57]. So our Diophantine estimate (3.38) holds true for a set of frequences $\alpha = (\alpha_1, \cdots, \alpha_n)$ of full Lebesgue measure in $\mathbb{R}^l$.

From estimate (3.38) we conclude

$$|S_\sigma(\cdot,\cdot/\varepsilon)|_H \leq C_0 \tag{3.41}$$

for all $\sigma = 1, \cdots, n$, provided that the sum

$$\sum_{k\in\mathbb{R}^l\setminus 0} |k_\sigma|^{-l_\sigma-1+\delta}|\Gamma_k|_H < \infty \tag{3.42}$$

converges. A sufficient condition in terms of Γ_k would be an algebraic decay

$$|\Gamma_k|_H \leq C(1+|k_\sigma|)^{-(l_\sigma-1+\delta)}(1+|k|)^{-(l+\delta')}, \tag{3.43}$$

for all $\sigma = 1, \cdots, n$ and some $\delta' > 0$. Analogously, to fulfil the second norm constraint in (3.37) it is sufficient that

$$\|\partial_{x_\sigma}\Gamma_k\|_{L_{6/5}(\Omega)} \leq C(1+|k_\sigma|)^{-(l_\sigma-1+\delta)}(1+|k|)^{-(l+\delta')}, \tag{3.44}$$

for $\sigma = 1, \cdots, n,\ k \in \mathbb{Z}^l \backslash 0,\ l = l_1 + \cdots + l_n$, and some $\delta' > 0$.

In the original notation of (3.14) for $\tilde{g}$-components $\tilde{g}_r$, we now have to replace γ by $\gamma_r = g_r$, to keep track of $r = 1, \cdots, n$ again. Putting

$$\tilde{g}_r = \tilde{\gamma}_r = \sum_{k \neq 0} \Gamma_{rk}(x) \exp(i \sum_j k_j \cdot \alpha_j z_j), \tag{3.45}$$

as in (3.31), conditions (3.43) and (3.44) read

$$|\Gamma_{rk}(\cdot)|_H \leq C(1 + |k_\sigma|)^{-(l_\sigma - 1 + \delta)}(1 + |k|)^{-(l + \delta')} \tag{3.46}$$

$$\|\partial_{x_\sigma} \Gamma_{rk}(\cdot)\|_{L_{6/5}(\Omega)} \leq C(1 + |k_\sigma|)^{-(l_\sigma - 1 + \delta)}(1 + |k|)^{-(l + \delta')}. \tag{3.47}$$

Analogously to the divergence representation (3.14) for $\tilde{g}r$, we now consider the divergence representation (3.12) for the coefficents $\tilde{b}_r^j(x, z)$ by functions $B_{r\sigma}^j(x, z)$. For simplicity of notation we abbreviate $\tilde{b}_r^j(x, z)$ by $\beta(x, z)$. Analogously to $\Gamma(x, \omega)$ in (3.28) above, let $\Gamma(x, \omega)$ now denote the 2π-periodic function in ω which corresponds to $\beta(x, z)$. As with $\tilde{g}_r$ above, we just replace γ by $\gamma_r^j = b_r^j$ to keep track of $r = 1, \cdots, n$, and $j = 1, \cdots, M$. Due to the different form (3.11), (3.13) of the modified norm constraints for $B_{r\sigma}^j$, the norm constraints (3.37) take the modified form

$$\begin{aligned} |S_\sigma(x, z)| &\leq C_0, \\ \|\partial_{x_\sigma}^1 S_\sigma(\cdot, \cdot/\varepsilon)\|_{L_q(\Omega)} &\leq C_0, \end{aligned} \tag{3.48}$$

where the first inequality is now required to hold uniformly for all x, z. Analogously to (3.30), (3.31) we expand

$$\tilde{b}_r^j = \tilde{\gamma}_r^j = \sum_{k \neq 0} \Gamma_{rk}^j \exp(i \sum_j k_j \cdot \alpha_j z_j) \tag{3.49}$$

with $\Gamma_{rk}^j(x)$ denoting the Fourier expansion of $\Gamma(x, \omega)$ with respect to ω. The two norm constraints in (3.48) are then both satisfied if

$$\|\Gamma_{rk}^j(\cdot)\|_{C^0(\bar{\Omega})} \leq C(1 + |k_\sigma|)^{-(l_\sigma - 1 + \delta)}(1 + |k|)^{-(l + \delta')} \tag{3.50}$$

$$\|\partial_{x_\sigma} \Gamma_{rk}^j(\cdot)\|_{L_3(\Omega)} \leq C(1 + |k_\sigma|)^{-(l_\sigma - 1 + \delta)}(1 + |k|)^{-(l + \delta')} \tag{3.51}$$

both hold. Again $r = 1, \cdots, n$, $k \in \mathbb{Z}^l \backslash \{0\}$, $l = l_1 + \cdots + l_n$, and $j = 1, \cdots, M$.

So we have proved the following

Theorem 3.2 *Let the coefficients $b_r^j(x,z)$ and the functions $g_r(x,z)$ in the reaction-diffusion system (3.1) satisfy the following conditions:*

- *all these functions are quasiperiodic with respect to z;*
- *their corresponding frequencies α_{ij} satisfy Diophantine condition (3.39)*
- *the coefficients $\Gamma_{rk}(x)$ in the series (3.45) of $\tilde{g}_r(x,z) = g_r(x,z) - g_r^0(x)$ satisfy the decay conditions (3.46), (3.47)*
- *the coefficients $\Gamma_{rk}^j(x)$ in the series (3.49) of $\tilde{b}_r^j(x,z) = b_r^j(x,z) - b_r^{0,j}(x)$ satisfy the decay conditions (3.50), (3.51).*

Then the divergence representations (3.12) for $\tilde{b}_r^j$ and (3.14) for $\tilde{g}_r$ hold true, together with their respective estimates (3.11), (3.13) and (3.15). Consequently the solutions of the Cauchy problem (3.1), (3.2) and the averaged system (3.8), (3.10) satisfy the global distance estimate (3.16) of theorem 3.1.

We repeat that the Diophantine condition (3.39) is satisfied for a set of frequencies $\alpha_{11}, \cdots, \alpha_{n,l_n}$ of full Lebesgue measure in $\mathbb{R}^l$. Moreover, these Diophantine frequencies are allowed to be different for b and g, and different even for different components b_r^j and g_r.

4 Quantitative homogenization of the global attractor: Fractional order convergence

In lemma 4.1 we isolate a simple abstract principle for fractional convergence of global attractors; see also [Vis00]. As an application, we obtain fractional order upper semicontinuity of the global attractors $\mathcal{A}^\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$, for reaction-diffusion systems of type (1.1). For $0 \le \varepsilon \le \varepsilon_0$ let $S_t^\varepsilon, t \ge 0$ denote a family of semiflows on a metric space H with distance function d. For $0 \le \varepsilon \le \varepsilon_0$ let $\mathcal{A}^\varepsilon \subseteq H$ be *negatively invariant* under S_t^ε : for any $\tilde{u}_0 \in \mathcal{A}^\varepsilon$ and any $t \ge 0$ we require that there exists $u_0 \in \mathcal{A}^\varepsilon$ such that

$$S_t^\varepsilon u_0 = \tilde{u}_0. \tag{4.1}$$

For example, the global attractor of a semigroup S_t^ε is negatively invariant, because it consists of all globally bounded trajectories $S_t^\varepsilon u_0, t \in \mathbb{R}$.

For $\varepsilon = 0$, let $\mathcal{A}^0 \subseteq H$ denote the global attractor of the semigroup S_t^0 in H, which we will in fact require to be *exponentially attracting* with exponential rate $\nu > 0$: we assume there exists a constant $C = C(\varepsilon_0)$ such that for all $t \ge 0$

$$d(S_t^0 u_0, \mathcal{A}_0) \le C e^{-\nu t} \tag{4.2}$$

holds, uniformly for all $u_0 \in \bigcup\limits_{0<\varepsilon\le\varepsilon_0} \mathcal{A}^\varepsilon$.

Concerning the perturbation S_t^ε of S_t^0, we assume a *fractional order convergence* rate $\gamma > 0$ in ε at an *exponential separation rate* $\rho > 0$ in time $t \ge 0$: we assume there exists a constant $C = C(\varepsilon_0)$ such that for all $0 < \varepsilon \le \varepsilon_0$ and all $t \ge 0$

$$d(S_t^\varepsilon u_0, S_t^0 u_0) \le C \varepsilon^\gamma e^{\rho t} \tag{4.3}$$

holds, uniformly for all $u_0 \in \bigcup\limits_{0<\varepsilon\le\varepsilon_0} \mathcal{A}^\varepsilon$. For the semigroups S_t^ε of section 1 on the Hilbert space $H = L_2(\Omega)$ with global attractors $\mathcal{A}^\varepsilon$, this is precisely the estimate shown to hold in theorem 3.1, (3.16) for our dissipative reaction-diffusion system (3.1), (3.2). Note that $\gamma = 1$ in that case.

Lemma 4.1 *Consider semigroups $S_t^\varepsilon, 0 \le \varepsilon \le \varepsilon_0 \le 1,\ t \ge 0$ on a metric space H with negatively invariant sets $\mathcal{A}^\varepsilon$, for $\varepsilon > 0$ and an exponential attractor $\mathcal{A}^0$ for $\varepsilon = 0$; see (4.1), (4.2). Finally assume fractional order* *convergence of semigroups S_t^ε, for $\varepsilon \searrow 0$, with at most exponential separation; see (4.3).*

Then the sets $\mathcal{A}^\varepsilon$ converge to $\mathcal{A}^0$ with fractional order in ε. More precisely, define the fractional convergence rate

$$\gamma' = \gamma/(1+\rho/\nu). \tag{4.4}$$

Then there exists a constant $C > 0$ such that

$$d(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \sup_{\tilde{u}_0 \in \mathcal{A}^\varepsilon} d(\tilde{u}_0, \mathcal{A}^0) \leq C\varepsilon^{\gamma'} \tag{4.5}$$

holds for all $0 < \varepsilon \leq \varepsilon_0$.

Proof:

The proof is a simple application of the triangle inequality. Let

$$\mathcal{B} := \bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}^\varepsilon. \tag{4.6}$$

Choose $C > 0$ such that exponential attraction and fractional convergence assumptions (4.2), (4.3) both hold for this set $\mathcal{B}$. Pick $0 < \varepsilon \leq \varepsilon_0$ and $\tilde{u}_0 \in \mathcal{A}^\varepsilon \subset \mathcal{B}$, arbitrarily. For $t \geq 0$ chosen below consider $u_0 \in \mathcal{A}^\varepsilon$ such that

$$S_t^\varepsilon u_0 = \tilde{u}_0. \tag{4.7}$$

This is possible by backwards invariance (4.1) of $\mathcal{A}^\varepsilon$. Then estimates (4.2) and (4.3) imply

$$\begin{aligned} d(\tilde{u}_0, \mathcal{A}^0) &= d(S_t^\varepsilon u_0, \mathcal{A}^0) \leq \\ &\leq d(S_t^\varepsilon u_0, S_t^0 u_0) + d(S_t^0 u_0, \mathcal{A}^0) \leq \\ &\leq C(\varepsilon^\gamma e^{\rho t} + e^{-\nu t}). \end{aligned} \tag{4.8}$$

Balancing terms, we choose $t \geq 0$ such that

$$\varepsilon^\gamma e^{\rho t} = e^{-\nu t}. \tag{4.9}$$

Clearly this is the case for

$$t = -\frac{\gamma \log \varepsilon}{\nu + \rho} \geq 0. \tag{4.10}$$

Substituting this choice of t back into (4.8) proves the lemma. $\bowtie$

For a specific example, let

$$u^\varepsilon(t) := S_t^\varepsilon u_0, \quad t \geq 0 \tag{4.11}$$

denote the global solution of

$$u_t^\varepsilon = a\Delta u^\varepsilon + f(x, x/\varepsilon, u^\varepsilon) + g(x, x/\varepsilon), \tag{4.12}$$

for $\varepsilon > 0$ and, for $\varepsilon = 0$, of the averaged system

$$u_t^0 = a\Delta u^0 + f^0(x, u^0) + g^0(x). \tag{4.13}$$

For simplicity, we consider $x \in \Omega$ and Dirichlet boundary conditions, as well as the dissipation, regularity, and convergence assumptions (2.5)–(2.13) and (3.3)–(3.7), (3.11)–(3.15) of theorems 2.1 and 3.1. Let $\mathcal{A}^\varepsilon,\ 0 \leq \varepsilon \leq \varepsilon_0$ denote the associated global attractors on $H = (L_2(\Omega))^N$. Uniform H-boundedness, and even H^2-boundedness, of the global attractors $\mathcal{A}^\varepsilon$ was established in estimate (2.18) of proposition 2.2. Negative invariance holds automatically by the general theory; see [BV92], [Tem88]. Fractional order convergence (4.3), subject to the above H^1-bound, was established in theorems 3.1, 3.2 with $\gamma = 1$, subject to Diophantine conditions (3.39) on the rapid quasiperiodic x-dependence of f and g, and subject to spatial regularity assumptions (3.46), (3.47), (3.50), (3.51). Exponential attractivity of $\mathcal{A}^0$, which is a property of the homogenized system (4.13) alone, independently of its rapidly oscillating approximations, will be deferred to section 5, under slightly stronger smoothness assumptions on the nonlinearity f; see (5.9), (5.10). In particular for $\varepsilon = 0$ we will only consider perturbations $f = f^1 + \eta f^2, |\eta| \leq \eta_0$, of the homogenized gradient nonlinearity f^1.

Summarizing, this proves

Theorem 4.1 *Let assumptions (2.5)–(2.13) of theorem 2.1 and proposition 2.2 be satisfied, as well as assumptions (3.3)–(3.7), (3.11)–(3.15), (3.39),* *(3.46), (3.47), (3.50), (3.51) of theorems 3.1, 3.2. Moreover we require Hölder continuity and at most quadratic growth of derivatives of the homogenized nonlinearities f^1, f^2 as specified in (5.9), (5.10) below. For $\eta = 0$, we assume all equilibria of the homogenized gradient system (4.13) to be hyperbolic; see (5.5).*

Then there exist $\varepsilon_0, \eta_0 > 0$ such that in the topology of $L_2(\Omega)$ the global attractors $\mathcal{A}^\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$, of reaction-diffusion systems (4.1), (4.13) on $\Omega \subset \mathbb{R}^3$ with Dirichlet boundary conditions and nonlinearity $f = f^1 + \eta f^2$, $|\eta| \le \eta_0$, satisfy an upper semicontinuity estimate of the form

$$d_{L_2(\Omega}(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \sup_{\tilde{u} \in \mathcal{A}^\varepsilon} d(\tilde{u}_0, \mathcal{A}^0) \le C\varepsilon^{\gamma'},$$

for some $C > 0$ and fractional order

$$\gamma' = \frac{1}{1 + \rho/\nu}.$$

Here $\nu > 0$ is the exponential attraction rate of $\mathcal{A}^0$ and ρ is the exponential separation rate between trajectories of the semiflows S_t^ε and S_t^0, both measured in $L_2(\Omega)$. The perturbation bound η_0 depends only on the homogenized data f^1, f^2, g.

5 Exponential attractor of the averaged equation

In section 3 we proved that the averaged equation for (3.1) has the form

$$\partial_t u = a\Delta u - f(x, u) + g(x), \quad u|_{\partial\Omega} = 0. \tag{5.1}$$

Here $f(x, u) = f^0(x, u)$, $g(x) = g^0(x)$ satisfy conditions (2.5)–(2.13) but do not depend on ε.

The matrix a is positive definite and symmetric, $g \in H = L_2(\Omega)$ and

$$f(x, w) = f^1(x, w) + f^2(x, w) \tag{5.2}$$

where $f_1^1(x,w) = (f_1^1, \cdots, f_N^1)$ is a potential vector function,

$$f^1(x,w) = \nabla_w F(x,w). \tag{5.3}$$

The functions $f^1(x,w)$ and $f^2(x,w)$ satisfy the inhomogeneous growth and dissipation conditions which are analogous to (2.7)–(2.13). First consider the case $f^2 = 0$. We then obtain a gradient system

$$\partial_t u = a\Delta u - f^1(x,u) + g(x), \quad u|_{\partial\Omega} = 0, \tag{5.4}$$

with global L_2-attractor $\mathcal{A}^0$ bounded in H^2; see proposition 2.2. We assume that this gradient system possesses only a finite number of equilibria $\varphi_i(x),\ i = 1, \cdots, \mu$,

$$a\Delta\varphi_i - f^1(x,\varphi_i) + g(x) = 0 \quad \varphi_i|_{\partial\Omega} = 0 \tag{5.5}$$

and that all these equilibria are hyperbolic. Then the exponential attraction property of the global attractor $\mathcal{A}^0$ of (5.5) in the phase space $H = (L_2(\Omega))^N$ was established in [BV92]. There it was proved, under the supplementary condition of Hölder continuity of $\partial_w f^1(x,w)$ specified in (5.10) below, that for any bounded set $\mathcal{B} \subseteq H$ there exist constants $C = C(\mathcal{B})$ and $\nu > 0$, such that for any trajectory $u(t) = u(t;u_0),\ t \geq 0$, starting at $u(0) = u_0 \in \mathcal{B}$ we have

$$dist_H(u(t), \mathcal{A}^0) \leq Ce^{-\nu t}. \tag{5.6}$$

It was also established, that for these trajectories $u(t;u_0)$ there exists a piecewise defined companion trajectory $\tilde{u}(t)$ on $\mathcal{A}^0$ such that

$$dist_H(u(t), \tilde{u}(t)) \leq Ce^{-\nu t}, \tag{5.7}$$

for all $t \geq 0$.

We recall that a *companion function* $\tilde{u}(t),\ t \geq 0$, is a piecewise continuous function with continuous trajectory pieces $\tilde{u}(t) \in \mathcal{A}^0, t_k \leq t < t_{k+1},\ k = 1, \cdots, \mu'$, for some $\mu' \leq \mu$. The jumps at $t = t_k$ occur within neighborhoods of individual equilibria φ_j, and the continuous pieces are solutions of (5.5) interconnecting these neighborhoods.

We now specify additional conditions such that the original exponential estimate (5.6) persists to hold for the perturbed global attractor $\mathcal{A}_\eta$ of the non-gradient system

$$\partial_t u = a\Delta u - f^1(x,u) - \eta f^2(x,u) + g(x), \tag{5.8}$$

containing an additional small perturbation parameter η, $0 \le \eta \le \eta_0$. Here $f^1, f^2, g(x)$ satisfy the above conditions (2.7)–(2.13). We restrict our presentation to the case $\dim x = 3$. We require at most quadratic growth of derivatives,

$$|\partial_w f^1(x,w)|, |\partial_w f^2(x,w)| \le C(1+|w|^2), \tag{5.9}$$

and Hölder continuity

$$|\partial_w f^1(x,w_1) - \partial_w f^1(x,w_2)| \le C(1+|w_1|+|w_2|)^s |w_1-w_2|^\sigma, \tag{5.10}$$

where $s+\sigma \le 2$ and $\sigma > 0$. For sufficiently small η, $0 \le \eta \le \eta_0$ the global attractor $\mathcal{A}_\eta$ of the system (5.8) then is exponentially attracting, analogously to (5.6),

$$dist_H(u(t), \mathcal{A}_\eta) \le Ce^{-\nu t}, \tag{5.11}$$

uniformly for $u(0) \in \mathcal{B}$ bounded in H^2 and for some constant $C = C(\mathcal{B})$. As before, the trajectories $u(t)$ possess companions $\tilde{u}(t) \in \mathcal{A}_\eta$ such that the more explicit exponential estimate (5.7) remains true for the distance $dist_H(u(t), \tilde{u}(t))$. These facts can be easily deduced from the paper [GV97], where similar facts have been established for abstract nonautonomous equations and scalar parabolic equations of the type

$$\partial_t u = -\mathcal{A}_0 u + R_0(u) + \eta R_1(u,t). \tag{5.12}$$

Further details concerning the exponential attraction properties (5.11) and (5.7) for the global attractor of the system (5.8), as well as more general examples will be treated in a forthcoming paper.

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