Can one determine a graph by counting walks?

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Abstract. Let G be a rooted graph with root x_0 and denote, for any positive integer n, by a_n the number of walks of length n which start at x_0. Is it possible to identify G up to isomorphism from the sequence (a_n)?

This problem can be thought of as a disrete variant of Kac' famous "Can you hear the shape of a drum?". In the present paper it is analysed in detail for a certain rather simple class of rooted trees.

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keywords: tree, identification problem, random walk, drum.

1 Introduction

This paper deals with walks in rooted graphs, i.e., walks in graphs (G, V, E) which have a distinguished vertex x_0, the root. We will only consider undirected graphs which have no self-loops and no parallel edges. A *walk of length* n in G is a sequence $y_0, \ldots, y_n$ of vertices such that $y_i \neq y_{i+1}$ and $\{y_i, y_{i+1}\} \in E$ for every i; it is *not* assumed that all y_i are different. (Here and in the sequel our terminology follows [6].)

Denote, for $n \in \mathbb{N}$, by a_n – or by $a_n(G)$ if G is not clear from the context – the number of walks of length n in G which start at the root. With the help of a computer it is rather easy to determine them:

> Initialize the procedure by assigning weight 1 to the root and weight 0 to all other vertices x. An "update" assigns to each vertex x the sum over the weights of the neighbours. Then a_n is the total weight after n updates.
>
> Alternatively one can find a_n as $\sum_i \alpha^n_{1i}$, where α^n_{ji} are the entries of the n'th power of the adjacency matrix and the vertices are labelled such that the root is the first vertex.

Now suppose that two graph theorists, $\mathbb{A}$ and $\mathbb{B}$, say, decide to play the following game: $\mathbb{A}$ chooses a connected rooted graph (G, x_0, V, E) where each vertex has finite degree and calculates the a_n for $n = 1, 2, \ldots$. Then $\mathbb{B}$ is asked to identify (G, x_0, V, E) up to isomorphism, the only information available to her or him is the sequence (a_n).

Does $\mathbb{B}$ have a chance to be successful?

For some trivial cases the answer is surely "yes", e.g., if the (a_n)-sequence is $(0, 0, \ldots)$ or $(1, 1, \ldots)$. Also, if they agree that $\mathbb{A}$ will choose as G a path, i.e., a tree of the type

fig. 1

it is easy for $\mathbb{B}$ (the root is depicted as a small black square). But what can be said in more complicated cases?

The problem arose when we investigated *a probabilistic variant of Kac's celebrated problem "Can you hear the shape of a drum?"* (see [8]; cf. [4], [7] for counterexamles and [1], [3] for some more recent investigations on isospectral sets).

Imagine a rooted graph where the degree d_x of each vertex x is bounded by a certain number d. Random walks on such graphs, in particular the rate of convergence to the equilibrium of the associated Markov chain, have been investigated in some detail (see chapter 11 in [2]). Here we are interested in a random walk subject to the following two rules:

- The walk starts at the root.
- If the walk is at time $k \geq 0$ at vertex x, stop with probability $(d - d_x)/d$; continue with probability d_x/d, in this case step with equal probability to one of the adjacent vertices.

E.g., the probability that the walk is not stopped immediately is $\mathbb{P}_1 := d_{x_0}/d$, and similarly the structure of the graph determines $\mathbb{P}_n$ (= the probability that the walk is not stopped before step n) for every n.
Can you recover the graph from the $\mathbb{P}_1, \mathbb{P}_2, \ldots$?

It is not hard to convince oneself that $\mathbb{P}_n$ is just a_n/d^n. (It is only necessary to embed G into a d-regular graph G'. The probability to survive is just the ratio between the number of walks starting at the root of length n in G by those in G', i.e., a_n/d^n.) We conclude that the $\mathbb{P}_1, \mathbb{P}_2, \ldots$ contain the same information as the sequence (a_n). Note, however, that the latter can reasonably be defined also in the case of unbounded degree.

Let us return to graph theory, can our identification problem be solved[1]?. A moment's reflection shows that $\mathbb{B}$ has no chance to succeed if $\mathbb{A}$ has no restrictions concerning the choice of (G, x_0, V, E). E.g., in each of the following non-isomorphic examples one has $a_n = 2^n$:

[1]It should be noted here that there is a famous ”‘reconstruction problem”’ in graph theory which is not related with our investigations; see the article of Nash-Williams in [5] for a survey.

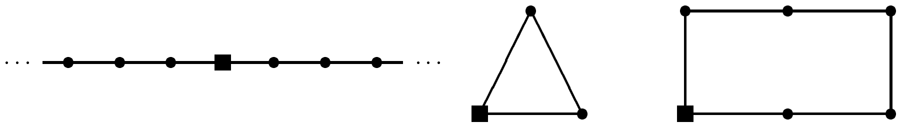


fig. 2

As a fairness condition it is therefore necessary to exclude such cases. Even more can be said, the game is *not fair* (i.e., $\mathbb{B}$ has no chance to identify the graph) as soon as $\mathbb{A}$ chooses a (G, x_0, V, E) which is *not a rooted tree.* This should be clear after the next definition and the following proposition.

Definition 1.1 Let (G, x_0, V, E) and (T, x_0', V', E') be a rooted graph and a rooted tree, respectively. T is called a *covering tree of* G if there is a mapping f from V' onto V with the following properties:

(i) For $(x', y') \in E'$ one has $(f(x'), f(y')) \in E$.

(ii) Let x', x be such that $f(x') = x$. Then, for every neighbour y of x, there exists a unique neighbour y' of x' such that $f(y') = y$.

(iii) $f(x_0') = x_0$.

In the preceding picture, the first graph is a covering tree for the other two, here you have another example:

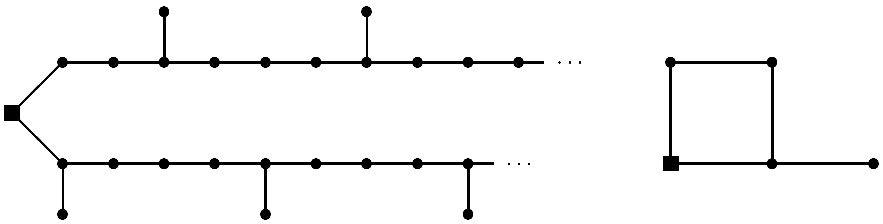


fig. 3

Proposition 1.2

(i) *Let T be a covering tree of G. Then $a_n(G) = a_n(T)$ for every n.*

(ii) *Every rooted graph has – up to isomorphism – precisely one covering tree.*

The easy proof of these assertions is left to the reader. For us they have the consequence that we have to modify the rules of the game in that $\mathbb{A}$ is allowed *to play with rooted trees* only. Even then it can happen that $\mathbb{B}$ has no winning strategy, in theorem 1.5(i) one can find non-isomorphic rooted trees with the same a_n's (for further examples cf. remark 4 at the end of this paper). However, in all of our examples at least one of the trees is infinite, the natural question whether $\mathbb{B}$ always is able to identify *a finite rooted tree* from the a_n

will not be settled here. It even has to be admitted that the positive answer we will provide concerns a rather simple situation, nevertheless, the proof is surprisingly complicated.

It will be convenient to consider also a variant of the original problem, we will call it *the decision problem.* Now $\mathbb{B}$ wins the game if he or she is able to decide correctly from what class of two possible candidates the tree (G, x_0, V, E) had been chosen. It is time to be a little bit more formal:

Definition 1.3 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be classes of rooted trees such that the members of these classes are mutually not isomorphic. Also, $\mathcal{G}$ denotes a class of rooted trees.

(i) We say that *the pair* $(\mathcal{G}_1, \mathcal{G}_2)$ *is decidable*, if one can – for any G from $\mathcal{G}_1 \cup \mathcal{G}_2$ – decide with certainty from the (a_n) whether $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$. (I.e., the function $G \mapsto (a_n(G))_n$ maps $\mathcal{G}_1$ and $\mathcal{G}_2$ to disjoint sets of sequences.)

(ii) It is said to be *decidable after at most* r *steps* if the decision is possible on the basis of $a_1, \ldots, a_r$.

(iii) By the assertion that *the identification problem is solvable in* $\mathcal{G}$ we mean the fact that one can recover a $G \in \mathcal{G}$ up to isomorphism from the associated sequence (a_n). (I.e., $G \mapsto (a_n(G))_n$ is one-to-one on $\mathcal{G}$ modulo isomorphism.)

The aim of this paper is the study of decision and identification problems for a class of certain simple rooted trees:

> Imagine a (finite or countable) connected tree where the degree of the vertices is 1, 2 or 3, but degree 3 occurs at most once. Then declare any of the vertices as the root.

Here are some examples (as before, the small black square stands for the root):

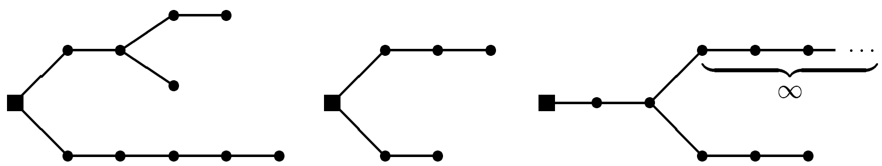


fig. 4

Trees of this type will be called *simple rooted* 3*-trees* in the sequel, they form the simplest class after the family of rooted paths; there the identification problem obviously is easily solvable. To fix notation we introduce the following

Definition 1.4 Let l_1, l_2, l_3, l_4 be numbers in $\{0, 1, 2, \ldots, \infty\}$ such that

(i) $l_3 \geq l_4 \geq 1$ or $l_3 = l_4 = 0$.
(ii) If $l_1 = \infty$, then $l_3 = l_4 = 0$.
(iii) If $l_3 = l_4 = 0$, then $l_1 \geq l_2$.
(iv) If $l_1 = 0$, then $l_3 \geq l_4 \geq l_2$.
(v) $l_1 = l_2 = 0$ is admissible only in the case $l_1 = l_2 = l_3 = l_4 = 0$.

Then $[l_1, l_2, l_3, l_4]$ denotes the rooted tree for which two paths of length l_1 and l_2 leave the root, and the first one (possibly) bifurcates into two paths of length l_3 and l_4. It has $1 + l_1 + l_2 + l_3 + l_4$ vertices, typically it looks as follows:

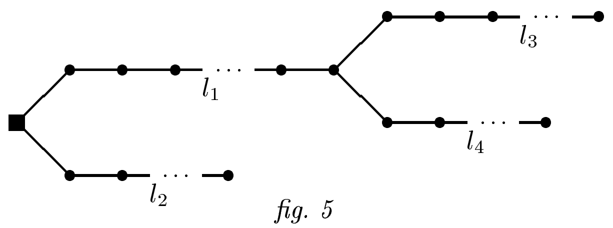


fig. 5

As an illustration, consider the trees in the preceding figure 4: they show $[2, 5, 2, 1]$, $[3, 2, 0, 0]$, and $[2, 0, \infty, 3]$. We note that simple rooted 3-trees are in one-to-one correspondence with the possible choices of the numbers l_1, l_2, l_3, l_4: the trees $[l_1, l_2, l_3, l_4]$ and $[l'_1, l'_2, l'_3, l'_4]$ are isomorphic iff $l_i = l'_i$ for $i = 1, 2, 3, 4$. (To have this uniqueness we have restricted the possible l's; for example, without assumption (iv) above $[0, 7, 5, 4]$ would be isomorphic with $[0, 4, 7, 5]$ and with $[0, 5, 7, 4]$).

The main result of this paper is the following

Theorem 1.5 *Let G be a simple rooted 3-tree.*

(i) *G is isomorphic with $[1, 1, 2, 2]$ or with $[\infty, \infty, 0, 0]$ if and only if $a_n(G) = 2^n$ for every n.*
(ii) *If (a_n) is* not *the sequence (2^n), then G can be uniquely determined from (a_n).*
(iii) *The identification problem is solvable in the class of all finite rooted simple 3-trees.*

The theorem will be proved in *section 4*. The idea is to use induction: if we know the structure of the "t-truncated subtree" G^t – by definition it contains the vertices such that the distance to the root is at most t – then the (a_n) enable us to identify G^{t+1}. The induction step is reduced to *three decision problems* which are studied in detail in *section 3*, they cover the non-obvious parts of the proof of the theorem.

Three methods will be developed to treat the decision problems. The first one is to calculate explicitly the a_n (more precisely: suitable related numbers) and to show by comparing them that they distinguish between the classes under consideration; this will be important for the discussion of decision problem I. The second method will be used when dealing with decision problems II and III: there one defines certain mappings which transform walks to walks and which are one-to-one, and in this way it is possible to investigate which one of two trees admits the larger number of walks.

However, there remain cases where these techniques do not suffice, one has to investigate also nonisomorphic rooted trees where the a_n coincide much longer than one would expect. These exceptional cases will be treated by nested induction, they are considered in *section 2*.

After these preparations it is then rather easy to complete the argument, it only remains to pass from the local isomorphisms which are found by the induction process to a global one.

2 The exceptional cases

Let G and G' be non-isomorphic rooted trees and t as large as possible such that G^t and $(G')^t$ coincide. If it happens that the (a_n)-sequences are different for G and G' one could expect that $a_n(G) \neq a_n(G')$ for some n which are "not much larger" than t. In fact, this is true for the majority of situations which are considered in this paper. We have, however, encountered a number of cases where $a_n(G) = a_n(G')$ for surprisingly many n. They are studied in the present section, the results are an essential building block for the proof of the main theorem.

As it is to be expected the trees under consideration are simple rooted 3-trees, we use the notation from definition 1.4, although we may sometimes violate our condition $l_3 \geq l_4$. If G and G' are clear from the context, we write a_n instead of $a_n(G)$ and a'_n for $a_n(G')$. Similarly, we use a''_n, a^*_n, a^{**}_n for graphs G'', G^*, G^{**}.

Our results are contained in the following propositions; the proofs necessitate some preliminary work, we will have to study some rather technical counting lemmas first.

Proposition 2.1 *Let t and c be integers with $t \geq 2$ and $G = [1, t, t+1, c]$, $G' = [1, \infty, t-1, c]$. (Note that the notation will meet the requirements of definition 1.4 only if $c < t$. However, it will be convenient here to treat arbitrary c.)*

(i) *If $c \geq t/2$ we have $a_n = a'_n$ for $n \leq 2t+1$ and $a_{2t+2} = a'_{2t+2} + 1$.*

(ii) *In the case $c = (t-1)/2$ one has to consider the "large" and the "small" t separately. For $t > 5$ we have $a_n = a'_n$ for $n \leq 2t+3$, but $a_{2t+4} = a'_{2t+4} + 2$. For $t = 5$ (so that $c = 2$) we also have $a_n = a'_n$ for $n \leq 2t+3$, but then only $a_{2t+4} = a'_{2t+4} + 1$. Finally, for $t = 3$ and $c = 1$, a_n and a'_n only coincide up to $n = 2t+2$. Then $a_{2t+3} = a'_{2t+3} - 1$, but then again $a_{2t+4} = a_{2t+4}$; note that this is the only situation where the first nonzero $a_n - a'_n$ is negative.*

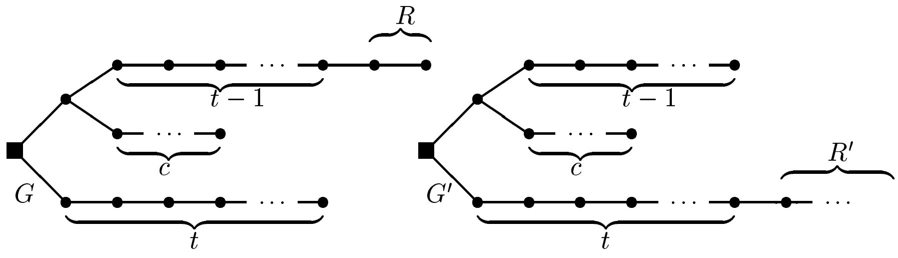


fig. 6

Proposition 2.2 *With t and c as in the preceding proposition we consider $G = [1, t, t+1, c]$ and $G' = [1, t+2c+2, t-1, c]$. If $c \le (t-2)/2$, then $a_n = a'_n$ for $1 \le n \le 2t+1$ and $a_{2t+2} \ne a'_{2t+2} = a_{2t+2} - 1$.*

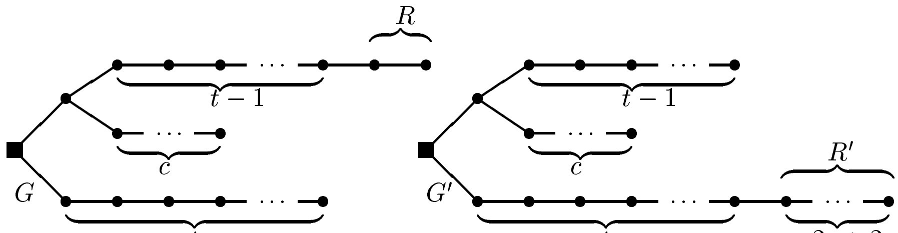


fig. 7

Proposition 2.3 *Let $G = [k, t, t-k+2, t-k+2]$ and $G' = [k, t+2, t+k+2, t-k]$, where $1 \le k \le t-1$. Then $a_n = a'_n$ for $n \le 2t-k+2$ and $a_{2t-k+3} \ne a'_{2t-k+3} = a_{2t-k+3} - 1$.*

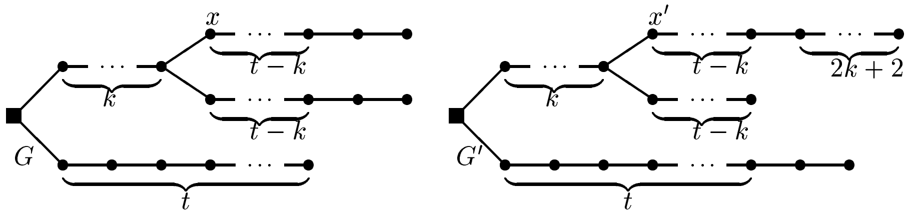


fig. 8

If it happens that 3 divides t and $k = t/3$, we would like to compare the preceding G of proposition 2.3 also with $G'' = [k, t+2, \infty, t-k]$.

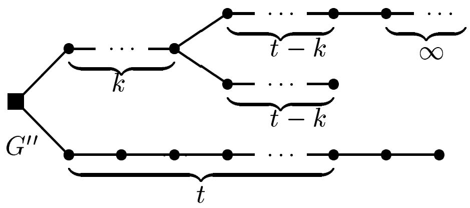


fig. 9

Proposition 2.4 *For $t = 3k \geq 3$ we consider $G = [k, t, t-k+2, t-k+2]$, i.e., $G = [k, 3k, 2k+2, 2k+2]$. We also put $G'' = [k, t+2, \infty, t-k] = [k, 3k+2, \infty, 2k]$, and we claim that $a_n = a''_n$ for $1 \leq n \leq 2t-k+4 = 5k+4$ and $a_{5k+5} = a''_{5k+5}+2$.*

Next we will compare, for arbitrary $k \geq 0$ and $t \geq 2$, the rooted tree $G = [1, t, t+k, t+1]$

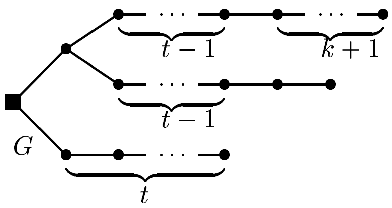


fig. 10

with the two trees $G' = [1, t+k+1, t+k+2, t-1]$ and $G'' = [1, \infty, t+k, t-1]$.

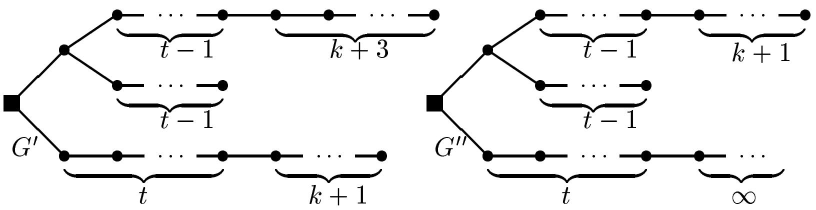


fig. 11

Proposition 2.5 *$a_n = a'_n = a''_n$ for $n \leq 2t+1$, and $a_{2t+2} = a'_{2t+2} + 1 = a''_{2t+2} + 1$.*

For $k = t-2$ (resp. $k = t$) we will compare the graph $G = [1, t, t+k, t+1]$ of prop. 2.5 also with the graph G^* (resp. G^{**}) which is obtained from the preceding G' (resp. G'') by extending the top branch infinitely long:

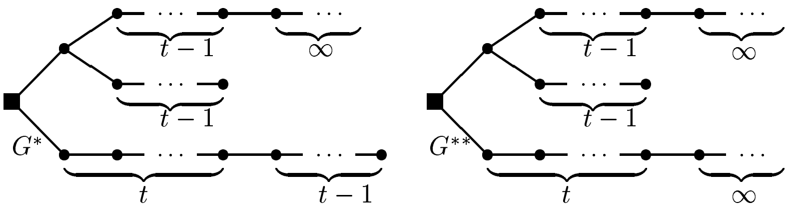


fig. 12

Proposition 2.6 *Let $G = [1, t, 2t-2, t+1]$, $G^* = [1, 2t-1, \infty, t-1]$, where $t \geq 3$. Then $a_n = a_n^*$ for $1 \leq n \leq 2t+3$ and $a_{2t+4} = a_{2t+4}^* + 1$.*

In the case $t = 2$ the a_n, a_n^* with $n \leq 2t+4$ do not suffice to distinguish G and G^*, one needs one step more: $a_n = a_n^*$ for $n \leq 8$, and $a_n = a_n^* - 1$ for $n = 9$ and $n = 10$.

Proposition 2.7 *Let $G = [1, t, 2t, t+1]$, $G^{**} = [1, \infty, \infty, t-1]$ (with $t \geq 2$). Then $a_n = a_n^{**}$ for $1 \leq n \leq 2t+3$ and $a_{2t+4} = a_{2t+4}^{**} + 1$.*

Next we investigate $G = [2, t, t, 1]$ and $G' = [2, \infty, t-2, 1]$ for $t \geq 3$:

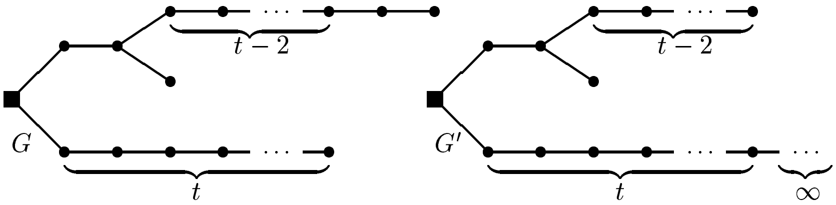


fig. 13

Proposition 2.8 *$a_n = a_n'$ for $n \leq 2t$, and $a_{2t+1} = a_{2t+1}' + 1$.*

Finally we study $G = [2, t, t+2, t]$ and $G' = [2, \infty, \infty, t-2]$ for $t \geq 3$.

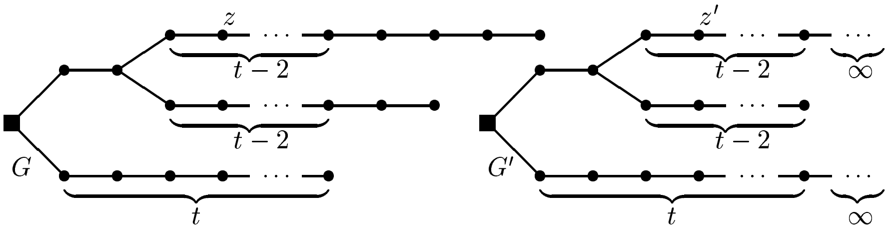


fig. 14

Proposition 2.9 *$a_n = a_n'$ for $1 \leq n \leq 2t$ and $a_{2t+1} = a_{2t+1}' + 1$.*

We now turn to the proofs of the preceding propositions. For any rooted tree (G, x_0, V, E) we introduce the following numbers:

Definition 2.10 Let x be a vertex of G and $R, S \subset V$. By $a_n(x, R, \overline{S})$ we denote the number of walks of length n from x_0 to x which meet R but do not step into S. (If $x \in S$, then we count the walks to x which meet R and omit S *with the exception of the last step.* Similarly, if $x_0 \in S$, the walks start in x_0, leave S immediately and must not return to S later.)
Some variants will also be in use:

$$a_n(x) := a_n(x, G, \overline{\emptyset}), \quad a_n(x, R) := a_n(x, R, \overline{\emptyset}),$$

$$a_n(x, \overline{S}) := a_n(x, G, \overline{S}), \quad a_n(R) := \sum_{x \in G} a_n(x, R, \overline{\emptyset}), \ldots.$$

If we deal with two trees G and G' the corresponding numbers will be denoted by $a_n(\ldots)$ and $a'_n(\ldots)$, respectively.

The following lemma will be fundamental:

Lemma 2.11 *Consider the following graph (the letter "K" stands for an arbitrary subgraph):*

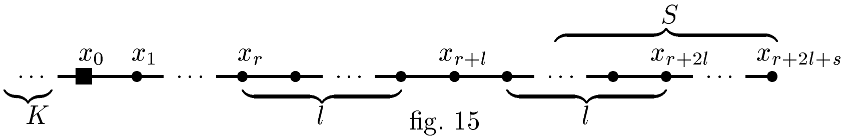


fig. 15

For $n \leq 3r + 2l + 1$ *one has*

$$a_n(x_{r+2l}) = a_n(x_r, \{x_{r+l}\}, \overline{\{x_{r+l+s+1}, \ldots, x_{r+2l+s}\}}) :$$

there are precisely as many walks of length n *from* x_0 *to* x_{r+2l} *as there are walks to* x_r *which meet* x_{r+l} *but never step to* $x_{r+l+s+1}$*.*

Proof. : All walks under consideration meet x_{r+l} and therefore we can condition on the first time where this vertex is visited: we will write the set of these walks as the disjoint union of the walks which are at x_{r+l} for the first time at the k'th step, the cardinalities of these subsets are easily calculated.

We put $S := \{x_{r+l+s+1}, \ldots, x_{r+2l+s}\}$, and we denote by $b_m(x_{r+2l})$ (resp. $b_m(x_r, \overline{S})$) the number of walks of length m which start at x_{r+l} and end at x_{r+2l} (resp. start at x_{r+l}, end at x_r and omit S). Then

$$a_n(x_{r+2l}) = \sum_{k=0}^{n} a_k(x_{r+l}, \overline{\{x_{r+l}\}}) b_{n-k}(x_{r+2l}),$$

$$a_n(x_r, \{x_{r+l}\}, \overline{S}) = \sum_{k=0}^{n} a_k(x_{r+l}, \overline{\{x_{r+l}\}}) b_{n-k}(x_r, \overline{S});$$

where, by definition, $a_0(\ldots) = b_0(\ldots) = 1$.

Since $a_k(x_{r+l}) = 0$ for $k < r + l$ we only have to sum over the k with $r + l \le k \le n$, hence – if we assume that $n \le 3r + 2l + 1$ – the relevant k satisfy $n - k \le 2r + l + 1$.

Now we observe that the number of walks of length $n - k$ from x_{r+l} to x_r which do not meet S is the number of walks from x_r to x_{r+l} avoiding S. Also, since $n - k \le 2r + l + 1$, these walks cannot meet K. It remains to shift the walks l steps to the right to see that their number coincides with the number of walks of length $n - k$ from x_{r+l} to x_{r+2l}, i.e., $b_{n-k}(x_{r+2l}) = b_{n-k}(x_r, \overline{S})$. □

Lemma 2.12 *Let $t \ge 2$, $c \ge 1$ and $G = [1, t, t+1, c]$, $G' = [1, \infty, t-1, c]$; see figure 6.*

(i) *If $t/2 \le c$, then*

$$a_n - a'_n = \begin{cases} 0, & \text{if } n \le 2t + 1 \\ 1, & \text{if } n = 2t + 2 \end{cases} .$$

(ii) *If even $(t+2)/2 \le c$ and $c \ge 3$, then also*

$$a_n - a'_n = \begin{cases} 2, & \text{if } n = 2t + 3 \\ 2t + 8, & \text{if } n = 2t + 4 \end{cases} .$$

Proof. The proof will be by induction on n, the case "$n = 1$" is clear. Denote by R and R' the vertices of $G \setminus G^t$ and $G' \setminus (G')^t$, respectively. Since $G^t = (G')^t$, we only have to compare walks which leave G^t: $a_n - a'_n$ coincides with $a_n(R) - a'_n(R')$.

Now observe that for $n \le 2t + 1$ there is no walk which first leaves G^t and then comes back to x_0 (or x'_0), whereas for $n = 2t + 2$ there is in either case exactly one. For $n = 2t + 3$ both G and G' admit exactly two such walks. For $n = 2t + 4$, however, there are two more of these walks in G than in G', namely those that make use of the c-branch of G on their way to R or back to x_0.

Next we compare the number of walks in G (resp. G'), which – after leaving G^t – do *not* return to x_0 (resp. x'_0). By conditioning on the last time such a walk meets x_0 (resp. x'_0) we conclude that there are

$$\sum_{m=0}^{n} a_m(x_0, \overline{R}) a_{n-m}(R, \overline{\{x_0\}}) \text{ resp. } \sum_{m=0}^{n} a'_m(x'_0, \overline{R'}) a'_{n-m}(R', \overline{\{x'_0\}})$$

of them. We note that – obviously – $a_m(x_0, \overline{R}) = a'_m(x'_0, \overline{R'})$ for all m, and it remains to study the $a_k(R, \overline{\{x_0\}}), a'_k(R', \overline{\{x'_0\}})$.

Claim: Let $r_k := a_k(R, \overline{\{x_0\}}) - a'_k(R', \overline{\{x'_0\}})$. Then

$$r_k = \begin{cases} 0, & \text{if } k \le 2t + 1 \\ 1, & \text{if } k = 2t + 2 \end{cases} .$$

If even $c \ge (t+2)/2$ and $c \ge 3$ hold we also have

$$r_k = \begin{cases} 2, & \text{if } k = 2t + 3 \\ 2t + 4, & \text{if } k = 2t + 4 \end{cases} .$$

Notice that once we have proved this claim we are done: we know that

$$a_n - a'_n = \sum_{m=0}^{n} a_m(x_0, \overline{R}) \left(a_{n-m}(R, \overline{\{x_0\}}) - a'_{n-m}(R', \overline{\{x'_0\}}) \right)$$

for $n \leq 2t+3$, and we have to add "2" for $n = 2t+4$; also $a_0(x_0, \overline{R}) = 1$, $a_1(x_0, \overline{R}) = a_3(x_0, \overline{R}) = 0$, $a_2(x_0, \overline{R}) = 2$ hold, and thus

$$\begin{aligned} a_n - a'_n &= \begin{cases} 0, & \text{if } n \leq 2t+1 \\ a_0(x_0, \overline{R}), & \text{if } n = 2t+2 \\ 2a_0(x_0, \overline{R}) + a_1(x_0, \overline{R}), & \text{if } n = 2t+3 \\ (2t+4)a_0(x_0, \overline{R}) + 2a_1(x_0, \overline{R}) + a_2(x_0, \overline{R}) + 2, & \text{if } n = 2t+4 \end{cases} \\ &= \begin{cases} 0, & \text{if } n \leq 2t+1 \\ 1, & \text{if } n = 2t+2 \\ 2, & \text{if } n = 2t+3 \\ 2t+8, & \text{if } n = 2t+4 \end{cases} \end{aligned}$$

(the identities for $n = 2t+3$ and $2t+4$ are valid only under the assumptions $c \geq (t+2)/2$ and $c \geq 3$).
Proof of the claim. $a_k(R, \overline{\{x_0\}})$, the number of walks of length k in G that exit G^t and never return to x_0, is exactly the number of walks of length $k-1$ in the graph H from figure 16 starting at x_1 which meet the set R.

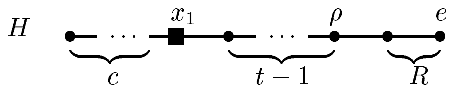


fig. 16

From now on we may assume that $c = \infty$, since a walk of length $\leq 2t+1$ (resp. $\leq 2t+3$) entering R can not make use of these new vertices added to H in case $c \geq \frac{t}{2}$ (resp. $c \geq \frac{t+2}{2}$ and $c \geq 3$).

An analogous assertion holds for $a'_k(R', \overline{\{x'_0\}})$, it coincides with the number of walks of length $k-1$ starting at x'_1 that meet the set R' in H' (figure 17):

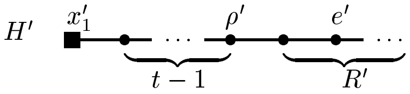


fig. 17

Walks of length n which start at x_1 resp. x'_1 are denoted by b_n and b'_n, and as above some modifications are in use. (E.g., $b_n(R)$ stands for the number of walks which visit the subset R.)

We have $r_k = b_{k-1}(R) - b'_{k-1}(R')$, the claim will now be proved by induction on k. The case "$k = 1$" is clear, we suppose that k with $1 \leq k \leq 2t+3$ is fixed such that r_k satisfies the claim.

$b_k(R)$ is related with $b_{k-1}(R)$ as follows:

$$b_k(R) = 2b_{k-1}(R) - b_{k-1}(e) + b_{k-1}(\rho, \overline{R}).$$

(Every walk which contributes to $b_{k-1}(R)$ can be continued into two different ways as a walk which counts for $b_k(R)$, except for those which end at e: they give rise to only one walk. Also we have to take into account the walks which enter R with the k-th step for the first time.) A similar consideration yields

$$b'_k(R') = 2b'_{k-1}(R') - b'_{k-1}(x'_1, R') + b'_{k-1}(\rho', \overline{R'}).$$

Subtracting the second from the first equality we get

$$r_{k+1} = 2r_k - b_{k-1}(e) + b_{k-1}(\rho, \overline{R}) + b'_{k-1}(x'_1, R') - b'_{k-1}(\rho', \overline{R'}), \tag{2.1}$$

and now lemma 2.11 comes into play. This lemma is applied with $r = t-1$, $l = 1$ and $s = \infty$, it follows that $b'_{k-1}(\rho', R') = b'_{k-1}(e')$ (note that $k-1 \leq 2t-2 \leq 3t$).

But since the number of walks from x'_1 to e' equals the number of walks from e' to x'_1 and since these walks can be identified with the walks from x_1 to e in H it follows that

$$b'_{k-1}(\rho', R') = b_{k-1}(e).$$

Furthermore, $b'_{k-1}(\rho')$ equals the number of walks from ρ' to x'_1, which in turn is the number of walks in H from x_1 to ρ which avoid R. Hence

$$b'_{k-1}(\rho') = b_{k-1}(\rho, \overline{R}).$$

Plugging these equalities into equation (2.1) we get

$$\begin{aligned} r_{k+1} &= 2r_k - b'_{k-1}(\rho', R') + b'_{k-1}(\rho') + b'_{k-1}(x'_1, R') - b'_{k-1}(\rho', \overline{R'}) && (2.2)\\ &= 2r_k + b'_{k-1}(x'_1, R'). && (2.3) \end{aligned}$$

It is not hard to see that

$$b'_k(x'_1, R') = \begin{cases} 0, & \text{if } k \leq 2t-1 \\ 1, & \text{if } k = 2t \\ 0, & \text{if } k = 2t+1 \\ 2t, & \text{if } k = 2t+2, \end{cases}$$

so that we have in fact proved that:

$$\begin{aligned} k \leq 2t &\Rightarrow r_{k+1} = 2\cdot 0 + 0 = 0 \\ k = 2t+1 &\Rightarrow r_{k+1} = 2\cdot 0 + 1 = 1 \\ k = 2t+2 &\Rightarrow r_{k+1} = 2\cdot 1 + 0 = 2 \\ k = 2t+3 &\Rightarrow r_{k+1} = 2\cdot 2 + 2t = 2t+4 \end{aligned}$$

This completes the proof of lemma 2.12. □

Proof of proposition 2.1. The case $c \geq t/2$ has been settled by the preceding lemma.

So assume that $c = (t-1)/2$, first we will consider the case that $t > 5$. As before put $R = G \setminus G^t$, $R' = G' \setminus (G')^t$. Let $a_n^c(R)$ (resp. $a_n'^c(R')$) be the number of walks in $[1, t, t+1, c)]$ (resp. $[1, \infty, t-1, c]$) which leave G^t; similarly, $a_n^{c+2}(R)$ (resp. $a_n'^{c+2}(R')$) stands for the number of walks in $[1, t, t+1, c+2)]$ (resp. $[1, \infty, t-1, c+2]$) which meet R (resp. R').

First we compare $a_n^c(R)$ with $a_n^{c+2}(R)$ for $n \leq 2t+4$. For $n \leq 2t+1$ walks in $[1, t, t+1, c+2]$ which visit R cannot make use of the two more vertices, so these numbers are equal. For $n > 2t+1$, however, they can. One easily calculates that

$$a_n^{c+2}(R) - a_n^c(R) = \begin{cases} 1 & \text{if } n = 2t+2 \\ 2 & \text{if } n = 2t+3 \\ 2t+7 & \text{if } n = 2t+4. \end{cases}$$

> Only the case $n = 2t+4$ is not trivial. Since by assumption $t > 5$ so that $c \geq 3$, the walks cannot first visit R and then one of the new vertices. The shortest walk in $[1, t, t+1, c+2]$ that enters R and does not stay in $[1, t, t+1, c]$ goes one step to the bifurcation point, then $c+1$ steps into the $(c+2)$-branch to the right (leaving $[1, t, t+1, c]$) and from there to R as quickly as possible. Its length is $1 + c + 1 + c + 1 + t = 2t + 2$. Let's call it the "optimal" route. Now for $n = 2t+4$ the situation is similar, but at the beginning or between two steps of the optimal route we may add two steps (back and forth). This accounts for $(2t+2)+2$ walks. Note that the optimal route hits the bifurcation point twice and that at this point there are *two* not optimal directions. It remains to add 3, the number of walks that first take the optimal route and make two arbitrary steps later.

Similarly, a comparison of $a_n'^c(R')$ with $a_n'^{c+2}(R')$ yields

$$a_n'^{c+2}(R') - a_n'^c(R') = \begin{cases} 0 & \text{if } n = 2t+2 \\ 0 & \text{if } n = 2t+3 \\ 1 & \text{if } n = 2t+4. \end{cases}$$

Therefore, since $c+2 = (t+3)/2$, it follows from lemma 2.12 that

$$a_n([1, t, t{+}1, c]) - a_n([1, \infty, t{-}1, c]) = \begin{cases} 0 & \text{if } n \leq 2t+1 \\ 1-1=0 & \text{if } n = 2t+2 \\ 2-2=0 & \text{if } n = 2t+3 \\ 2t+8-(2t+7)+1=2 & \text{if } n = 2t+4 \end{cases}$$

so that proposition 2.1 is proved in the case $t > 5$.
For $t = 5$ and $t = 3$ one can check the assertion by direct calculation.

> Why has the preceding argument which has worked successfully in the case $t > 5$ to be modified? The proof is essentially the same except for the comparison of a_n^{c+2} and a_n^c. We now have to take into account walks in $[1, t, t{+}1, c{+}2]$ that first visit R and then leave the subgraph $[1, t, t{+}1, c]$.

For $t = 5$, $c = 2$ there is only one such walk of length $2t + 4$. For $t = 3$, $c = 1$, however, there are two of these walks of length $2t + 4$ and one of length $2t + 3$. So for $t = 5$ and $c = 2$ the argument from above now shows that $a_n = a'_n$ for $n \leq 2t + 3$, but $a_{2t+4} = a_{2t+4} + 1$. For $t = 3$ and $c = 1$ it yields $a_n = a'_n$ up to $n = 2t + 2$, $a_{2t+3} = a'_{2t+3} - 1$ and $a_{2t+4} = a_{2t+4}$.

□

The c with $c < (t-1)/2$ are considered in proposition 2.2, one has to compare $[1, t, t+1, c]$ with $[1, t+2c+2, t-1, c]$. Our notation is as above. First, we need another lemma comparing two graphs H, H'. The walks therein start at x_1, x'_1 and will be counted by b, b', etc. as in the proof of lemma 2.12.

Lemma 2.13 *Consider the graphs H and H':*

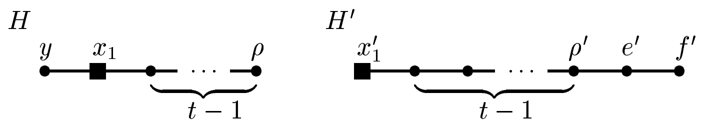


fig. 18

Then, for $k \leq 3t$, we have $b_k(\rho, \{y\}) = b'_k(f')$.

Proof. Lemma 2.11 with $l = 1$, $s = 0$, $r = t - 1$ and $K = \emptyset$ shows that $b'_k(f') = b'_k(\rho', \{e'\}, \overline{\{f'\}})$, the number of walks from x'_1 to ρ' which meet e' but not f'; this is true for $k \leq 3r + 2l + 1 = 3t$. But going these walks backwards we are in the same situation as going in H from x_0 to ρ via y. □

Proof of proposition 2.2. Again we only have to consider walks that enter $R := G \setminus G^t$ (resp. $R' := G' \setminus (G')^t$), for $G^t = (G')^t$.

For $n \leq 2t + 1$ there are no walks which leave G^t before returning to x_0 (resp. x'_0), and for $n = 2t + 2$ there is in each graph exactly one such walk. So, as in the proof of lemma 2.11, we get by conditioning on the last time a walk visits x_0 (resp. x'_0)

$$a_n - a'_n = \sum_{m=0}^{n} a_m(x_0, \overline{R}) \left(a_{n-m}(R, \overline{\{x_0\}}) - a'_{n-m}(R', \overline{\{x'_0\}}) \right).$$

Analogously to the proof of lemma 2.12 we now compare walks in H and H' which start at x_1, x'_1, as in the the proof of this lemma we use the notation $b_j(\ldots)$, $b'_j(\ldots)$ to count them.

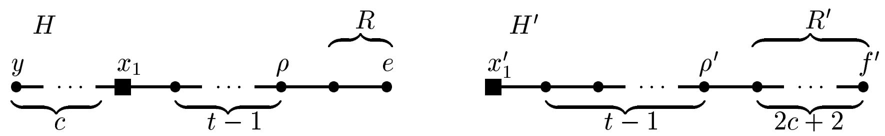


fig. 19

We have to show that $b_k(R) = b'_k(R')$ for $k \leq 2t$, and $b_{2t+1}(R) = b'_{2t+1}(R')+1$, this will be done by induction on k. The case "$k = 0$" is clear, assume that $b_k(R) = b'_k(R')$ for some $k \leq 2t$ has been shown. Then

$$b_{k+1}(R) = 2b_k(R) - b_k(e) + b_k(\rho, \overline{R}) :$$

every walk of length k which contributes to $b_k(R)$ gives rise to two walks for $b_{k+1}(R)$, except for those ending at e, plus the walks that enter R in step number $k+1$ for the first time (note that for $k \leq 2t$ there are no walks to y which meet R). Analogously one gets for H':

$$b'_{k+1}(R') = 2b'_k(R') - b'_k(f') - b'_k(x'_1, R') + b'_k(\rho', \overline{R'}).$$

Thus, to prove the assertion for $k+1$, it remains to show that

$$-b_k(e) + b_k(\rho, \overline{R}) = -b'_k(f') - b'_k(x'_1, R') + b'_k(\rho', \overline{R'});$$

and that one has to add "+1" at the right hand side in the case $k = 2t$. Since $b'_k(x'_1, R') = 0$ for $k \leq 2t-1$ and $b'_k(x'_1, R') = 1$ for $k = 2t$, this is equivalent to

$$-b_k(e) + b_k(\rho, \overline{R}) = -b'_k(f') + b'_k(\rho', \overline{R'}) \tag{2.4}$$

for $k \leq 2t$.
Now consider $b'_k(f')$, the walks from x'_1 to f'.

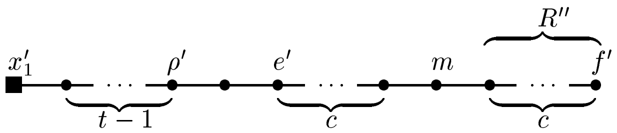


fig. 20

By lemma 2.11 – with $l = c$, $s = 0$, $r = t+1$ and $K = \emptyset$ – this is just $b'_k(e', \{m\}, \overline{R''})$ for $k \leq 3(t+1)+2c+1$, and this number coincides with $b_k(e, \{y\})$. (Note that a walk from x'_1 to e' which visits m and avoids R'' can be identified with a walk from x_1 which on its way to ρ meets y: simply let it run backwards.) Also we may split the walks from x_1 to e on the left hand side of equation (2.4) into those which make use of y and those that do not, and in this way it remains to show that

$$-b_k(e, \{y\}) - b_k(e, \overline{\{y\}}) + b_k(\rho, \overline{R}) = -b_k(e, \{y\}) + b'_k(\rho', \overline{R'}), \quad k \leq 2t,$$

i.e.,

$$-b^c_k(e, \overline{\{y^c\}}) + b^c_k(\rho, \overline{R}) = b'^c_k(\rho', \overline{R'^c}), \quad k \leq 2t, \tag{2.5}$$

where we have added superscripts c to show the c-dependence.

Equation (2.5) will now be shown by induction on c. Note that in the case "$c = 1$" one has the identity $b^c_k(\rho, \overline{R}) - b'^c_k(\rho', \overline{R'^c}) = b^c_k(\rho, \{y^c\}, \overline{R})$, and therefore equation (2.5) holds in this case by lemma 2.13.

Assume that (2.5) has been shown for c. Since

$$b_k^{c+1}(\rho,\overline{R})-b_k^{c+1}(e,\overline{\{y^{c+1}\}}) = b_k^c(\rho,\overline{R})+b_k^{c+1}(\rho,\{y^{c+1}\},\overline{R})-b_k^c(e,\overline{\{y^c\}})-b_k^c(e,\{y^c\})$$

and

$$b_k'^{c+1}(\rho',\overline{R'^{c+1}}) = b_k'^c(\rho',\overline{R'^c})$$

hold, the induction hypothesis reduces our claim to

$$b_k^{c+1}(\rho,\{y^{c+1}\},\overline{R}) = b_k^c(e,\{y^c\}). \tag{2.6}$$

This equality will be verified using the auxiliary graphs H^* and H'^*.

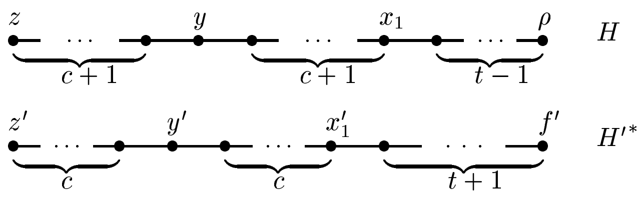


fig. 21

The left hand side of (2.6) can be identified with the number of walks in H^* which start at ρ, end at x_1, and which on their way visit y but avoid the vertices left to y (simply reverse the direction of the walks). By lemma 2.11 (with $l = c+1$, $s = 0$, $r = t-1$ and $K = \emptyset$) there are as many such walks as there are walks from ρ to z in H^*.

Similarly, the right hand side of (2.6) is the number of walks from f' to z' in H'^* (use lemma 2.11 with $l = c$ and $s = 0$). But these numbers are obviously equal, since H^* and H'^* are paths of the same length and we count walks between the extreme vertices. □

Proof of proposition 2.3. We have to consider walks of length $\leq 2t-k+3$. Walks in G and G' that do *not* meet x resp. x' are walks in isomorphic trees and thus their numbers are equal (for notation, cf. figure 8). It remains to compare those which do meet x, x'. Conditioning on the first time they do we calculate that their numbers are

$$\sum_{m=0}^{n} a_m(x,\overline{\{x\}})b_{n-m} \quad \text{resp.} \quad \sum_{m=0}^{n} a'_m(x',\overline{\{x'\}})b'_{n-m},$$

where $b_j(\ldots)$ resp. $b'_j(\ldots)$ means the number of walks starting at x resp. x' of length j.

Now observe that walks from x_0 resp. x'_0 to x resp. x' of length at most $2t-k+2$ cannot leave G^t resp. $(G')^t$, hence their numbers are equal. However, in the case of walks of length $2t-k+3$ there is one more in G (that one which first uses the k-branch, then turns to the right to leave G^t for one step and then returns to x; this necessitates $k+(t-k+1)+(t-k+1)+1 = 2t-k+3$ steps).

Also note that $\mathrm{dist}(x_0, x) = \mathrm{dist}(x_0', x') = k + 1$, and therefore only the m with $k + 1 \le m \le n$ have to be considered. It follows that

$$a_n - a_n' = \sum_{m=k+1}^{n} a_m(x, \overline{\{x\}}) \left(b_{n-m} - b_{n-m}'\right)$$

for $n \le 2t - k + 2$, and for $n = 2t - k + 3$ one has to add "1" at the right hand side.

We are done if we can prove that $b_m = b_m'$ for $m \le 2t - 2k + 2$, and this leads us to another decision problem: if we define x, x' as the new roots of the graphs G, G',

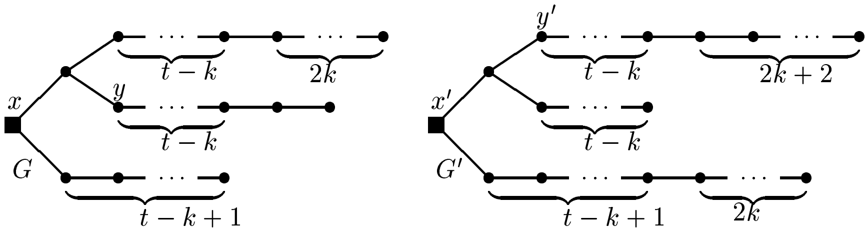


fig. 22

it has to be shown that $b_m = b_m'$ for $m \le 2t - 2k + 2$. Again we proceed by induction. We have $b_0 = b_0' = 1$, and it is clear that $b_1 = b_1'$. Now let $2 \le m \le 2t - 2k + 2$ be fixed and assume that $b_l = b_l'$ holds for $l \le m - 1$. If we erase in G resp. G' the branch which contains y resp. y' the remaining subgraphs are isomorphic; in particular the numbers of walks from the roots coincide. Thus we may concentrate on the walks of length m which meet y resp. y', and if we condition on the first time a walk from the root hits these vertices we arrive at the expressions

$$\sum_{i=0}^{m} b_i(y, \overline{\{y\}}) c_{m-i} \quad \text{resp.} \quad \sum_{i=0}^{m} b_i'(y', \overline{\{y'\}}) c_{m-i}'$$

for their total number; here c_j and c_j' stand for the walks of length j starting at y resp. y'.

Since walks of length $\le 2t - 2k + 2$ from x (resp. x') to y (resp. y') cannot leave $G^{t-k+1} = (G')^{t-k+1}$, the $(t - k + 1)$-truncations with respect to the new roots x, x', we have $b_i(y, \overline{\{y\}}) = b_i'(y', \overline{\{y'\}})$. Also, a walk cannot hit y, y' already with the first step, and thus

$$b_m - b_m' = \sum_{i=2}^{m} b_i(y, \overline{\{y\}}) \left(c_{m-i} - c_{m-i}'\right).$$

It remains to show that $c_l = c_l'$ for $l \le m - 2$. As above this leads to a new decision problem where y resp. y' are declared as the new roots:

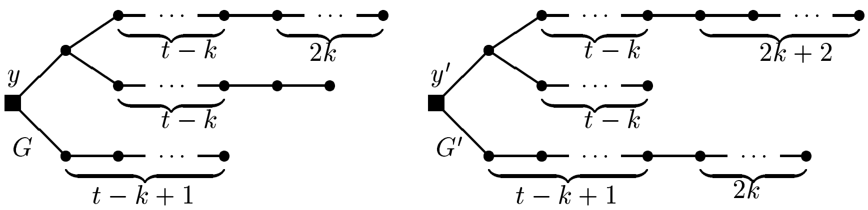


fig. 23

But this is exactly the situation from figure 22 above. *There* $b_l = b_l'$ for $l \leq m-1$ by assumption so that *here* $c_l = c_l'$ for $l \leq m-1$. This shows that the assertion holds for m, and the the proof of proposition 2.3 is complete. □

Remark: It will be necessary to discuss a variant of the preceding proof which will be important for the proof of proposition 2.4. We use proposition 2.5 – the proof of which does not need prop. 2.3 – to compare the b and b', the numbers of walks in the graphs of figure 22. Setting $t_0 = t-k+1$ and $k_0 = 2k-1$ we deduce from proposition 2.5 that $b_n = b_n'$ up to $n = 2t_0 + 1 = 2t - 2k + 3$, and $b_{2t-2k+4} = b'_{2t-2k+4} + 1$. Furthermore we saw that $a_m(x, \overline{\{x\}}) - a_m'(x', \overline{\{x'\}})$ is zero for $m \leq 2t-k+2$ and one for $m = 2t-k+3$. For $m = 2t-k+4$ it will again be zero since the $a_m(x, \overline{\{x\}})$, $a_m'(x', \overline{\{x'\}})$ vanish if m is *not* $k+1$ plus an even number.

Finally, for $m = 2t-k+5$, the difference is $2t-k+4$. This can be seen by a similar counting procedure as in the quote of page 14. Here, however, walks are not continued after reaching x. Also, when passing the bifurcation point for the first time, there is only one step off the "optimal" route: note that the walk must not meet x before the last step. Since the optimal route has length $2t-k+3$, we get $a_{2t-k+5}(x, \overline{\{x\}}) - a'_{2t-k+5}(x', \overline{\{x'\}}) = (2t-k+3)+1 = 2t-k+4$. It follows that, for $n \leq 2t-k+5$,

$$a_n - a_n' = \sum_{m=k+1}^{n} \left(a_m(x, \overline{\{x\}}) - a_m'(x', \overline{\{x'\}}) \right) b_{n-m} \quad (+1 \text{ if } n = 2t-k+5),$$

from which we can deduce that

$$\begin{aligned} a_n - a_n' &= \begin{cases} 0, & \text{if } n \leq 2t-k+2 \\ 1 \cdot b_0, & \text{if } n = 2t-k+3 \\ 1 \cdot b_1 + 0 \cdot b_0, & \text{if } n = 2t-k+4 \\ 1 \cdot b_2 + 0 \cdot b_1 + (2t-k+4) \cdot b_0 + 1, & \text{if } n = 2t-k+5 \end{cases} \\ &= \begin{cases} 0, & \text{if } n \leq 2t-k+2 \\ 1, & \text{if } n = 2t-k+3 \\ 2+0=2, & \text{if } n = 2t-k+4 \\ 5+0+(2t-k+4)+1 = 2t-k+10, & \text{if } n = 2t-k+5. \end{cases} \end{aligned} \tag{2.7}$$

We are now prepared for the

Proof of prop. 2.4. Let $t = 3k$. Our strategy is to compare the tree $G'' = [k, t+2, \infty, t-k]$ $=[k, 3k+2, \infty, 2k]$ with the tree G' of proposition 2.3, i.e., with $G' = [k, t+2, t+k+2, t-k] = [k, 3k+2, 4k+2, 2k]$. By formula (2.7) we are done if we can show that

$$a''_n - a'_n = \begin{cases} 0, & \text{if } n \leq 2t-k+2 = 5k+2 \\ 1, & \text{if } n = 2t-k+3 = 5k+3 \\ 2, & \text{if } n = 2t-k+4 = 5k+4 \\ 2t-k+8 = 5k+8 & \text{if } n = 2t-k+5 = 5k+5 \end{cases}. \tag{2.8}$$

We have to count the walks in G'' of length up to $5k+5$ that leave G'. The shortest of these walks is of length $5k+3$. So it is easy to see that $a''_n - a'_n$ is zero for $n \leq 5k+2$, one for $n = 5k+3$ and two for $n = 5k+4$. For $n = 5k+5$ a consideration similar to the one given on page 14 shows that $a''_n - a'_n = (5k+3)+1+4 = 5k+8$. □

Now we turn to the proofs of proposition 2.5, 2.6 and 2.7. We start with the following

Lemma 2.14 *Let G', G'' be as in figure 11 ($t \geq 2$). Then $a'_n = a_n(G')$ and $a''_n = a_n(G'')$ are equal up to $n = 2t+4$ for $k > 0$. For $k = 0$ they are equal up to $n = 2t+3$, and $a'_{2t+4} = a''_{2t+4} + 1$.*

Proof. Let $s := t+k+1$, $c = t-1$ ($< s-1$). We have to compare $[1, s, s+1, c]$ and $[1, \infty, s-1, c]$. If $k \leq t-2$ (in particular, if $k = 0$), then $s \leq 2t-1$, so that $s \leq 2c+1$. By proposition 2.1, the numbers of walks agree up to $n = 2s+1 = 2t+2k+3$ and $a'_{2t+2k+4} = a''_{2t+2k+4} + 1$. If $k \geq t-1$, then $s \geq 2c+2$. Considering walks of length $\leq s+2c+2$, we can replace the ∞-branch in G'' by a branch of length $s+2c+2$. By prop. 2.2 the number of walks in this graph is equal to the number of walks in G' up to $n = 2s+1$. So a_n and a'_n agree up to $n = \min\{s+2c+2, 2s+1\} \geq 2t+4$. □

Proof of prop. 2.5. By the preceding lemma we only have to compare G with G''. But by flipping the second branch to the top in each graph and setting $c = t+k$ we have reduced the claim directly to lemma 2.12. □

Note that we have shown more, we have in fact established the identities

$$a_n - a'_n = a_n - a''_n = \begin{cases} 0, & \text{if } n \leq 2t+1 \\ 1, & \text{if } n = 2t+2 \\ 2, & \text{if } n = 2t+3 \\ 2t+8, & \text{if } n = 2t+4 \end{cases}$$

(except that for $k = 0$ we have $a_{2t+4} - a'_{2t+4} = 2t+7$ instead of $2t+8$).

We can now prove the next two propositions:

Proof of prop. 2.7. In light of the preceding remark we only have to compare $G^{**} = [1, \infty, \infty, t-1]$ and $G'' = [1, 2t, \infty, t-1]$. For $n \leq 2t+1$ we clearly have $a^{**}_n = a''_n$. It is also easy to see that $a^{**}_{2t+2} = a''_{2t+2} + 1$ and $a^{**}_{2t+3} = a''_{2t+3} + 2$. For $n = 2t+4$ one can argue as follows: The shortest walk in G^{**} to leave the subgraph G'' is of length $2t+2$. Now there are two steps left. These could be a step forth and back on or before this way which accounts for $2t+3$ possibilities.

(Note that there are *two* "wrong" directions at the bifurcation point.) It is also possible to add the two steps at the end of the optimal route, this amounts in four more possibilities. So $a^{**}_{2t+4} = a''_{2t+4} + 2t + 7$. By the remark preceding this proof it follows that $a_n([1, t, 2t, t+1]) = a_n([1, \infty, \infty, t-1])$ for $n \le 2t+3$, and $a_{2t+4} = a^{**}_{2t+4} + 1$. $\square$

Proof of prop. 2.6. The proof is analogous to the proof of proposition 2.7 just given. We settle the case $t = 2$ by direct calculation so that we may assume that $t \ge 3$.

The same techniques as above apply to show that

$$a^*_n - a'_n = \begin{cases} 0, & \text{if } n \le 2t+1 \\ 1, & \text{if } n = 2t+2 \\ 2, & \text{if } n = 2t+3 \\ 2t+7, & \text{if } n = 2t+4 \end{cases}$$

So again by the remark following the proof of prop. 2.5 we are done (note that $k > 0$ since $t \ge 3$). $\square$

Proof of proposition 2.8. No walk of length $\le 2t+1$ can first leave G^t (resp. $(G')^t$) and then return to its starting point. So, as in the earlier proofs, we are led to compare the number of walks of length $\le 2t$ which start at x_1 and x'_1 in the graphs H and H' and which visit R and R', respectively (see figure 18), as before we will use the letters $b_n(\ldots)$, $b'_n(\ldots)$ for the number of the walks of length n.

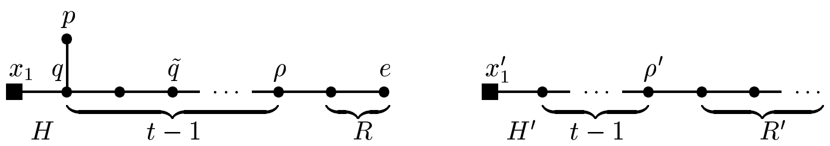


fig. 24

In order to show that $b_n(R) = b_n(R')$ for $n \le 2t-1$ and $b_{2t}(R) = b_{2t}(R')+1$, we will proceed by induction on n. The case "$n = 1$" is clear since no walk of length 1 can visit R or R'. Suppose that $b_n(R) = b_n(R')$ has been proved for some $n \le 2t-1$. As in the proof of lemma 2.12 we calculate $b_{n+1}(R)$ as

$$b_{n+1}(R) = 2b_n(R) - b_n(e) + b_n(q, R) + b_n(\rho, \overline{R}) :$$

every walk of length n which counts for $b_n(R)$ contributes with two walks to $b_{n+1}(R)$, except for those which end at e (one walk) or q (three walks). For $n \le 2t-1$, no walk of length n can enter R and end at x_1 or p; also we have to add those which enter R in step number $n+1$ for the first time.

A similar consideration yields

$$b'_{n+1}(R') = 2b'_n(R') + b'_n(\rho', \overline{R'}).$$

Subtracting the second from the first equality we get from our induction hypothesis that

$$b_{n+1}(R) - b'_{n+1}(R') = -b_n(e) + b_n(\rho, \overline{R}) - b'_n(\rho', \overline{R'}) + b_n(q, R).$$

Note that $b_n(q, R)$ equals 0 for $n < 2t-1$ and 1 for $n = 2t-1$.

Obviously, $b_n(\rho, \overline{\{p\} \cup R}) = b'_n(\rho', \overline{R'})$, so it remains to show that

$$b_n(e) = b_n(\rho, \{p\}, \overline{R}).$$

We condition on the last time a walk meets p: if the number of walks of length j which start at p is denoted by $c_j(\ldots)$ and those starting at q resp. $\tilde{q}$ by $d_j(\ldots)$ resp. $\tilde{d}_j(\ldots)$, then

$$\begin{aligned}
b_n(\rho, \{p\}, \overline{R}) &= \sum_{m=2}^{n-1} b_m(p) c_{n-m}(\rho, \overline{\{p\} \cup R}) \\
&= \sum_{m=2}^{n-1} b_{m-1}(q) d_{n-m-1}(\rho, \overline{\{p\} \cup R}) \\
&= \sum_{m=2}^{n-1} b_{m-1}(q) \tilde{d}_{n-m-1}(e, \overline{\{q\}}) \\
&= \sum_{m=2}^{n-1} b_{m-1}(q) d_{n-m+1}(e, \overline{\{q\}}) \\
&= \sum_{m=1}^{n-2} b_m(q) d_{n-m}(e, \overline{\{q\}}).
\end{aligned}$$

But this is just $b_n(e)$, the number of walks from x_1 to e, conditioned on the last time they meet q. $\square$

Proof of proposition 2.9. We will proceed similarily as in the proof of proposition 2.3. The cases $n = 0, 1, 2, 3$ are clear, since walks of length 3 cannot leave $G^t = (G')^t$. Fix n, suppose that $a_m = a'_m$ has been proved for all $m \leq n-1$ and consider walks of length n. Divide the set of walks into those that meet z, resp. z' and those which do not. We first compare the number of walks that do meet z resp. z': Conditioning on the first time a walk is at z, resp. z' we get

$$a_n(\{z\}) = \sum_{k=0}^{n} a_k(z, \overline{\{z\}}) a_{n-k}, \quad a'_n(\{z\}) = \sum_{k=0}^{n} a'_k(z, \overline{\{z\}}) a'_{n-k}. \tag{2.9}$$

(Note that, by symmetry, starting the walks in G at x_0 is just as good as starting the walks in z; the same holds true in G' for x'_0 and z'.) Since walks of length $\leq 2t+1$ from x_0 resp. x'_0 to z resp. z' cannot exit G^t resp. $(G')^t$ their numbers are equal, i.e., $a_k(z, \overline{\{z\}}) = a'_k(z', \overline{\{z'\}})$. Since these numbers are zero for $k \leq 3$, we have to sum over k with $4 \leq k \leq n$ only. But then by induction hypothesis $a_{n-k} = a'_{n-k}$, so the numbers in (2.9) are equal.

Now consider the walks of length n that do *not* meet z, resp. z'. These are just walks in the graphs H, H':

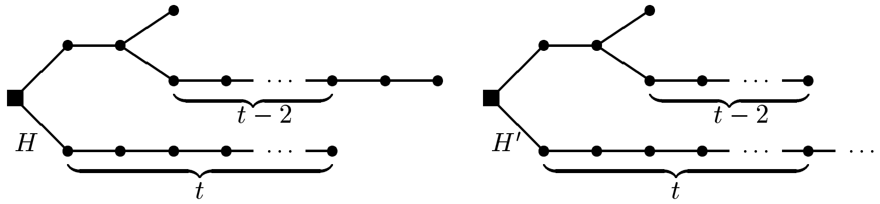


fig. 25

Flipping the top branches of these graphs we arrive precisely at the situation of proposition 2.8. The proposition is proved. □

3 Three decision problems

In this section we study three decision problems. The classes under consideration are rather special, but the results are of crucial importance for the proof of our main result.

In all of these three cases we want to distinguish between two classes – $\mathcal{G}_1$ and $\mathcal{G}_2$, say – of simple rooted 3-trees. There is a fixed t such that for all $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ the t-truncated tree G^t is the same.

Let $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ be given. Our *strategy number one* to decide whether it is in $\mathcal{G}_1$ or in $\mathcal{G}_2$ will be as follows. First, we consider all $(t+1)$-truncations of the trees in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. There will be only finitely many of them, denote them by $T_1^1, \ldots, T_1^r$ resp. $T_2^1, \ldots, T_2^{r'}$.

We note that G^{t+1} is one of these finitely many trees and also that for the calculation of $a_{t+1}(G)$ the tree G is as good as G^{t+1}. Thus, if it happens that $a := a_{t+1}(G)$ is a number which does *does not occur in both of the sets*

$$\{a_{t+1}(T_1^1), \ldots, a_{t+1}(T_1^r)\}, \{a_{t+1}(T_2^1), \ldots, a_{t+1}(T_2^{r'})\},$$

then it is already possible on the basis of $a_{t+1}(G)$ to decide whether $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$.

Suppose that there are certain T_1's and also certain T_2's for which a_{t+1} equals a; as an example, let us assume that these are T_1^1, T_1^2, T_2^1. Consider all possible $(t+2)$-truncations of trees T in $\mathcal{G}_1$ for which the $(t+1)$-truncation is T_1^1 or T_1^2 and calculate the associated a_{t+2}. Call this *table 1*.

Similarly, calculate in *table 2* the a_{t+2}'s of all $(t+2)$-truncations of the $T \in \mathcal{G}_2$ for which the $(t+1)$-truncation is T_2^1.

The number $a_{t+2}(G)$ can be found somewhere in one of these lists, and if it occurs only in one of them we are done. If not one has to continue.

We will see that in the case of decision problem I this method works, rather soon an $a_{t+r}(G)$ will allow us to decide with certainty whether $G \in \mathcal{G}_1$ or $G \in \mathcal{G}_2$.

For problem II and problem III we will use *another strategy, the method of one-to-one embeddings* which will be discussed below. However, there will remain situations which demand some how-to-count-walks-proofs, these have been prepared in section 2.

3.1 Decision problem I

We want to distinguish by counting between simple rooted 3-trees of the following two kinds:

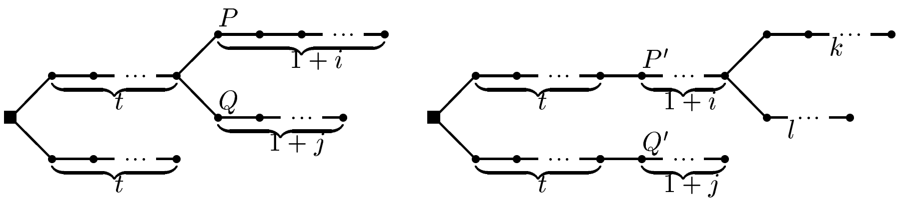


fig. 26

(The vertices P, Q, P', Q' will play a role in the following investigations, they all have the same distance $t+1$ to the root.)

More formally, we fix a $t \geq 1$, the classes we are interested in are

$$\mathcal{G}_1^t := \{[t, t, i+1, j+1] \mid i \geq j \geq 0\}$$

and

$$\mathcal{G}_2^t := \{[t+1+i, t+1+j, k, l] \mid i, j, k, l \geq 0\}$$

(as above, we tacitly assume for the sake of uniqueness that $i \geq j$ if $k = l = 0$ and that $k \geq l \geq 1$ if $(k, l) \neq (0, 0)$).

Proposition 3.1

(i) *If* $t \geq 2$, *then the pair* $(\mathcal{G}_1^t, \mathcal{G}_2^t)$ *is decidable after at most* $t + 13$ *steps. For* $t \geq 6$ *fewer steps are not sufficient. The minimal number of steps in the cases* $t = 2, 3, 4, 5$ *are* $11, 14, 15, 18$*, respectively.*

(ii) $[1, 1, 2, 2]$ *and* $[\infty, \infty, 0, 0]$ *have the same* (a_n)*-sequence* (2^n)*. Consequently,* $(\mathcal{G}_1^1, \mathcal{G}_2^1)$ *is not decidable.*

(iii) *Put* $\mathcal{G}_1^* := \mathcal{G}_1^1 \setminus \{[1, 1, 2, 2]\}$ *and* $\mathcal{G}_2^* := \mathcal{G}_2^1 \setminus \{[\infty, \infty, 0, 0]\}$*.*
Then $(\mathcal{G}_1^*, \mathcal{G}_2^1)$ *and* $(\mathcal{G}_1^1, \mathcal{G}_2^*)$ *are also decidable, the first one after* 12 *steps (which is the best possible number). For the second problem there does not exist a uniform bound: for every* n_0 *there are* $G_1 \in \mathcal{G}_1^1$ *and* $G_2 \in \mathcal{G}_2^*$ *for which the* a_n *with* $n = 1, 2, \ldots, n_0$ *are identical.*

3.2 Problem I: preparations of the proof

Later we will have to distinguish the cases $t \geq 6$ and $t < 6$. We start the investigations by fixing a $t \geq 1$, first we introduce some further notation. In $[t, t, 0, 0]$, the common initial segment of all trees under consideration here, we will label the vertices as follows:

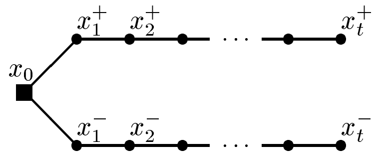


fig. 27

For $i = 0, 1, \ldots, t$, α_i stands for the number of walks of length t in $[t, t, 0, 0]$ from the root x_0 to x_i^+ (to x_0 in the case $i = 0$); by symmetry this is also the number of walks to x_i^-. The α_i could be explicitly determined (they are zero or certain binomial coefficients), but this will not be of importance in the sequel.

We are interested in walks in the trees selected from from $\mathcal{G}_1^t \cup \mathcal{G}_2^t$ of length $t + r$ for $r = 1, 2, \ldots, 13$. Fix such a walk which leaves $[t, t, 0, 0]$. After t steps it will be in x_0 or in some $x_i^\pm$, and the r further steps have to be used to leave $[t, t, 0, 0]$. Suppose that the exit takes place in step number $t + s$, then the walk will be at vertex P or Q (resp. at vertex P' or Q') in the case of trees in $\mathcal{G}_1^t$ (resp. in $\mathcal{G}_2^t$). Now $r - s$ steps are left, each of the possible choices gives rise to a walk of interest.

These observations motivate the following definitions, here G resp. G' are any fixed trees in $\mathcal{G}_1^t$ resp. in $\mathcal{G}_2^t$:

Definition 3.2

(i) For $s \geq 1$ the number $\beta_{s,i}^{++}$ denotes the number of walks of length $s-1$ in $[t, t, 0, 0]$ from x_i^+ to x_t^+, i.e., the number of walks of length s starting at x_i^+ which can leave $[t, t, 0, 0]$ precisely at the s'th step at the "northern" end. By definition, $\beta_{1,t}^{++} = 1$, and the other $\beta_{1,i}^{++}$ vanish.
Similarly we define $\beta_{s,i}^{+-}$ (= the walks from x_i^+ to x_t^-) and $\beta_{s,i}^{-\pm}$; $\beta_{s,0}^{\pm}$ stand for the numbers of walks of length $s - 1$ from x_0 to $x_t^\pm$.

(ii) $\gamma_{s,i} := \beta_{s,i}^{++} + \beta_{s,i}^{-+}$ (the walks which come from x_i^+ *or* from x_i^- with a northern destination); by symmetry, this number coincides with $\beta_{s,i}^{+-} + \beta_{s,i}^{--}$. $\gamma_{s,0} := \beta_{s,0}^{+} = \beta_{s,0}^{-}$.

(iii) $d_\rho(G)$ is the number of walks in G of length ρ which start in P or Q (we set $d_0(G) := 2$). Similary, $d_\rho(G')$ counts the walks of length ρ in G' which start in P' or in Q'.

The calculation of the d_ρ's can be cumbersome. What is, for example, the number $d_3(G)$ for $G = [5, 5, 2, 1] \in \mathcal{G}_1^5$?

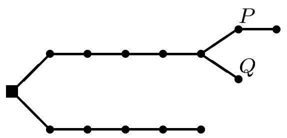


fig. 28

Starting at Q, there are five possible walks of length 3: *(lu,l,l)*, *(lu,l,r)*, *(lu,ru,r)*, *(lu,ru,ld)*, *(lu,rd,lu)*; here *l*, *r*, *lu*, *ld*, *ru*, *rd* stand for left, right, left up, left down, right up, right down, respectively. Similarly one finds the walks starting at P: *(r,l,ld)*, *(r,l,r)*, *(ld,ru,ld)*, *(ld,ru,r)*, *(ld,rd,ru)*, *(ld,l,l)*,*(ld,l,r)*. Summing up, we arrive at $d_3(G) = 12$.

Already for moderate values of i and j the calculations can be lengthy. Fortunately it is easy to write a computer program, it is only necessary to modify the idea how to calculate the a_n slightly (cf. the beginning of the introduction). Assign initially - step 0 - weigth 1 to the vertices P and Q and weigth 0 to all others. Update from step n to get step $n+1$ by assigning to each vertex the sum of the weights of the adjacent vertices, the number d_n we are looking for is just the total weight after the n'th step.

Lemma 3.3 *The number $a_r^t := a_{t+r}(G) - a_{t+r}(G^t)$ of walks of length $t+r$ in G which start at the root and leave $[t,t,0,0]$, is precisely*

$$\sum_{s=1}^{r}(\sum_{i=0}^{t}\alpha_i\gamma_{s,i})d_{r-s}(G).$$

The corresponding number for G' is

$$\sum_{s=1}^{r}(\sum_{i=0}^{t}\alpha_i\gamma_{s,i})d_{r-s}(G').$$

Proof. First we consider G and a walk π (resp. π^*) which after t steps is in x_i^+ (resp. x_i^-). For fixed s this π gives rise to $\beta_{s,i}^{++}$ walks of length $t+s$ which for the first time leave $[t,t,0,0]$ with a step to P and also to $\beta_{s,i}^{++}$ ones which have their exit at Q. The numbers for the walks associated with π^* are $\beta_{s,i}^{-+}$ for both a first exit step at P and at Q. There are $r-s$ steps left, from now on every walk counts. Thus, if we define d_ρ^P (resp. d_ρ^Q) to be the number of all walks of length ρ in G which start at P (resp. at Q), we have shown that the contribution to $a_r^t(G)$ of one pair of walks from the i-level which leave $[t,t,0,0]$ for the first time at step number s is

$$\begin{aligned}\beta_{s,i}^{++}d_{r-s}^P + \beta_{s,i}^{++}d_{r-s}^Q + \beta_{s,i}^{-+}d_{r-s}^P + \beta_{s,i}^{-+}d_{r-s}^Q &= (\beta_{s,i}^{++} + \beta_{s,i}^{-+})(d_{r-s}^P + d_{r-s}^Q)\\ &= \gamma_{s,i}d_{r-s}(G).\end{aligned}$$

Since there are precisely α_i such pairs, we have established the first formula.

Next we consider G'. Again we fix a pair π,π^*, and we consider an exit after s steps. From π one gets $\beta_{s,i}^{++}$ walks which, after leaving $[t,t,0,0]$, are at P' and $\beta_{s,i}^{+-}$ ones which are at Q'. Each of the first ones contributes with $d_{r-s}^{P'}$ to $a_r^t(G')$, each of the second ones with $d_{r-s}^{Q'}$ (it should be clear how $d_\rho^{P'}$ and $d_\rho^{Q'}$ are defined). Altogether, there are $\beta_{s,i}^{++}d_{r-s}^{P'} + \beta_{s,i}^{+-}d_{r-s}^{Q'}$ walks which are derived from π.

The corresponding number for walks which are extensions of π^* is $\beta^{-+}_{s,i} d^{P'}_{r-s} + \beta^{--}_{s,i} d^{Q'}_{r-s}$, and therefore the pair π, π^* contributes with

$$\begin{aligned} \beta^{++}_{s,i} d^{P'}_{r-s} + \beta^{+-}_{s,i} d^{Q'}_{r-s} + \beta^{-+}_{s,i} d^{P'}_{r-s} + \beta^{--}_{s,i} d^{Q'}_{r-s} &= (\beta^{++}_{s,i} + \beta^{-+}_{s,i})(d^{P'}_{r-s} + d^{Q'}_{r-s}) \\ &= \gamma_{s,i} d_{r-s}(G'); \end{aligned}$$

here we have made use of the fact that $\beta^{-+}_{s,i} = \beta^{+-}_{s,i}$ and $\beta^{++}_{s,i} = \beta^{--}_{s,i}$.

The calculations for the walks from x_0 are similar, and it remains to multiply the above expressions with α_i – this is the number of pairs (π, π^*) – and to sum up. □

The lemma has an interesting consequence:

Lemma 3.4

(i) *For any G in $\mathcal{G}^t_1 \cup \mathcal{G}^t_2$ it is possible to determine the $d_1(G), d_2(G), \ldots$ if the sequence $(a_n(G))$ is known.*

(ii) *For $G, G' \in \mathcal{G}^t_1 \cup \mathcal{G}^t_2$ one has $a_n(G) = a_n(G')$ for $n = 1, \ldots, t+1+r$ iff $d_\rho(G) = d_\rho(G')$ for $\rho = 1, \ldots, r$.*

Proof. (i) Since we are working with a fixed and known t, all α_i and all $\gamma_{s,i}$ can be determined, and also one may calculate the a^t_r from the sequence (a_n). Now note that $d_0(G) = a^t_1 = 2$. Also, by lemma 3.3, $d_{k+1}(G)$ can be expressed by the α_i, the $\gamma_{s,i}$ the $d_1(G), \ldots, d_k(G)$, and by $a^t_{k+2}(G)$; here it is important to note that $\gamma_{1,i}$ is 1 for $i = t$ and all other $\gamma_{s,i}$ vanish. In this way one gets $d_1(G), d_2(G), \ldots$ recursively.
This argument also settles assertion (ii). □

Remark: The proof reveals also that $a_{t+r+1}(G) - a_{t+r+1}(G') = d_r(G) - d_r(G')$ if $d_\rho(G) = d_\rho(G')$ for $\rho = 0, \ldots, r-1$.

We conclude that, in order to check whether G and G' have the same a_n's, it suffices to investigate the much smaller $d_\rho(G), d_\rho(G')$. Also it is important to observe that the latter are in a sense independent of t for "large" t and "small" ρ. More precisely: if $\rho \leq 2t+1$, then d_ρ is the same for $G = [t, t, i+1, j+1]$ and for $G = [t+s, t+s, i+1, j+1]$ for arbitrary $s \geq 0$, and similarly the d_ρ coincide for $G' = [t+1+i, t+1+j, k, l]$ and $G' = [t+s+1+i, t+s+1+j, k, l]$.

The reason is simple: for the calculation of the d_ρ with $\rho \leq 2t+1$ it is of no importance whether there are $2t+1$ or $2(t+s)+1$ vertices available to the left of P, Q, P', Q'.
This makes the following strategy possible:

- Prove that for a suitable t_0 and all $G \in \mathcal{G}^{t_0}_1$, $G' \in \mathcal{G}^{t_0}_2$ there is an $r \leq 2t_0+1$ such that $d^G_r \neq d^{G'}_r$. (Of course it is not clear that such a t_0 exists, but we will see that $t_0 = 6$ has this property.) By lemma 3.4 this will solve the decision problem for t_0.

- By the preceding discussion the pair $(\mathcal{G}_1^t \cup \mathcal{G}_2^t)$ is then also decidable for arbitrary $t \geq t_0$.
- Treat the remaining cases $t < t_0$ separately.

3.3 Problem I: the proof for the case $t \geq 6$

Fix any $t \geq 6$ and let $G \in \mathcal{G}_1^t \cup \mathcal{G}_2^t$ be given. We claim that we can decide on the basis of $a_{t+1}(G), \ldots, a_{t+13}(G)$, i.e., on the basis of $d_0(G), \ldots, d_{12}(G)$, whether G belongs to $\mathcal{G}_1^t$ or to $\mathcal{G}_2^t$.

The idea of the proof has been explained in section 3.1. First we fix notation: The "typical" tree $[t, t, 1+i, 1+j]$ in $\mathcal{G}_1^t$ (resp. $[t+1+i, t+1+j, k, l]$ in $\mathcal{G}_2^t$) will be denoted by $L_{i,j}$ (resp. by $R_{i,j,k,l}$). The proof starts with a systematic enumeration of the $(t+2)$-truncations of all trees in $\mathcal{G}_1^t \cup \mathcal{G}_2^t$ and the calculation of their d_1. We have to take into account $L_{0,0}$, $L_{1,0}$, $L_{1,1}$, $R_{0,0,0,0}$, $R_{1,0,0,0}$, $R_{1,1,0,0}$, $R_{0,0,1,1}$, and $R_{0,1,1,1}$.

These trees and the corresponding d_1 can be found in the table **c 1** below, the trees from $\mathcal{G}_1^t$ are on the left, those from $\mathcal{G}_2^t$ on the right-hand side. **c 1** gives also rise to the first branch of the "decision tree" depicted in figure 30 below. Only trees from $\mathcal{G}_2^t$ have $d_1 = 5$, and therefore a decision (between $G \in \mathcal{G}_1^t$ or $G \in \mathcal{G}_2^t$) is possible already now if $d_1(G) = 5$. This is indicated in figure 30 by a black rectangle under the "5" in the correponding box. Cases where the number d_1 is produced by $\mathcal{G}_1^t$- *and* $\mathcal{G}_2^t$-representatives have to be investigated further. These are here the cases $d_1 = 2, 3, 4$, the investigation is continued in **c 2**, **c 3**, and **c 8**, respectively. In this way we proceed until we arrive at leaves where all numbers have their small black rectangle.

Sometimes it happens that there is only one tree from each class which has to be taken into account. For example, in **c 2**, we only have to treat $L_{0,0}$ and $R_{0,0,0,0}$: we know that $G^{t+3} = L_{0,0}$ if $G^{t+2} = L_{0,0}$, and similarly for $R_{0,0,0,0}$. This fact has also made it sometimes possible to combine two tables into one (see **c 7**, **c 13** and **c 23**). And, on the other hand, we have to investigate so many cases in **c 10** that this table has been been split into two parts.

The tables associated with problem I, the case $t \geq 6$

c 1	$L_{0,0}$	$L_{1,0}$	$L_{1,1}$	$R_{0,0,0,0}$	$R_{1,0,0,0}$	$R_{1,1,0,0}$	$R_{0,0,1,1}$	$R_{0,1,1,1}$
d_1	2	3	4	2	3	4	4	5

c 2	$L_{0,0}$	$R_{0,0,0,0}$
d_2	6	4

c 3	$L_{1,0}$	$L_{2,0}$	$R_{1,0,0,0}$	$R_{2,0,0,0}$	$R_{1,0,1,1}$
d_2	7	8	5	6	7

c 4	$L_{1,0}$	$R_{1,0,1,1}$	$R_{1,0,2,1}$	$R_{1,0,2,2}$
d_3	12	11	12	13

c 5	$L_{1,0}$	$R_{1,0,2,1}$	$R_{1,0,3,1}$
d_4	28	27	28

c 6	$L_{1,0}$	$R_{1,0,3,1}$	$R_{1,0,4,1}$
d_5	49	49	50

c 7	$L_{1,0}$	$R_{1,0,3,1}$
d_6	114	114
d_7	202	201

c 8	$L_{1,1}$	$L_{2,1}$	$L_{2,2}$	$R_{1,1,0,0}$	$R_{2,1,0,0}$	$R_{2,2,0,0}$	$R_{1,1,1,1}$	$R_{1,2,1,1}$	$R_{0,0,1,1}$	$R_{0,0,2,1}$	$R_{0,0,2,2}$
d_2	8	9	10	6	7	8	8	9	6	7	8

c 9	$L_{1,1}$	$R_{2,2,0,0}$	$R_{3,2,0,0}$	$R_{3,3,0,0}$	$R_{2,2,1,1}$	$R_{2,3,1,1}$	$R_{1,1,1,1}$	$R_{1,1,2,1}$	$R_{1,1,2,2}$	$R_{0,0,2,2}$	$R_{0,0,3,2}$	$R_{0,0,3,3}$
d_3	16	14	15	16	16	17	14	15	16	14	15	16

c 10	$L_{1,1}$	$R_{3,3,0,0}$	$R_{4,3,0,0}$	$R_{4,4,0,0}$	$R_{3,3,1,1}$	$R_{3,4,1,1}$	$R_{2,2,1,1}$	$R_{2,2,2,1}$
d_4	34	30	31	32	32	33	30	31

c 10	$L_{1,1}$	$R_{2,2,2,2}$	$R_{1,1,2,2}$	$R_{1,1,3,2}$	$R_{1,1,3,3}$	$R_{0,0,3,3}$	$R_{0,0,4,3}$	$R_{0,0,4,4}$
d_4	34	32	32	33	34	30	31	32

c 11	$L_{1,1}$	$R_{1,1,3,3}$	$R_{1,1,4,3}$	$R_{1,1,4,4}$
d_5	68	66	67	68

c 12	$L_{1,1}$	$R_{1,1,4,4}$	$R_{1,1,5,4}$	$R_{1,1,5,5}$
d_6	146	146	147	148

c 13	$L_{1,1}$	$R_{1,1,4,4}$
d_7	292	292
d_8	628	630

c 14	$L_{2,1}$	$L_{3,1}$	$R_{1,2,1,1}$	$R_{1,2,2,1}$	$R_{1,3,1,1}$	$R_{1,2,2,2}$	$R_{1,3,2,1}$	$R_{1,3,2,2}$
d_3	17	18	15	16	16	17	17	18

c 15	$L_{2,1}$	$R_{1,2,2,2}$	$R_{1,2,3,2}$	$R_{1,2,3,3}$	$R_{1,3,2,1}$	$R_{1,3,3,1}$	$R_{1,4,2,1}$	$R_{1,4,3,1}$
d_4	39	36	37	38	36	37	37	38

c 16	$L_{3,1}$	$L_{4,1}$	$R_{1,3,2,2}$	$R_{1,3,3,2}$	$R_{1,4,2,2}$	$R_{1,3,3,3}$	$R_{1,4,3,2}$	$R_{1,4,3,3}$
d_4	40	41	37	38	38	39	39	40

c 17	$L_{3,1}$	$R_{1,4,3,3}$	$R_{1,4,4,3}$	$R_{1,5,3,3}$	$R_{1,4,4,4}$	$R_{1,5,4,3}$	$R_{1,5,4,4}$
d_5	80	77	78	78	79	79	80

c 18	$L_{3,1}$	$R_{1,5,4,4}$	$R_{1,5,5,4}$	$R_{1,6,4,4}$	$R_{1,5,5,5}$	$R_{1,6,5,4}$	$R_{1,6,5,5}$
d_6	177	174	175	175	176	176	177

c 19	$L_{3,1}$	$R_{1,6,5,5}$	$R_{1,6,6,5}$	$R_{1,7,5,5}$	$R_{1,6,6,6}$	$R_{1,7,6,5}$	$R_{1,7,6,6}$
d_7	354	351	352	352	353	353	354

c 20	$L_{3,1}$	$R_{1,7,6,6}$	$R_{1,7,7,6}$	$R_{1,8,6,6}$	$R_{1,7,7,7}$	$R_{1,8,7,6}$	$R_{1,8,7,7}$
d_8	781	779	780	780	781	781	782

c 21	$L_{3,1}$	$R_{1,7,7,7}$	$R_{1,7,8,7}$	$R_{1,7,8,8}$	$R_{1,8,7,6}$	$R_{1,8,8,6}$	$R_{1,9,7,6}$	$R_{1,9,8,6}$
d_9	1562	1560	1561	1562	1560	1561	1561	1562

c 22	$L_{3,1}$	$R_{1,7,8,8}$	$R_{1,7,9,8}$	$R_{1,7,9,9}$	$R_{1,9,8,6}$	$R_{1,9,9,6}$	$R_{1,10,8,6}$	$R_{1,10,9,6}$
d_{10}	3439	3440	3441	3442	3439	3440	3440	3441

c 23	$L_{3,1}$	$R_{1,9,8,6}$
d_{11}	6878	6878
d_{12}	15119	15125

Here are the record holders: $L_{3,1}$ and $R_{1,9,8,6}$ have the same same $d_1, \ldots, d_{11}$, namely

$$4, 9, 18, 40, 80, 177, 354, 781, 1562, 3439, 6878,$$

but a different d_{12}-value; thus they have identical $a_{t+1}, \ldots, a_{t+12}$, and only a_{t+13} enables one to distinguish them:

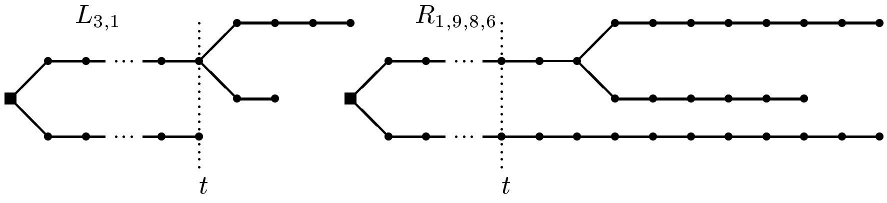


fig. 29

It should be recalled that the a_n grow exponentially fast. On the other hand, by the remark after lemma 3.4, the first $a_n(G) - a_n(G')$ which is different from zero coincides with the first nontrivial $d_r(G) - d_r(G')$, and – as the preceding table indicates – these numbers are *remarkably small*. For example, in the case of $L_{3,1}$ and $R_{1,9,8,6}$ with $t = 10$ the first nonzero difference of the d-numbers is $15119 - 15125 = -6$, it occurs at $r = 12$. The corresponding a_n-values have to be considered at $n = 10+12+1 = 23$:

$$a_{23}(L_{3,1}) = 8\,481\,720, \;\; a_{23}(R_{1,9,8,6}) = 8\,481\,726.$$

We have encountered similarly small first nonzero differences in all cases under consideration, we have no idea how this could be explained.

Our proof can also be visualized by the "decision tree" of figure 30.

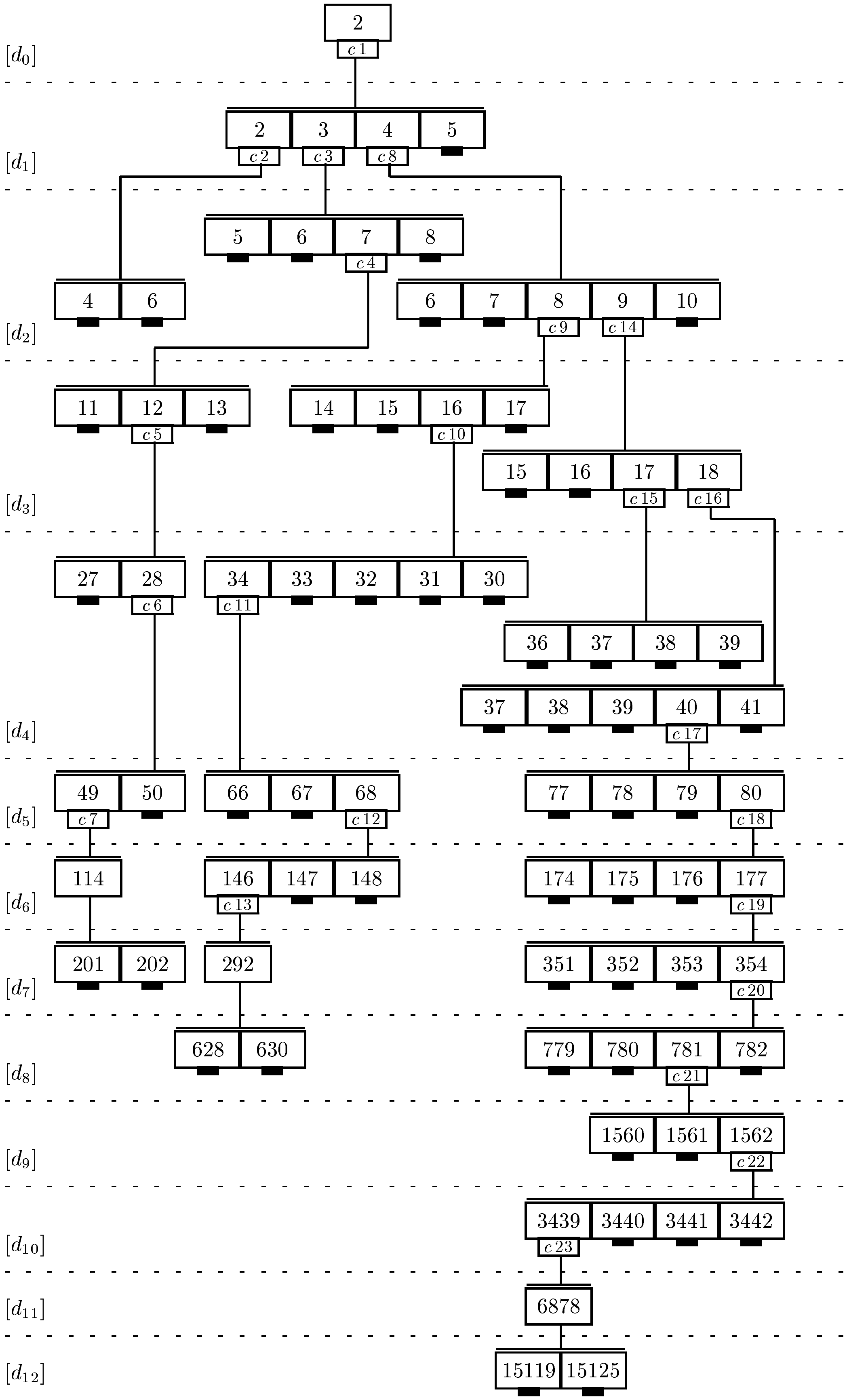
3.4 Problem I: the proof for the case $t = 1$

Our strategy is the same as before, here, however, we will work with the a_n directly. Let G be a tree in $\mathcal{G}_1^1$ or in $\mathcal{G}_2^1$, we first assume that $G \neq [1, 1, 2, 2]$.

The investigation starts as above. We calculate a table of the a_3-values of all possible G^3, where those coming from $\mathcal{G}_1^1$ resp. $\mathcal{G}_2^1$ are in the left resp. right part:

c 1	[1, 1, 1, 1]	[1, 1, 2, 1]	[1, 1, 2, 2]	[2, 2, 0, 0]	[3, 2, 0, 0]	[2, 2, 1, 1]	[3, 3, 0, 0]	[2, 3, 1, 1]
a_3	6	7	8	6	7	8	8	9

(Do you wonder why you find $[1, 1, 2, 2]$ here which has been excluded some lines before? This tree is a G^3 for some $G \in \mathcal{G}_1^*$ – e.g., for $G = [1, 1, 17, 15]$ – and thus we must not forget it.) The number 9 occurs only on the right hand side. Thus, if $a_3(G) = 9$, it is clear that $G \in \mathcal{G}_2^1$. The cases $a_3 = 6, 7, 8$ necessitate further investigation. We have calculated tables until all leaves of the decision tree have had their small black rectangle.



[d0]
2
c 1
[d1]
2 3 4 5
c 2 c 3 c 8
5 6 7 8
c 4
4 6
6 7 8 9 10
c 9 c 14
[d2]
11 12 13
c 5
14 15 16 17
c 10
15 16 17 18
c 15 c 16
[d3]
27 28
c 6
34 33 32 31 30
c 11
36 37 38 39
37 38 39 40 41
c 17
[d4]
49 50
c 7
66 67 68
c 12
77 78 79 80
c 18
[d5]
114
146 147 148
c 13
174 175 176 177
c 19
[d6]
201 202
292
351 352 353 354
c 20
[d7]
628 630
779 780 781 782
c 21
[d8]
1560 1561 1562
c 22
[d9]
3439 3440 3441 3442
c 23
[d10]
6878
[d11]
15119 15125
[d12]

fig. 30

For the sake of saving space the tables are not reproduced here, we only mention the "hard" cases: $[1,1,4,2]$ and $[3,7,2,2]$ have the same $a_1,\ldots,a_{11}$, but the a_{12}-values are different (5311 vs. 5313), and also $[1,1,10,2]$, $[3,9,4,2]$ compete until a_{12} (with values 5797 vs. 5800).

Let us indicate where the assumption $G \neq [1,1,2,2]$ plays a role. It is crucial when one arrives at the table of all possible 4-truncations of trees T under consideration with $T^3 = [1,1,2,2]$. By assumption the tree $[1,1,2,2]$ has not to be taken into account (which has the consequence that we are not running into an infinite branch of the decision tree).

Next we consider the $(\mathcal{G}_1^1, \mathcal{G}_2^*)$-case, that is, we start with a G which is not the tree $[\infty,\infty,0,0]$. This time our successful strategy seems not to work, regardless of how long we calculate we always arrive at tables of the form

	$[1,1,2,2]$	$[k+3,k+3,0,0]$	$[k+2,k+2,1,1]$	$[k+1,k+1,2,2]$	$[k,k,2,2]$
a_{k+3}	2^{k+3}	2^{k+3}	2^{k+3}	2^{k+3}	2^{k+3}

(These are the only cases which resist to die out.)

It is therefore clear that a uniform bound to decide between the classes under consideration cannot exist. But is it a decision pair? The problem we are faced with can be reformulated as follows:

Is it true that no $G' \in \mathcal{G}_2^*$ has (2^n) as its (a_n)-sequence?

For the proof we use the fact that $a_n([\infty,\infty,0,0]) = 2^n$ which is obvious. It follows that no proper subtree of $[\infty,\infty,0,0]$, i.e., no $[i,j,0,0]$ with $\min\{i,j\} < \infty$, can have the same a_n. Thus we only have to treat the $[i,j,k,l]$, where i is ≥ 2 and finite, and $k,l \geq 1$.

But $[i,j,k,l]$ and $[\infty,\infty,0,0]$ have different a_n for some n, we argue as follows. If $i \neq j$, then obviously $a_n([i,j,k,l]) \neq a_n([\infty,\infty,0,0])$ for $n = \min\{i,j\}+1$. In the case $i = j =: r$, however, there must be an $n > r$ with $a_n([i,j,k,l]) \neq a_n([\infty,\infty,0,0])$ since $(\mathcal{G}_1^r, \mathcal{G}_2^r)$ is a decision pair. (This is already established for $t \geq 6$, the remaining cases which might be of importance here, the cases $t = 2,3,4,5$, will be treated in the next subsection.)

This completes the investigations in the case $t = 1$, and assertion (iii) of the proposition is also proved if we anticipate the calculations of

3.5 Problem I: the proof for the cases $t = 2,3,4,5$

Similarly systematically as with $t = 1$ we have dealt with the cases $t = 2$ and $t = 3$ where the investigation stopped with a_{14} or earlier (the champions for $t = 2$ are $[2,2,4,2]$, $[4,10,4,4]$, $[4,8,6,4]$ with different a_n-values only at $n = 11$, and in the case $t = 3$ the record holders are $[3,3,4,2]$ and $[5,11,6,6]$ with the different a_{14}-values 20015 and 20055).

With $t = 4,5$ the situation was somewhat simpler. In view of the observations following lemma 3.4 one only has to check d_r's where $r > 9$ (for $t = 4$) and > 11 (for $t = 5$): if two graphs are distinguished by a d_r with a smaller r, then the calculations of section 3.3 can be used. This we have done systematically, for $n = 4$ the associated a_n always distinguish between the two classes not later than $n = 15$ (for $[4, 4, 4, 2]$ and $[6, 15, 8, 6]$ no $n \leq 14$ suffices). And $n = 18$ is admissible for $t = 5$, in fact the a_{18}-value is the first a_n where $[5, 5, 4, 2]$ and $[7, 15, 8, 6]$ behave differently.

3.6 Problem I: the remaining parts of the proof

We have proved so far the assertions (i) and (iii) of proposition 3.1, only (ii) is still open. It is plain that $a_n([\infty, \infty, 0, 0]) = 2^n$ for every n, for the calculation of the $a_n([1, 1, 2, 2])$ we use induction on n. To this end we label the vertices of $[1, 1, 2, 2]$ as follows:

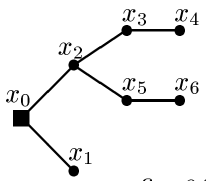


fig. 31

Let $e_{n,i}$ be the number of walks of length n from x_0 to x_i. Then $a_n = \sum_i e_{n,i}$ and – as we have already noted – $e_{n+1,i}$ is the sum over the $e_{n,j}$ with j running through the neighbours x_j of x_i. With this remark it is easy to prove by induction on n that

i	0	1	2	3	4	5	6
$e_{n,i}$	$1 + (2^n - 1)/3$	0	0	$(2^n - 1)/3$	0	$(2^n - 1)/3$	0

for n even and

i	0	1	2	3	4	5	6
$e_{n,i}$	0	$1 + (2^{n-1} - 1)/3$	2^{n-1}	0	$(2^{n-1} - 1)/3$	0	$(2^{n-1} - 1)/3$

for n odd. This implies that $a_n([1, 1, 2, 2]) = 2^n$ as claimed.

As a variant of the proof one could show by induction on n the following assertion: "$a_n = 2^n$, and if n is odd, half of the walks end at x_2." The induction step uses the observation that for odd n the walks which end after n steps in x_2 have three possibilities to proceed, whereas the remaining only have one choice. In the case of even n each walk can be continued into two ways, and half of the extended walks end at x_2.

3.7 Decision problem II

Consider the following two classes of trees which consist of $[t-b, t, b, c]$ plus a path of length at least one:

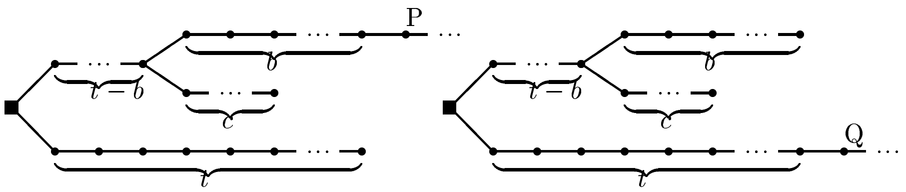


fig. 32

More precisely, we fix $t \geq 2$ and $t > b \geq c \geq 1$, the classes we are interested in are

$$\mathcal{H}_1^{t,b,c} := \{L_\mu \mid 0 \leq \mu \leq \infty\},$$
$$\mathcal{H}_2^{t,b,c} := \{R_\nu \mid 0 \leq \nu \leq \infty\},$$

where $L_\mu := [t-b, t, b+1+\mu, c]$ and $R_\nu := [t-b, t+1+\nu, b, c]$.

Proposition 3.5 *$(\mathcal{H}_1^{t,b,c}, \mathcal{H}_2^{t,b,c})$ is decidable after at most $2t+4$ steps. For certain values of t, b, c this is best possible.*

Note the difference between problem I and problem II: in problem I all trees have the simple t-truncation $[t, t, 0, 0]$ which has nice symmetry properties, but there can be complicated extensions to trees in $\mathcal{G}_1^t$ or $\mathcal{G}_2^t$; here, however, the extensions are simple, but the common G^t, i.e., $[t-b, t, b, c]$ has a more complicated structure than $[t, t, 0, 0]$.

3.8 Problem II: the proof of proposition 3.5

We have to produce, for arbitrary μ and ν, an n such that $a_n(L_\mu) \neq a_n(R_\nu)$. All L_μ, R_ν have the t-truncation $[t-b, t, b, c]$, and thus $a_n(L_\mu) = a_n(R_\nu)$ for $n = 1, \ldots, t$. The difference between the number of walks in L_μ resp. R_ν can only be caused by walks which leave $[t-b, t, b, c]$. Therefore it is reasonable to work with the $a_r^t(L_\mu), a_r^t(R_\nu)$. (Recall that $a_r^t(G) = a_{t+r}(G) - a_{t+r}(G^t)$; it should be stressed that – for all $G \in \mathcal{H}_1^{t,b,c} \cup \mathcal{H}_2^{t,b,c}$ – the information contained in the sequence $(a_n(G))_n$ is as good as that in $(a_r^t(G))_r$ since G^t is known.)

Denote the collection of walks of length $t+r$ in L_μ (resp. R_ν) which start at the root and which leave $[t-b, t, b, c]$ by $\Lambda_r(L_\mu)$ (resp. $\Lambda_r(R_\mu)$). The cardinalities of these sets, i.e., the numbers $a_r^t(L_\mu), a_r^t(R_\nu)$, will be compared by studying suitable embeddings. In order to be able to define these mappings unambiguously we will regard the L_μ, R_ν as subtrees of $[t-b, \infty, \infty, \infty]$ the vertices of which will be labelled as follows.

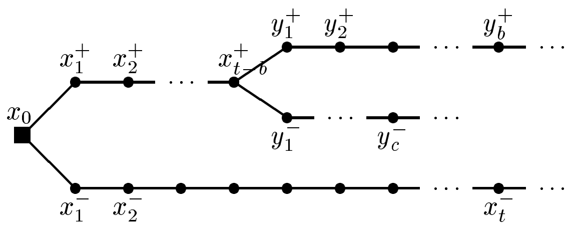


fig. 33

Let π be any walk in $[t-b, \infty, \infty, \infty]$ which starts at x_0 and which *does not* enter the $(y_1^-, y_2^-, \ldots)$-branch. Denote by $\Phi_1(\pi)$ the *reflection of* π: a step of π to a vertex x_i^+ resp. y_j^+ is replaced by one to x_i^- resp. x_{t-b+j}^-, and the x_i^- have to be replaced by x_i^+ for $i \le t-b$ and by $y_{i-(t-b)}^+$ otherwise. It is obvious that Φ_1 is a well-defined bijection on the collection of these walks.

The first steps of our investigations are simple, it will suffice to work with suitable restrictions of Φ_1 (which will also be called Φ_1).

$\boxed{L_0 \textit{ vs. } R_\nu}$

Consider first R_ν with $\nu \ge 1$. The restriction of Φ_1 to $\Lambda_2(L_0)$ is a well-defined injection to $\Lambda_2(R_\nu)$ since no $\pi \in \Lambda_2(L_0)$ can visit y_1^-. Thus $a_2^t(L_0) \le a_2^t(R_\nu)$. This inequality is even strict since $(x_0 \to x_{t+2}^-)$ (= the walk which moves directly from x_0 to x_{t+2}^-) is not in the range of this map.

How performs L_0 against R_0? Consider $\Phi_1 : \Lambda_3(R_0) \to \Lambda_3(L_0)$. Again, this is a well-defined map, and now $(x_0 \to y_1^- \to y_{b+1}^+)$ is not in the range[2].

Summing up, we have shown that $a_2^t(R_\nu) = a_2^t(L_0) + 1$ for $\nu \ge 1$, and $a_3^t(L_0) = a_3^t(R_0) + 1$.

$\boxed{L_\mu \textit{ with } \mu \ge 2 \textit{ vs. } R_\nu}$

Fix $\mu \ge 2$ and consider, for arbitrary ν,

$$\Phi_1 : \Lambda_3(R_\nu) \to \Lambda_3(L_\mu).$$

This well-defined map fails to have $(x_0 \to y_1^- \to y_{b+1}^+)$ in its range so that $a_3^t(L_\mu) \ge a_3^t(R_\nu) + 1$.

$\boxed{L_1 \textit{ vs. } R_\nu}$

This is the most difficult part of this proof. We will be able to show that there is an r such that $a_r^t(L_1) > a_r^t(R_\infty)$, the choice of r will depend on b and c. Therefore, a fortiori, $a_r^t(L_1) > a_r^t(R_\nu)$ for all finite ν (we note that the argument in the case d) below will be slightly different).

It will not be sufficient to work with the reflection Φ_1, we will need *another embedding which is defined by suitably "folding" walks.*

[2]The notation should be self-explaining: $(x_0 \to y_1^- \to y_{b+1}^+)$ is the walk which leads from x_0 directly to y_1^- and then directly to y_{b+1}^-. In the sequel we will define various walks in this way.

Let $\Lambda_r^*(R_\infty)$ denote the collection of all $\pi \in \Lambda_r(R_\infty)$ which do not visit y_1^-, a map $\Phi_2 : \Lambda_r^*(R_\infty) \to \Lambda_r(L_1)$ will be defined as follows.

Suppose that $r \le 7$. If $\pi \in \Lambda_r^*(R_\infty)$ is given, then we first pass to $\pi' := \Phi_1(\pi)$. If π' lies in $\Lambda_r(L_1)$, put $\Phi_2(\pi) := \pi'$, otherwise there is a unique $k \ge 1$ such that π' meets $\{y^+_{b+2k+1}, y^+_{b+2k+2}\}$ but *not* y^+_{b+2k+3}. Since π' is the reflection of a walk in $\Lambda_r^*(R_\infty)$ this walk does not visit y_1^-. We will modify it in such a way that π'-visits of y_j^+ with $j \ge b+3$ are replaced by steps into the $(y_1^-, \ldots)$-branch.

More precisely, we observe first that in the sequence of vertices visited by π' there necessarily will appear blocks of the form $(x^+_{t-b}, y_1^+, y_2^+)$. Replace the *last* one of these blocks by $(x^+_{t-b}, y_1^-, x^+_{t-b})$ and shift the remaining vertices by two units to the left. (E.g.,

$$(\ldots, x^+_{t-b}, y_1^+, y_2^+, y_1^+, y_2^+, y_1^+, y_2^+, y_3^+, \ldots)$$

has to be modified to

$$(\ldots, x^+_{t-b}, y_1^-, x^+_{t-b}, x^+_{t-b-1}, x^+_{t-b}, x^+_{t-b-1}, x^+_{t-b}, y_1^+, \ldots).)$$

Call the resulting walk π'', it will meet $\{y^+_{b+2k-1}, y^+_{b+2k}\}$ but *not* y^+_{b+2k+1}. If π'' is still not contained in $\Lambda_r(L_1)$ (i.e., if $k \ge 2$), repeat this procedure: the last $(x^+_{t-b}, y_1^+, y_2^+)$-block of π'' is replaced by $(x^+_{t-b}, y_1^-, x^+_{t-b})$ followed by a translation of two for the remaining steps. After k "foldings" of this kind we necessarily arrive at a walk in $\Lambda_r(L_1)$, and *this* walk will be called $\Phi_2(\pi)$.

The number k can be recovered from $\Phi_2(\pi)$, it is just the number of the $(x^+_{t-b}, y_1^-, x^+_{t-b})$-blocks in this walk. It follows easily that Φ_2 is one-to-one so that the cardinality of $\Lambda_r^*(R_\infty)$ is bounded by $a_r^t(L_1)$. It will remain to relate $\Lambda_r^*(R_\infty)$ with $\Lambda_r(R_\infty)$.

We will consider the following cases:

a) $t - b \ge 2$, $c \ge 2$.
b) $t - b > 2$, $c = 1$.
c) $t - b = 2$, $c = 1$.
d) $t - b = 1$.

a) Suppose that $t - b \ge 2$ and that $c \ge 2$. The first condition implies that $\Lambda_5^*(R_\infty) = \Lambda_5(R_\infty)$, and we note that $\Phi_2 : \Lambda_5^*(R_\infty) \to \Lambda_5(L_1)$ is *not* onto: the walk $(x_0 \to y_2^- \to y^+_{b+1})$ is not in the range. It follows that $a_5^t(R_\infty) < a_5^t(L_1)$.
b) The preceding argument breaks down if $c = 1$, then $(x_0 \to y_2^- \to y^+_{b+1})$ is not an element of $\Lambda_5(L_1)$. We will pass to walks of length $t+7$. Unfortunately, one has $\Lambda_7^*(R_\infty) = \Lambda_7(R_\infty)$ only if $t - b > 2$, we first assume that this is the case. Again it suffices to provide a $\pi \in \Lambda_7^*(L_1)$ which is not in $\Phi_2(\Lambda_7^*(R_\infty))$: clearly $(x_0 \to y_1^- \to x^+_{t-b-2} \to y^+_{b+1})$ has this property, and thus $a_7^t(R_\infty) < a_7^t(L_1)$.
c) The case $c = 1$, $t - b = 2$ is more interesting, it has been discussed already in section 2. With the notation introduced for the trees in problem II proposition 2.8 just asserts that $a_{2t+1}(R_\infty) = a_{2t+1}(L_1) - 1$; also, the a_n for L_1 and R_∞ coincide for $n \le 2t$.
d) The proof for the remaining case $t - b = 1$ has also been prepared in section 2. Suppose first that $c > (t-1)/2$. Then, by proposition 2.1, the first n where L_1

and R_∞ have different a_n is $n = 2t + 2$:

$$a_{2t+2}(L_1) = a_{2t+2}(R_\infty) + 1.$$

By the same proposition one has $a_{2t+4}(L_1) = a_{2t+4}(R_\infty) + 2$ if it happens that $c = (t-1)/2$, and $2t + 4$ cannot be replaced by a smaller number.

(Only *this* case provides the worst-case lower bound $2t+4$ of proposition 3.5; if t is even it cannot occur, then already $a_1, \ldots, a_{2t+2}$ suffice to distinguish between $\mathcal{H}_1^{t,b,c}$ and $\mathcal{H}_2^{t,b,c}$.)

It also follows that $a_r(L_1) > a_r(R_\nu)$ for every ν, where $r = 2t + 2$ resp. $r = 2t + 4$ if $c > (t-1)/2$ resp. $c = (t-1)/2$.

In view of proposition 2.1(ii) the preceding discussion has to be slightly modified in the cases $t = 5$ (with $c = 2$) and $t = 3$ (with $c = 1$). For $t = 5$ one only has to replace "$2t + 4$" by "$2t + 3$", for $t = 3$ the easiest way to prove the assertion is by direct calculation. In fact, all R_ν have an a_n different from $a_n(L_1)$ not later than $n = 10 = 2t + 4$; this worst case occurs with $R_4 = [1, 8, 2, 1]$.

It remains to discuss the case $c < (t-1)/2$, here proposition 2.2 comes into play. It asserts that $a_{2t+2}(L_1) > a_{2t+2}(R_{2c+1})$ and that the a_n for $n < 2t + 2$ coincide. Now let ν be arbitrary with $\nu \neq 2c + 1$, suppose first that $\nu < 2c+1$. Then R_ν is a proper subset of R_{2c+1}, and the a_{t+2c+2} for both trees are different (the latter admits at least one walk more than the other). Thus, since $t + 2c + 2 < 2t + 2$,

$$a_{t+2c+2}(R_\nu) < a_{t+2c+2}(R_{2c+1}) = a_{t+2c+2}(L_1),$$

i.e., the (a_n)-sequence distinguishes between R_ν and L_1.

If $\nu > 2c + 1$ we argue similarly. We consider walks of length $t + 2c + 3$ and note that $t + 2c + 3 \leq 2t + 1$. There is a walk in R_ν of length $t + 2c + 3$ (namely $(x_0 \to x^-_{t+2c+3})$) which is not admissible in R_{2c+1}, and we conclude that

$$a_{t+2c+3}(R_\nu) > a_{t+2c+3}(R_{2c+1}) = a_{t+2c+3}(L_1).$$

3.9 Decision problem III

Finally, we want to distinguish between trees of the following kind:

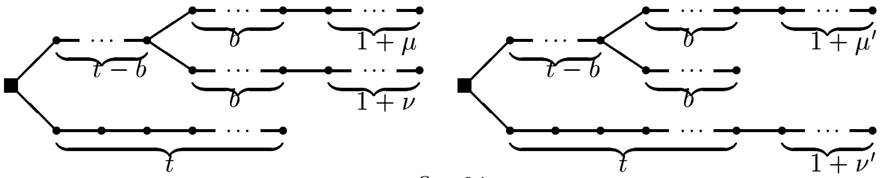


fig. 34

The situation is similar to that of the preceding subsection, but now we have to deal with *two* branches which leave $[t-b,t,b,b]$. We fix t and b with $1 \leq b < t$, our notation will be as follows:

– $L_{\mu,\nu} := [t-b, t, b+1+\mu, b+1+\nu]$, $\mu,\nu = 0,1,\ldots,\infty, \mu \geq \nu$,
– $R_{\mu',\nu'} := [t-b, t+1+\nu', b+1+\mu', b]$, $\mu',\nu' = 0,1,\ldots,\infty$,
– $\mathcal{I}_1^{t,b} := \{L_{\mu,\nu} \mid \mu,\nu = 0,1,\ldots,\infty, \mu \geq \nu\}$,
– $\mathcal{I}_2^{t,b} := \{R_{\mu',\nu'} \mid \mu',\nu' = 0,1,\ldots,\infty\}$.

Note that the $L_{\mu,\nu}$ are considered only under the restriction $\mu \geq \nu$, in this way the trees $L_{\mu,\nu}$ are pairwise non-isomorphic. For the $R_{\mu',\nu'}$ the situation is different, they are pairwise non-isomorphic without any additional assumption.

Proposition 3.6 *The pair $(\mathcal{I}_1^{t,b}, \mathcal{I}_2^{t,b})$ is decidable after at most $2t+6$ steps. For the case $t=2$ this is best possible, for $t \geq 3$ the bound can be replaced by $2t+4$ but by no smaller numeber.*

Let us briefly compare problem II with problem III. One aspect of the first one was more complicated, an additional parameter (the number c) had to be taken into account. On the other hand, now the classes depend on two parameters.

3.10 Problem III: the proof of proposition 3.6

The proof is similar to the preceding one for problem II, here, however, the walks which are going to leave $[t-b,t,b,b]$ have *two* possibilities.

The appropriate modifications of the Λ_r read as follows (the notation is as in section 3.8):

- $\Lambda_r^1(L_{\mu,\nu})$ (resp. $\Lambda_r^2(L_{\mu,\nu})$) denotes the collection of the walks of length $t+r$ in $L_{\mu,\nu}$ which start at x_0 and which leave $[t-b,t,b,b]$ for the first time with a step to y_{b+1}^+ (resp. to y_{b+1}^-).
- $\Lambda_r^1(R_{\mu',\nu'})$ and $\Lambda_r^3(R_{\mu',\nu'})$ are defined similarly: the first exit step is to y_{b+1}^+ and x_{t+1}^-, respectively. (The exponent refers to the branch of the exit: we write "1" and "3" since we count first exits in the first or third branch.)

Then $a_r^t(L_{\mu,\nu})$ (resp. $a_r^t(R_{\mu',\nu'})$) is the cardinality of $\Lambda_r^1(L_{\mu,\nu}) \cup \Lambda_r^2(L_{\mu,\nu})$ (resp. of $\Lambda_r^1(R_{\mu',\nu'}) \cup \Lambda_r^3(R_{\mu',\nu'})$), and – as in the case of problem II – we will compare the a_r^t by constructing suitable one-to-one maps between certain Λ's.

$\boxed{L_{0,0} \textit{ vs. } R_{\mu',\nu'}}$

First we introduce another reflection Φ_3 which is defined similarly as Φ_1. It applies to walks in $[t-b,\infty,\infty,\infty]$ which start at x_0 and do not visit y_1^+. For such a walk π we replace steps to $x_1^+,\ldots,x_{t-b}^+, y_1^-, y_2^-,\ldots$ by those to $x_1^-,\ldots,x_{t-b}^-, x_{t-b+1}^-, x_{t-b+2}^-,\ldots$ and vice versa.

Let $R_{\mu',\nu'}$ be given, we suppose first that $\mu' > 0$ or $\nu' > 0$. Then $\Lambda_2^1(L_{0,0})$ is a subset of $\Lambda_2^1(R_{\mu',\nu'})$, and $\Lambda_2^2(L_{0,0})$ embeds with Φ_3 into $\Lambda_2^3(R_{\mu',\nu'})$. Consequently $a_2^t(L_{0,0}) \leq a_2^t(R_{\mu',\nu'})$.

But at least one of the embeddings is not onto since $(x_0 \to y_{b+2}^+)$ or $(x_0 \to x_{t+2}^-)$ is not in the range, and thus the inequality is even strict.

To compare $L_{0,0}$ with $R_{0,0}$ we calculate the a_3^t. The set $\Lambda_3^1(L_{0,0})$ coincides with $\Lambda_3^1(R_{0,0})$ and $\Lambda_3^3(R_{0,0})$ embeds with Φ_3 into $\Lambda_3^2(L_{0,0})$. Since $(x_0 \to y_1^+ \to y_{b+1}^-)$ is in $\Lambda_3^2(L_{0,0})$ but not in $\Phi_3(\Lambda_3^3(R_{0,0}))$ it follows that $a_3^t(R_{0,0}) < a_3^t(L_{0,0})$.

$\boxed{L_{\mu,\nu} \textit{ with } \mu,\nu \geq 2 \textit{ vs. } R_{\mu',\nu'}}$

Let μ',ν' be arbitrary. Then, if $\mu \geq 2$, the set $\Lambda_3^1(R_{\mu',\nu'})$ is a subset of $\Lambda_3^1(L_{\mu,\nu})$, and $\Lambda_3^3(R_{\mu',\nu'})$ can be reflected by Φ_3 into $\Lambda_3^2(L_{\mu,\nu})$ if $\nu \geq 2$. The walk $(x_0 \to y_1^+ \to y_{b+1}^-)$ is not in the range of Φ_3, hence $a_3^t(R_{\mu',\nu'}) < a_3^t(L_{\mu,\nu})$. Note that it only remains to treat the $L_{\mu,0}$ and the $L_{\mu,1}$, where $\mu \geq 1$.

$\boxed{L_{\mu,0} \textit{ with } \mu \geq 1 \textit{ vs. } R_{\mu',\nu'}}$

We will consider the following cases separately:

a) $R_{\mu',\nu'}$ with $\mu',\nu' > 0$,
b) $R_{0,\nu'}$, $\mu \geq 2$,
c) $R_{0,\nu'}$ vs. $L_{1,0}$, $t-b > 1$,
d) $R_{0,\nu'}$ vs. $L_{1,0}$, $t-b = 1$,
e) $R_{\mu',0}$, $\mu \geq 2$,
f) $R_{\mu',0}$ vs. $L_{1,0}$, $\mu' \geq 3$,
g) $R_{\mu',0}$ vs. $L_{1,0}$, $\mu' \leq 1$,
h) $R_{2,0}$ vs. $L_{1,0}$, $t-b \geq 2$,
i) $R_{2,0}$ vs. $L_{1,0}$, $t-b = 1$.

a) Fix $\mu \geq 1$ and consider an $R_{\mu',\nu'}$ where both μ' and ν' are strictly positive. $\Lambda_2^1(L_{\mu,0}) = \Lambda_2^1(R_{\mu',\nu'})$, and $\Lambda_2^2(L_{\mu,0})$ embeds with Φ_3 into $\Lambda_2^3(R_{\mu',\nu'})$. The second map is not onto, the walk $(x_0 \to x_{t+2}^-)$ is not in the range so that $a_2^t(R_{\mu',\nu'}) > a_2^t(L_{\mu,0})$.
b) Assume that $\mu \geq 2$. Then Φ_1 embeds $\Lambda_3^3(R_{0,\nu'})$ into $\Lambda_3^1(L_{\mu,0})$, and the reflection Φ_4 – by definition, it transforms steps to y_i^+-vertices to those to y_i^- and vice versa – is a one-to-one map from $\Lambda_3^1(R_{0,\nu'})$ into $\Lambda_3^2(L_{\mu,0})$. The first reflection is not onto, it does not produce $(x_0 \to y_1^- \to y_{b+1}^+) \in \Lambda_3^1(L_{\mu,0})$, and consequently $a_3^t(R_{0,\nu'}) < a_3^t(L_{\mu,0})$.
c) For the case $\mu = 1$ the argument will depend on the size of $t-b$ (this is similar to the corresponding situation in section 3.8). First, let $t-b > 1$. The reflection Φ_4 provides an injection of $\Lambda_5^1(R_{0,\nu'})$ into $\Lambda_5^2(L_{1,0})$, and Φ_2 – this map has already been defined in section 3.8 – embeds $\Lambda_5^3(R_{0,\nu'})$ into $\Lambda_5^1(L_{1,0})$. Since $\pi = (x_0 \to y_2^- \to y_{b+1}^+)$ is not in the range of both maps – π lies in $\Lambda_5^1(L_{1,0'})$ for $b > 1$ and in $\Lambda_5^2(L_{1,0'})$ for $b = 1$ – it follows that $a_5^t(R_{0,\nu'}) < a_5^t(L_{1,0})$.
d) We will apply proposition 2.1. If we put $c := t$ there, then the trees G and G' are isomorphic with $L_{1,0}$ and $R_{0,\infty}$, respectively. Hence $a_{2t+2}(R_{0,\infty}) < a_{2t+2}(L_{1,0})$ and thus, a fortiori, $a_{2t+2}(R_{0,\nu'}) < a_{2t+2}(L_{1,0})$ for every ν'.

e) We begin the comparison of walks in $L_{\mu,0}$ and $R_{\mu',0}$ with the case $\mu \geq 2$. For arbitrary μ' the set $\Lambda_3^1(R_{\mu',0})$ is a subset of $\Lambda_3^1(L_{\mu,0})$, and Φ_3 is an embedding of $\Lambda_3^3(R_{\mu',0})$ into $\Lambda_3^2(L_{\mu,0})$ which fails to have $(x_0 \to y_1^+ \to y_{b+1}^-)$ in its range. Thus $a_3^t(R_{\mu',0}) < a_3^t(L_{\mu,0})$.
f) How performs $L_{1,0}$ against $R_{\mu',0}$ with $\mu' \geq 3$? For an answer we have to investigate our embeddings more carefully. We embed $\Lambda_4^3(R_{\mu',0})$ with Φ_3 into $\Lambda_4^2(L_{1,0})$, this is possible since no walk in $\Lambda_4^3(R_{\mu',0})$ visits y_1^+. We note that there is *only one* walk which is not in the range (namely $(x_0 \to y_1^+ \to y_{b+1}^- \to y_b^-)$). Also, $\Lambda_4^1(L_{1,0})$ is a subset of $\Lambda_4^1(R_{\mu',0})$, but the complement of $\Lambda_4^1(L_{1,0})$ contains *two* walks ($(x_0 \to y_{b+4}^+)$ and $(x_0 \to y_{b+3}^+ \to y_{b+2}^+)$). Hence $a_4^t(R_{\mu',0}) > a_4^t(L_{1,0})$.

> The complete truth is a little bit more complicated: in our argument we have tacitly assumed that $b > 1$: if $b = 1$ holds then the complement of the embedded $\Lambda_4^3(R_{\mu',0})$ contains one element more, namely $(x_0 \to y_2^- \to y_1^+)$. In fact, we have to work with walks of length $t + 5$, the argument is as follows.
>
> Let us assume that $b = 1$ and $t \geq 3$, we will show that $a_5^t(R_{\infty,0}) < a_5^t(L_{1,0})$ so that also $a_5^t(R_{\mu',0}) < a_5^t(L_{1,0})$.
>
> Since $t - b \geq 2$ no π in $\Lambda_5^3(R_{\infty,0})$ visits y_1^+, and therefore we can embed this set with Φ_3 into $\Lambda_5^2(L_{1,0})$. The complement of the range, i.e., the set of all $\pi \in \Lambda_5^2(L_{1,0})$ which meet y_1^+, will be denoted by $\Lambda_5^{2,*}(L_{1,0})$.
>
> Let $\pi_0, \ldots, \pi_6$ be the walks $(x_0 \to y_2^+ \to y_2^-)$, $(x_0 \to y_2^- \to y_2^+)$, $(x_0 \to y_2^- \to y_1^+ \to x_{t-1}^+)$, $(x_0 \to y_1^+ \to x_{t-1}^+ \to y_1^+ \to y_2^-)$, $(x_0 \to y_6^+)$, $(x_0 \to y_5^+ \to y_4^+)$ and $(x_0 \to y_1^- \to y_4^+)$, respectively. All $\pi \in \Lambda_5^1(L_{1,0})$ which are different from π_0 lie in $\Lambda_5^1(R_{\infty,0})$. We put
>
> $$\Lambda_5^{1,*}(R_{\infty,0}) := \Lambda_5^1(R_{\infty,0}) \setminus (\Lambda_5^1(L_{1,0}) \cup \{\pi_4, \pi_5, \pi_6\})$$
>
> and embed $\Lambda_5^{1,*}(R_{\infty,0})$ into $\Lambda_5^{2,*}(L_{1,0})$. This is achieved by a map Φ_5 which will be defined in a slightly more general setting below (cf. part h) of this proof).
>
> The embedding does not meet $\{\pi_1, \pi_2, \pi_3\}$ and therefore, when counting the total number of walks, we have shown that $a_5^t(R_{\infty,0}) \leq a_5^t(L_{1,0}) - 1$; this implies $a_5^t(R_{\nu',0}) < a_5^t(L_{1,0})$ for all ν'.
>
> In the remaining case $b = 1$, $t = 2$ the comparison of $L_{1,0}$ with $R_{\nu',0}$ leads to a comparison of $[1, 2, 3, 2]$ with $[1, 3, 2 + \nu', 1]$. A direct calculation reveals that the worst case appears with $\nu' = 5$ where $n = 10$ is the first n with different a_n (1558 vs. 1557). In all other cases a smaller n can be used. *This* case provides the bound $2t + 6$ of proposition 3.6.

g) This is simple: $\Lambda_3^1(R_{\mu',0})$ is a subset of $\Lambda_3^1(L_{1,0})$ (which might be proper), and Φ_3 maps $\Lambda_3^3(R_{\mu',0})$ into $\Lambda_3^2(L_{1,0})$ such that $(x_0 \to y_1^+ \to y_{b+1}^-)$ is not in the range. This implies $a_3^t(R_{\mu',0}) < a_3^t(L_{1,0})$.
h) We work with walks of length $t + 5$, and we embed $\Lambda_5^3(R_{2,0})$ with Φ_3 into $\Lambda_5^2(L_{1,0})$ (this is possible since, due to $t - b \geq 2$, the walks under consideration omit y_1^+). The complement of the collection of embedded walks will be called $\Lambda_5^{2,*}(L_{1,0})$, it consists of all π in $\Lambda_5^2(L_{1,0})$ which visit y_1^+. The set $\Lambda_5^1(L_{1,0})$ is a

subset of $\Lambda_5^1(R_{2,0})$, the complementary set will be denoted by $\Lambda_5^{1,*}(R_{2,0})$. We put $\pi_0 := (x_0 \to y_1^- \to y_{b+3}^+)$ and note that the set $\Lambda_5^{1,*}(R_{2,0}) \setminus \{\pi_0\}$ can be embedded into $\Lambda_5^{2,*}(L_{1,0})$ by a folding/translation-operation which will be called Φ_5 and which is defined similarly as Φ_2:

> Consider any $\pi \in \Lambda_5^{1,*}(R_{2,0})$ which is different from π_0, such a π does not visit y_1^-. Φ_4, the reflection from the $(y_1^+, \ldots)$- to the $(y_1^-, \ldots)$-branch, maps π to a walk π' in $[t-b, \infty, \infty, \infty]$ which will not yet be in $\Lambda_5^{2,*}(L_{1,0})$. Replace in π' the last block of the form $(x_{t-b}^+, y_1^-, y_2^-)$ by $(x_{t-b}^+, y_1^+, x_{t-b}^+)$ and shift the remaining vertices by two indices. This new walk will be called $\Phi_5(\pi)$.

In this way we have embedded the walks in $\Lambda_5^1(R_{2,0}) \cup \Lambda_5^3(R_{2,0})$ which are different from π_0 into $\Lambda_5^1(L_{1,0}) \cup \Lambda_5^2(L_{1,0})$. But there are *two* walks which are not in the range of this embedding: $(x_0 \to y_2^+ \to y_{b+1}^-)$ and $(x_0 \to y_1^+, x_{t-b}^+, y_1^+ \to y_{b+1}^-)$, and this shows that $a_5^t(R_{2,0}) < a_5^t(L_{1,0})$.

> Again the argument has to be slightly different in the case $b = 1$. It suffices to consider walks of length $t + 4$, no new ideas are necessary to prove that $a_4^t(R_{2,0}) < a_4^t(L_{1,0})$.

i) We already know that $L_{1,0}$ and $R_{0,\infty}$ have different a_{2t+2} (see part d) of this proof). But $a_{2t+2}(R_{2,0}) = a_{2t+2}(R_{0,\infty})$, this is contained in lemma 2.14 as the special case $k = 0$. Hence $a_{2t+2}(R_{2,0}) < a_{2t+2}(L_{1,0})$.

$\boxed{L_{\mu,1} \text{ with } \mu \geq 1 \text{ vs. } R_{\mu',\nu'}}$

The investigation will be organized as follows:

a) $R_{\mu',\nu'}$ with $\mu' = 0$ or $\nu' = 0$,
b) $\mu \geq 4$, $t - b > 1$
c) $\mu = 3$, $t - b > 2$,
d) $\mu = 3$, $t - b = 2$,
e) $\mu = 2$, $t - b > 1$,
f) $\mu = 1$, $t - b > 1$,
g) $t - b = 1$,

a) If μ' or ν' vanishes, then $a_2^t(R_{\mu',\nu'}) < a_2^t(L_{\mu,1})$ (since $\Lambda_2^1(R_{\mu',\nu'}) \subset \Lambda_2^1(L_{\mu,1})$, Φ_3 embeds $\Lambda_2^3(R_{\mu',\nu'})$ into $\Lambda_2^1(L_{\mu,1})$, and at least one of these embeddings has a range which is a proper subset). Thus we may restrict ourselves in the sequel to the investigation of the cases $\mu', \nu' \geq 1$.

b) We prove that $a_5^t(R_{\infty,\infty}) < a_5^t(L_{\mu,1})$ in this case so that also – a fortiori – $a_5^t(R_{\mu',\nu'}) < a_5^t(L_{\mu,1})$ for all $R_{\mu',\nu'}$.

$\Lambda_5^1(R_{\infty,\infty})$ is a subset of $\Lambda_5^1(L_{\mu,1})$, and $\Lambda_5^3(R_{\infty,\infty})$ can be embedded into $\Lambda_5^2(L_{\mu,1})$: use a folding/translation-procedure which is defined similarly as Φ_2 with the only difference that the $(y_1^-, \ldots)$- and the $(y_1^+, \ldots)$-branch have to be exchanged (this variant of Φ_2 will be called Φ_6). The range of Φ_6 does not contain $(x_0 \to y_2^+ \to y_{b+1}^-)$, and thus $a_5^t(R_{\infty,\infty}) < a_5^t(L_{\mu,1})$.

In the case $b = 1$ the argument has to be slightly different: then $\Lambda_5^1(R_{\infty,\infty})$ is a *proper* subset of $\Lambda_5^1(L_{\mu,1})$, it does not contain $(x_0 \to y_2^+ \to y_{b+1}^-)$; the mapping Φ_6 is an embedding as before.

c) Suppose first that $b > 2$, we claim that $a_7^t(R_{\infty,\infty}) < a_7^t(L_{3,1})$ in the case under consideration. As in part "b" of this proof we embed $\Lambda_7^3(R_{\infty,\infty})$ into $\Lambda_7^2(L_{3,1})$ with the Φ_6-map; here it is important to know that $t-b > 2$ so that no walks occur which visit y_1^-. Note that the range is disjoint with $\Lambda_7^{2,**}(L_{3,1})$ which we define to be the set of walks in $\Lambda_7^2(L_{3,1})$ which visit y_2^+. The range also does not contain $\pi^* = (x_0 \to y_1^+ \to x_{t-b-2}^+ \to y_{b+1}^-)$.
$\Lambda_7^1(L_{3,1})$ is a subset of $\Lambda_7^1(R_{\infty,\infty})$, the complement will be denoted by $\Lambda_7^{1,**}(R_{\infty,\infty})$. Now another map, Φ_7, comes into play, it embeds the $\pi \in \Lambda_7^{1,**}(R_{\infty,\infty})$ which are different from $\pi_0 := (x_0 \to y_{b+7})$ into $\Lambda_7^{2,**}(L_{3,1})$:

Let π be in $\Lambda_7^{1,**}(R_{\infty,\infty})$ such that $\pi \neq \pi_0$. Define $\Phi_7(\pi)$ as follows: at the first π-visit at y_2^+ the walk is reflected at y_2^+ and directed into the $(y_1^-, \ldots)$-branch, i.e., steps to y_2^+, y_3^+, y_4^+, … are replaced by steps to y_1^+, x_{t-b}^+, y_1^-, … respectively.

$\Phi_7(\pi)$ is well-defined since after the y_2^+-visit a step to x_{t-b-1}^+ is not possible, and it really lies in $\Lambda_7^{2,**}(L_{3,1})$ since $\pi \neq \pi_0$. It is also clear that Φ_7 is one-to-one.

Thus, with the exception of π_0, we have embedded all walks which contribute to $a_7^t(R_{\infty,\infty})$ into $\Lambda_7^1(L_{3,1}) \cup \Lambda_7^2(L_{3,1})$. But there are *two* walks which do not occur as embedded walks, namely π^* and $(x_0 \to y_2^+ \to x_{t-b-1}^+ \to y_{b+1}^-)$; consequently, $a_7^t(R_{\infty,\infty}) < a_7^t(L_{3,1})$.

It remains to treat the cases $b = 1$ and $b = 2$. If $b = 1$, we claim that already $a_4^t(R_{\infty,\infty}) < a_4^t(L_{3,1})$: in fact, $\Lambda_4^1(L_{3,1}) = \Lambda_4^1(R_{\infty,\infty})$, and with Φ_6 the set $\Lambda_4^3(R_{\infty,\infty})$ embeds into $\Lambda_4^2(L_{3,1})$ without meeting $(x_0 \to y_2^- \to y_1^-)$.

For $b = 2$ we have to consider walks of length $t + 5$. $\Lambda_5^1(L_{3,1})$ is a subset of $\Lambda_5^1(R_{\infty,\infty})$, there is *one* element in the complement (namely $(x_0 \to y_6^+)$). On the other hand, the Φ_6-embedding of $\Lambda_5^3(R_{\infty,\infty})$ into $\Lambda_5^2(L_{3,1})$ fails to contain the *two* walks $(x_0 \to y_2^+ \to y_3^-)$ and $(x_0 \to y_3^- \to y_1^+)$.

d) We have proved in proposition 2.9 above that $a_n(L_{3,1}) = a_n(R_{\infty,\infty})$ for $n \leq 2t$ and that $a_{2t+1}(L_{3,1}) > a_{2t+1}(R_{\infty,\infty})$. Thus $a_{2t+1}(L_{3,1}) > a_{2t+1}(R_{\mu',\nu'})$ for arbitrary μ', ν'.

e) With the standard techniques it is easy to see that $a_3^t(L_{2,1}) > a_3^t(R_{\mu',\nu'})$ if $\mu' \leq 1$ or $\nu' \leq 1$ and that $a_4^t(L_{2,1}) < a_4^t(R_{\mu',\nu'})$ if it happens that $\mu', \nu' \geq 3$. (For $b = 1$ one has to argue differently: then $a_5^t(L_{2,1}) > a_5^t(R_{\infty,\infty})$, cf. "f)" of the preceding part of the proof.)

Thus we only have to check how $L_{2,1}$ competes against the $R_{2,\nu'}$ and the $R_{\mu',2}$. We will prove that $a_5^t(R_{2,\infty})$ and $a_5^t(R_{\infty,2})$ are strictly smaller than $a_5^t(L_{2,1})$.

We consider $R_{2,\infty}$ first. Φ_6 embeds $\Lambda_5^3(R_{2,\infty})$ into a proper subset of $\Lambda_5^2(L_{2,1})$ ($(x_0 \to y_2^+ \to y_{b+1}^-)$ is in $\Lambda_5^2(L_{2,1})$ but not in the range), and $\Lambda_5^1(R_{2,\infty})$ coincides with $\Lambda_5^1(L_{2,1})$. Thus $a_5^t(L_{2,1}) > a_5^t(R_{2,\infty})$.

Next we investigate $R_{\infty,2}$. The folding/translation-map Φ_6 embeds $\Lambda_5^3(R_{\infty,2})$ into $\Lambda_5^2(L_{2,1})$, the *range does not contain the three walks* $(x_0 \to y_2^+ \to y_{b+1}^-)$, $(x_0 \to y_1^+, x_{t-b}^+, y_1^+ \to y_{b+1}^-)$, and $(x_0 \to y_1^+ \to y_{b+2}^-, y_{b+1}^-)$.

On the other hand, $\Lambda_5^1(L_{2,1})$ is a subset of $\Lambda_5^1(R_{\infty,2})$, but the latter set contains *only two* walks more than the former: $(x_0 \to y_{b+5}^+)$ and $(x_0 \to y_{b+4}^+, y_{b+3}^+)$. This proves that also $a_5^t(L_{2,1}) > a_5^t(R_{\infty,2})$.

We note that in the case $b = 1$ the argument has to be slightly modified; cf. part b) above.

f) How competes $L_{1,1}$ against the $R_{\mu',\nu'}$ in the case $t - b > 1$? We recall what has been shown in proposition 2.3 and proposition 2.4 above, where $k := t - b$:

> A. For $1 \le k < t$ the a_n of the trees $L_{1,1}$ and $R_{2k+1,1}$ coincide for $n = 1, \ldots, 2t - k + 2$, and $a_{2t-k+3}(L_{1,1}) > a_{2t-k+3}(R_{2k+1,1})$.
>
> B. In the special case $k = t/3$ one has $a_n(L_{1,1}) = a_n(R_{\infty,1})$ for $n \le 5k + 4$ and $a_{5k+5}(L_{1,1}) > a_{5k+5}(R_{\infty,1})$.

Case 1: $k < t/3$

α) Suppose first that $\mu' < 1$ or $\nu' < 1$, we consider walks of length $t + 2$. This number is smaller than $2t - k + 2$ so that $a_{t+2}(L_{1,1}) = a_{t+2}(R_{2k+1,1})$. On the other hand, the $(t+2)$-truncation of $R_{\mu',\nu'}$ is a proper subset of $R_{2k+1,1}$ so that $a_{t+2}(R_{\mu',\nu'}) < a_{t+2}(R_{2k+1,1})$, and this shows that $a_{t+2}(L_{1,1}) > a_{t+2}(R_{\mu',\nu'})$.

β) Consider next the case where both μ' and ν' exceed one.
Then the $(t + 3)$-truncation of $R_{2k+1,1}$ is properly contained in $R_{\mu',\nu'}$, hence $a_{t+3}(R_{\mu',\nu'}) > a_{t+3}(R_{2k+1,1})$. Since the a_{t+3}-values coincide for $L_{1,1}$ and $R_{2k+1,1}$ it follows that $a_{t+3}(R_{\mu',\nu'}) > a_{t+3}(L_{1,1})$.

γ) To study the case of the $R_{1,\nu'}$ we first note that $a_5(R_{1,\infty}) < a_5(L_{1,1})$. (The sets $\Lambda_5^1(L_{1,1})$ and $\Lambda_5^1(R_{1,\infty})$ coincide and $\Lambda_5^3(R_{\infty,1})$ embeds with Φ_6 into $\Lambda_5^2(L_{1,1})$ in such a way that $(x_0 \to y_2^+ \to y_{b+1}^-)$ is not in the range; it is *here* that it is important that $t - b > 1$ to be sure that the walks in $\Lambda_5^3(R_{1,\infty})$ avoid y_1^-.) Therefore also $a_5^t(L_{1,1}) > a_5(R_{1,\nu'})$ holds for every ν'.

δ) It remains to discuss the $R_{\mu',1}$. If $\mu' \le 2k + 1$ we have $R_{\mu',1} \subset R_{2k+1,1}$ so that

$$a_{2t-k+3}(R_{\mu',1}) \le a_{2t-k+3}(R_{2k+1,1}) < a_{2t-k+3}(L_{1,1}).$$

In the case $\mu' > 2k + 1$ we consider walks of length $t + 2k + 3$. Since the $(t + 2k + 3)$-truncation of $R_{2k+1,1}$ is a proper subset of $R_{\mu',1}$ it follows that $a_{t+2k+3}(R_{2k+1,1}) < a_{t+2k+3}(R_{\mu',1})$. But $t + 2k + 3 \le 2t - k + 2$ since $3k < t$ so that $a_{t+2k+3}(R_{2k+1,1}) = a_{t+2k+3}(L_{1,1})$, and we conclude that $a_{t+2k+3}(R_{\mu',1}) > a_{t+2k+3}(L_{1,1})$.

Case 2: $k = t/3$

Trees $R_{\mu',\nu'}$ which are *not* of the form $R_{\mu',1}$ are treated as in the preceding case (cf. the discussion of α, β, γ). But $R_{\mu',1}$ is contained in $R_{\infty,1}$ so that, by "B",

$$a_{5k+5}(R_{\mu',1}) \le a_{5k+5}(R_{\infty,1}) < a_{5k+5}(L_{1,1}).$$

Case 3: $k > t/3$

Again the only interesting trees are the $R_{\mu',1}$ (as several times before the case $b = 1$ needs special attention, new ideas, however, are not necessary to treat it). Their $(2t - k + 3)$-truncation is contained in $R_{2k+1,1}$, this is true since $t < 3k$ implies $2t - k + 3 \leq t + 2k + 2$. But then

$$a_{2t-k+3}(R_{\mu',1}) \leq a_{2t-k+3}(R_{2k+1,1}) < a_{2t-2k+3}(L_{1,1}).$$

g) We fix $\mu \geq 1$ and we have to discuss the $L_{\mu,1}$ under the assumption $t - b = 1$. First we recall three results from section 2 (cf. proposition 2.5, 2.6, and 2.7):

A. The a_n coincide for $L_{\mu,1}$, $R_{\mu,\infty}$ and $R_{\mu+2,\mu}$ for $n \leq 2t + 1$, and $a_{2t+2}(L_{\mu,1})$ exceeds both $a_{2t+2}(R_{\mu,\infty})$ and $a_{2t+2}(R_{\mu+2,\mu})$.

B. If $\mu = t$, then $a_n(L_{\mu,1}) = a_n(R_{\infty,\infty})$ for $n \leq 2t+3$, but $a_{2t+4}(L_{\mu,1}) > a_{2t+4}(R_{\infty,\infty})$.

C. In the case $\mu = t - 2$ the a_n coincide for $L_{\mu,1}$ and $R_{\infty,\mu}$ for $n = 1, \ldots, 2t + 3$, and $a_{2t+4}(L_{\mu,1}) > a_{2t+4}(R_{\infty,\mu})$; note that only μ with $\mu \geq 1$ have to be considered so proposition 2.7 is applied here with $t \geq 3$.

(It should be emphasized that pairs of trees as in "B" and " C" give rise to the bound $2t + 4$ in proposition 3.6.)

We have to compare the a_n of $L_{\mu,1}$ and $R_{\mu',\nu'}$, we will have to consider several cases.

Case 1: $\mu < t - 2$

α) If $\mu' < \mu$ or $\nu' < \mu$ we investigate walks of length $t + \mu + 1$. The $(t + \mu + 1)$-truncation of $R_{\mu',\nu'}$ is properly contained in $R_{\mu+2,\mu}$ so that $a_{t+\mu+1}(R_{\mu',\nu'}) < a_{t+\mu+1}(R_{\mu+2,\mu})$. But $a_{t+\mu+1}(L_{\mu,1}) = a_{t+\mu+1}(R_{\mu+2,\mu})$ since $t + \mu + 1 < 2t + 1$ so that $a_{t+\mu+1}(L_{\mu,1}) > a_{t+\mu+1}(R_{\mu',\nu'})$.

β) Suppose that both μ' and ν' exceed μ, this time we discuss walks of length $t+\mu+2$. This number is bounded by $2t+1$, hence $a_{t+\mu+2}(L_{\mu,1}) = a_{t+\mu+2}(R_{\mu+2,\mu})$. On the other hand, the $(t + \mu + 2)$-truncation of $R_{\mu+2,\mu}$ is properly contained in $R_{\mu',\nu'}$ which provides $a_{t+\mu+2}(R_{\mu',\nu'}) > a_{t+\mu+2}(R_{\mu+2,\mu})$. We conclude that $a_{t+\mu+2}(R_{\mu',\nu'}) > a_{t+\mu+2}(L_{\mu,1})$.

γ) The case of the $R_{\mu,\nu'}$ is easy. They are contained in $R_{\mu,\infty}$, and consequently

$$a_{2t+2}(R_{\mu,\nu'}) \leq a_{2t+2}(R_{\mu,\infty}) < a_{2t+2}(L_{\mu,1}).$$

δ) It remains to treat the $R_{\mu',\mu}$. If $\mu' \leq \mu + 2$ we have $R_{\mu',\mu} \subset R_{\mu+2,\mu}$, and we conclude from "A" that

$$a_{2t+2}(R_{\mu',\mu}) \leq a_{2t+2}(R_{\mu+2,\mu}) < a_{2t+2}(L_{\mu,1}).$$

If $\mu' > \mu + 2$ we consider walks of length $t + \mu + 4$. The $(t + \mu + 4)$-truncation of $R_{\mu+2,\mu}$ is a proper subset of $R_{\mu',\mu}$, hence

$$a_{t+\mu+4}(R_{\mu',\mu}) > a_{t+\mu+4}(R_{\mu+2,\mu}) = a_{t+\mu+4}(L_{\mu,1});$$

here it is important that $t+\mu+4 \leq 2t+1$.

Case 2: $\mu = t-2$
The $R_{\mu',\nu'}$ which are *not* of the form $R_{\mu',\mu}$ can be treated as in α, β, γ of the preceding case 1. But $R_{\mu',\mu} \subset R_{\infty,\mu}$, and we conclude from "C" that

$$a_{2t+4}(R_{\mu',\mu}) \leq a_{2t+4}(R_{\infty,\mu}) < a_{2t+4}(L_{\mu,1}).$$

Case 3: $\mu = t-1$
Trees $R_{\mu',\nu'}$ which are not of type $R_{\mu',\mu}$ with $\mu' > \mu+2$ are dealt with as in case 1. If $\mu' > \mu+2$ we consider walks of length $t+\mu+3$. The $(t+\mu+3)$-truncation of $R_{\mu',\mu}$ coincides with $R_{\mu+2,\mu}$, and $t+\mu+3 = 2t+2$. Hence, by "A",

$$a_{2t+2}(R_{\mu',\mu}) = a_{2t+2}(R_{\mu+2,\mu}) < a_{2t+2}(L_{\mu,1}).$$

Case 4: $\mu = t$
This is easy since by "B" $a_{2t+4}(L_{\mu,1})$ exceeds $a_{2t+4}(R_{\infty,\infty})$, and $R_{\infty,\infty}$ contains all $R_{\mu',\nu'}$.

Case 5: $\mu > t$
Here we have $t+1+\mu \geq 2t+2$, and it follows that the $(t+\mu+1)$-truncation of every $R_{\mu',\nu'}$ is contained in $R_{\mu+2,\mu}$. Hence

$$a_{2t+2}(R_{\mu',\nu'}) \leq a_{2t+2}(R_{\mu+2,\mu}) < a_{2t+2}(L_{\mu,1}).$$

4 Proof of the main result

We want to identify simple rooted 3-trees G from the $(a_n(G))$-sequence and this will be done by proving that the $a_n(G)$ determine all truncations G^t. We proceed by induction on t, the induction step can be reformulated as follows:

> Let G be a simple rooted 3-tree and $t \geq 0$. Suppose that G^t and the sequence $(a_n(G))$ are known. It is then possible to determine G^{t+1}.

For the proof of this assertion we will have to consider several cases. It will be convenient to introduce the following notation:

– If G is any rooted tree and $t \geq 0$, denote by $A_t(G)$ the number of vertices for which the distance to the root is precisely t.

– For a simple rooted 3-tree G and $t \geq 0$ we will write $\tau_t(G) = 0$ resp. $\tau_t(G) = 1$ if G^t is of the form $[l_1, l_2, 0, 0]$ or not; therefore the τ-value measures whether or not a bifurcation has occurred in G up to the t-level. To illustrate these notions we give some *examples:*

$$A_5([2,17,\infty,1]) = 2,\ A_{101}([3,4,5,6]) = 0,$$

$$\tau_1([2,17,\infty,1]) = 0,\ \tau_6([3,4,5,6]) = 1.$$

Now we start with the proof of theorem 1.5. It amounts to the discussion of a tree of cases[3]. Some of them are simple to investigate, the others have been prepared in the last section.

Let G be a simple rooted 3-tree for which the (a_n)-sequence is given. a_1 is the number $A_1(G)$, i.e., we know how many branches leave the root. The cases $A_1(G) \in \{0, 1, 3\}$ are easy to handle. If this happens it is very simple to identify G^{t+1} from G^t and the (a_n), even a_{t+1} suffices to derive G^{t+1} from G^t successfully. Therefore we may restrict the following discussion to the case $A_1 = 2$.

We assume that $t > 0$ is a fixed number such that G^t is known. How does G^{t+1} look like?

Let us suppose for the moment that $(a_n(G))$ is *not* the sequence (2^n), we then argue as follows. Since we are dealing with simple rooted 3-trees, only the numbers $0, 1, 2, 3$ can occur as $A_t(G)$. If $A_t(G) = 0$ then necessarily $G^{t+1} = G^t$, and G is already now completely identified. Suppose that $A_t(G) = 1$. It is important to observe that the number A_{t+1} can be derived form the a_n if G^t is known, it coincides with $a_1^t(G) = a_{t+1}(G) - a_{t+1}(G^t)$, the number of walks of length $t + 1$ which leave G^t.

It follows that we can use the value of $A_{t+1}(G)$ in our proof. In particular this implies that in the present situation – when $A_t(G) = 1$ – we can use $A_{t+1}(G)$ to determine G^{t+1}. (Suppose, e.g., that $G^{14} = [3, 14, 2, 6]$ and $A_{15}(G) = 1$; then there is only one possibility for G^{15}, namely $G^{15} = [3, 15, 2, 6]$. Similarly, $A_{15}(G) = 0$ would imply that $G^{15} = [3, 14, , 2, 6]$, and in the same way situations of the form $[l_1, l_2, 0, 0]$ with $l_1 < l_2$ and $[l_1, l_2, l_3, l_4]$ with $l_1 + l_3 > l_2$ can be treated.)

Let us turn to the case $A_t = 2$, now at level t there are two "active" branches. If $\tau_t(G) = 1$ (i.e., if G^t has a bifurcation), it depends on $A_{t+1}(G)$ how to proceed. If $A_{t+1}(G) = 0$ or $= 2$ it is easy to deduce G^{t+1}: in the first case $G^t = G^{t+1}$, and in the latter both "active" branches of G^t have to be extended by one vertex. (E.g., $G^4 = [1, 4, 3, 1]$ and $A_5(G) = 2$ imply that $G^5 = [1, 5, 4, 1]$.) More difficult is the case $A_{t+1} = 1$: it is not clear which of the two branches has to be extended. More precisely, if $\tau_t(G) = 1$, $A_t = 2$ and $A_{t+1} = 1$ there are the following possibilities:

– $G^t = [t{-}b, s, b, b]$ with $s < t$ and $1 \le b < t$. Then necessarily $G^{t+1} = [t{-}b, s, b + 1, b]$.

– Or $G^t = [t{-}b, t, b, c]$ with $1 \le c < b < t$. Then it is not clear whether $G^{t+1} = [t{-}b, t, b + 1, c]$ or $G^{t+1} = [t{-}b, t + 1, b, c]$.

Do you recognize decision problem II? Proposition 3.5 asserts that we can decide from the a_n which of the alternatives hold. Therefore the induction step is proved in this special case.

Consider now the case $\tau_t(G) = 0$ (= no bifurcation). G^t has the form $[t, t, 0, 0]$, and if $A_{t+1}(G) = 0, 1$ or 3 we can immediately identify G^{t+1} as $[t, t, 0, 0]$, $[t + 1, t, 0, 0]$ or $[t, t + 1, 1, 1]$, respectively. If $A_{t+1} = 2$, however, there are two possibilities for G^{t+1}: $[t, t, 1, 1]$ and $[t + 1, t + 1, 0, 0]$. Here *deci-*

[3] Below we have visualized the structure of the proof graphically.

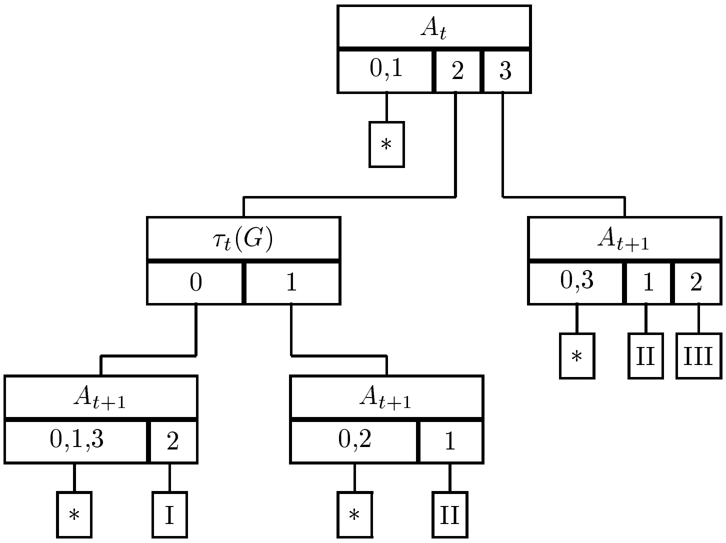


fig. 35

sion problem I comes into play: proposition 3.1 guarantees that we can choose between the alternatives on the basis of the $(a_n(G))$.

It remains to treat the case $A_t(G) = 3$ (which implies that $\tau_t(G) = 1$). We know that $G^t = [t-b, t, b, b]$ for a suitable $1 \leq b < t$, what can be said about G^{t+1}? If $A_{t+1}(G) = 0$ or $= 3$, the answer is simple: G^{t+1} will be $[t-b, t, b, b]$ or $[t-b, t+1, b+1, b+1]$ respectively. However, the situation is more complicated if $A_{t+1}(G) = 1$ or $= 2$. In the first case we must choose between $G^{t+1} = [t-b, t, b+1, b]$ and $G^{t+1} = [t-b, t+1, b, b]$, and in the second the alternatives are $G^{t+1} = [t-b, t, b+1, b+1]$ and $G^{t+1} = [t-b, t+1, b+1, b]$. But in both cases we can decide unambigously on the basis of the $a_n(G)$, this is just a reformulation of the decision problem II in the special case $c = b$ and decision problem III.

In this way we have *always* identified G^{t+1}, the structure of our proof can be visualized as in figure 35 (for the "simple" steps of the argument we have written a "$*$", the references to the decision problems are also indicated).

We recall that we have assumed that $(a_n(G))$ is not the sequence (2^n): this assumption was necessary in order to apply proposition 3.1 in the case $t = 1$. What happens if $a_n(G) = 2^n$ for all n? Then $A_1(G) = 2$, and we arrive immediately at decision problem I with $t = 1$. From proposition 3.1(i) and (iii) we conclude that $G = [1, 1, 2, 2]$ or $G = [\infty, \infty, 0, 0]$. Thus part (i) of the theorem is established. Since only one of these trees is finite, part (iii) is also proved.

We conclude the proof of the theorem with the remark that the identification of the G^t is in fact sufficient. Clearly one finds a tree G with the correct G^t – it suffices to build G recursively for the increasing levels t –, but is G unique up to isomorphism? The answer is in the positive, it can be found in part (i) of the following lemma (the assertion of part (ii) is not important here, it is included for the sake of completeness).

Lemma 4.1 *Let G and H be connected rooted trees such that G^t and H^t are isomorphic*[4] *for $t = 1, 2, \ldots$.*

(i) *If all G^t, H^t are finite, then it follows that G and H are isomorphic.*

(ii) *Without the finiteness assumpotion this is not true.*

Proof. (i) Denote, for $t = 1, 2, \ldots$, by M_t the finite set of all mappings from G^t to H^t and by M the product of the M_t provided with the product topology.

M is compact, we consider for every t_0 the subset Δ_{t_0} which is defined as the collection of all (I_t) such that

- I_t is an isomorphism for $t \leq t_0$, and
- the restriction of I_{t+1} to G^t is I_t for all t.

The Δ_{t_0}, $t_0 = 1, 2, \ldots$ are a decreasing family of closed subsets of M, and by assumption all of them are nonempty. Thus there is an element in the intersection by the Tychonoff theorem, and it is then rather easy to derive an isomorphism from such an element.

(ii) Let B_0 denote the tree which consists of the root and an unbounded path starting there. For $n \in \mathbb{N}$, B_n is derived from B_0 by attaching a branch of length 1 to the n'th vertex of B_0.

We define G to be the tree where the roots of $B_1, B_2, \ldots$ are identified: the root thus has countable degree, and at the n'th branch there is at distance n a bifurcation to a short branch of length 1. H is defined similarly, but this time we use $B_0, B_1, \ldots$.

It is then obvious that G and H are nonisomorphic trees for which nevertheless all G^t, H^t are isomorphic. □

Remarks: 1) It has to be stressed that the lengthy calculations in the sections 3.3, 3.4 and 3.5 which have been crucial in the proof of the main theorem cannot be achieved "by hand" with the necessary reliability within reasonable time. Thus we are – as in the case of other much more famous examples – in the somewhat unsatisfactory situation that the truth of a proof depends on the reliability of computers.

2) Let us briefly recall the strategy which we have used to prove our theorem. We wanted to solve an identification problem for a special class $\mathcal{G}$ of trees, the approach was as follows:

[4] "Isomorphic" means "isomorphic as rooted trees", i.e., isomorphisms map the root to the root.

For $G \in \mathcal{G}$ and $t \geq 1$ denote by G^t the t-truncated tree. Fix a family $G_1^{t+1}, \ldots, G_r^{t+1}$ in $\mathcal{G}$ such that for every $G \in \mathcal{G}$ there is a ρ such that G^{t+1} is isomorphic with G_ρ^{t+1}; put

$$\mathcal{G}_\rho^t := \{G \in \mathcal{G} \mid G^{t+1} \text{ is isomorphic with } G_\rho^{t+1}\}.$$

Prove that the $(\mathcal{G}_1^t, \ldots, \mathcal{G}_r^t)$-decision problem is decidable for every t.

By the special structure of the simple rooted 3-trees the number r is rather small here.

It is easy to find further classes of trees where this idea can be applied similarly successfully. However, it seems to be difficult to find examples which are at the same time natural and nontrivial.

3) The question "Is the identification problem for the class of finite rooted trees solvable?" remains widely open. We have treated a certain case of "small" trees by brute force computation:

Let $v, v_0, v_1, \ldots, v_{v_0}$ be positive integers such that $v_0 \leq v$ and $v \geq v_1 \geq v_2 \geq \cdots v_{v_0} \geq 0$. By $[v_0; v_1, \ldots, v_{v_0}]$ we denote the tree for which the root has degree v_0, and the v_0 edges coming from the root bifurcate into $v_1, \ldots, v_{v_0}$ further edges. Here are $[2; 1, 1]$ and $[3; 2, 1, 0]$:

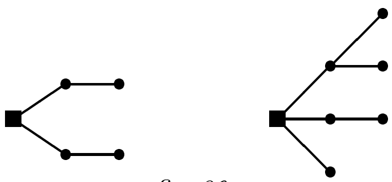


fig. 36

By $\mathcal{T}_v$ we denote the class of these trees.

With the help of a computer we have checked that the identification problem is solvable for $\mathcal{T}_7$. Since $\mathcal{T}_7$ has 6435 elements this asked for the discussion of about $20 \cdot 10^6$ pairs. In each of these cases the a_n are different not later than at $n = 8$. (Several times 8 steps really are necessary, e.g., to distinguish between $[6; 7, 5, 4, 3, 1, 0]$ and $[6; 6, 6, 5, 1, 1, 1]$.)

The number of elements in $\mathcal{T}_v$ grows extremely fast with v – $\mathcal{T}_8$ already has 24 310 members – and thus only rather small v can be treated in this way.

4) Let G be a tree and $t \geq 0$. Suppose that $G^t = G$ and that $A_t(G) > 0$: there are branches which lead to vertices of distance t to the root. It is easy to convince oneself that $a_{t+1} \leq a_t^2$ so that one may, for arbitrary $b > a_t^2$, extend G to a tree H in such a way that $a_{t+1}(H) = b$: simply attach $b - a_{t+1}(G)$ edges arbitrarily to the vertices of G with maximal distance to the root.

Therefore we can build up recursively trees G such that $(a_n(G))$ is any prescribed sequence (b_n) provided that $b_{n+1} > b_n^2$ for every n. The construction also shows that we have many choices, one gets an abundance of non-isomorphic

G with this property. (We are grateful to Vladimir Kadets for communicating this idea to us.)

The construction indicates that one needs a condition of the type "the a_n do not grow too fast" in order to be able to prove identification theorems. What "too fast" here precisely means is completely open.

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