ANALYTIC DETERMINACY AND $\sharp$'S

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Martin [6] showed that all Borel games are determined. However, this cannot be extended in ZFC. In this article we show that already the determinacy of analytic games implies the existence of large cardinals. More precisely, we present a proof of

Theorem 1 (Harrington [3])**.** *Analytic determinacy implies the existence of $x^\sharp$ for all $x \subseteq \omega$.* □

This theorem is an initial segment of the famous Martin-Steel Theorem (see [8] and [9]) that established a deep connection between the existence of certain large cardinals and the determinacy of certain classes of sets of reals.

Like in Harrington's original paper we will only show the theorem for $0^\sharp$ since the proof relativizes to every $x \subseteq \omega$ giving the existence of $x^\sharp$.

For an introduction to $\sharp$'s and to determinacy see [4].

1. Admissible sets and $0^\sharp$

For the proof of Theorem 1 we will need the notion of an admissible set. Probably the most comprehensive account on admissible sets is Barwise' book [1].

Definition 2. *Kripke-Platek set theory* (*KP*) is the theory axiomatized by the universal closures of the following formulas:

(i) $\forall x(x \in a \leftrightarrow x \in b) \leftrightarrow a = b$ (Extensionality)
(ii) $\exists x \phi(x) \to \exists x(\phi(x) \wedge \forall y \in x \neg\phi(y))$ for all formulas $\phi(x)$ without a free occurrence of y. (Foundation)
(iii) $\exists x(a \in x \wedge b \in x)$ (Pairing)
(iv) $\exists x \forall y \in a \forall z \in y(z \in x)$ (Union)
(v) $\exists x \forall y(y \in x \leftrightarrow (y \in a \wedge \phi(y)))$ for all Δ_0-formulas $\phi(y)$ without a free occurence of x. (Δ_0-comprehension)
(vi) $\forall x \in a \exists y \phi(x, y) \to \exists z \forall x \in a \exists y \in z \phi(x, y)$ for all Δ_0-formulas $\phi(x, y)$ without a free occurence of z. (Δ_0-collection)

A set M is *admissible* (*a-admissible* for a set a) if M is transitive and $(M, \in)$ $((M, a \cap M, \in))$ is a model of KP. An ordinal α is admissible (a-admissible

Date: July 31, 2002.

for a set a) if L_α ($L_\alpha[a]$) is admissible (a-admissible). Note that in some places in the literature admissible ordinals are assumed to be larger than ω. For $a \subseteq \omega$ let $\omega_1(a)$ be the smallest a-admissible ordinal $> \omega$. □

For all $a \subseteq \omega$ the existence of $\omega_1(a)$ is guaranteed by

Lemma 3. *An infinite cardinal κ is a-admissible for all bounded $a \subseteq \kappa$.* □

KP is slightly stronger than one would guess at first sight:

Lemma 4. *Models of KP satisfy Σ_1-collection and Δ_1-comprehension.* □

KP is sufficient to carry out the construction of L:

Lemma 5. *If M is a-admissible, then $(L[a])^M = L_{M\cap \mathrm{ON}}[a]$.* □

The proof of Theorem 1 relies on the following observation due to Silver:

Theorem 6. *Let $a \subseteq \omega$. Suppose for all ordinals α the following holds:*

$(*)$ *If α is a-admissible, then α is a cardinal in L.*

Then $0^\sharp \in L[a]$. □

According to Harrington, Silver used Jensen's Covering Lemma to prove this theorem. We will sketch a more elementary proof, which is similar to Paris' proof given in Harrington's article. We first adapt the proof of Theorem 4.3 in Devlin's book [2] in order to get

Lemma 7. *Let α and β be limit ordinals and $\gamma < |\alpha|^L$ an ordinal. Suppose that $L_\alpha[a]$ is closed under countable sequences. If there is an elementary embedding $j : L_\alpha[a] \to L_\beta[a]$ with $j(\gamma) \neq \gamma$, then $0^\sharp$ exists.*

Proof. The modification of the proof in Devlin's book is the following: In the proof, an ultrapower of some L_λ is formed using the L-ultrafilter $\mathcal{U} := \{X \in L : X \subseteq \gamma \wedge \gamma \in j(X)\}$. We use the assumption that $L_\alpha[a]$ is closed under countable sequences to show that this ultrapower is wellfounded.

Suppose not. Then there is a sequence $(g_n)_{n\in\omega}$ of constructible functions from γ to L_λ such that for all $n \in \omega$, $U_n := \{\beta \in \gamma : g_{n+1}(\beta) \in g_n(\beta)\} \in \mathcal{U}$. Since $\mathcal{P}(\gamma) \cap L \subseteq L_\alpha$ by $\gamma <|\alpha|^L$, the sequence $(U_n)_{n\in\omega}$ is a sequence of elements of L_α. Thus, $(U_n)_{n\in\omega} \in L_\alpha[a]$. Since $\gamma \in j(U_n)$ for every $n \in \omega$ and by elementarity of j, $\gamma \in j(\bigcap_{n\in\omega} U_n)$. Therefore, $\bigcap_{n\in\omega} U_n$ is nonempty. Let $\beta \in \bigcap_{n\in\omega} U_n$. Now $(g_n(\beta))_{n\in\omega}$ is an $\in$-decreasing sequence in L_λ. A contradiction. □

Proof of Theorem 6. We work in $L[a]$. Let M be an elementary submodel of $L_{\aleph_3}[a]$ of size $\aleph_1$ which is closed under countable sequences. This is possible since CH holds in $L[a]$. By the condensation lemma, the transitive collapse of M is some $L_\alpha[a]$. (See [2] for some information on the condensation lemma in the $L_\gamma[a]$ hierarchy.) Clearly, $L_\alpha[a]$ is closed under countable sequences. Let $j : L_\alpha[a] \to M \subseteq L_{\aleph_3}[a]$ be the inverse of the Mostowski collapse from M onto $L_\alpha[a]$. Clearly, $L_\alpha[a]$ is admissible and therefore, using our assumption $(*)$, α is a cardinal in L. Since $L_\alpha[a]$ is of size $\aleph_1$, α is less than $\aleph_2$. Since $\aleph_2 \in M$, there is $\gamma < \alpha$ such that $j(\gamma) > \gamma$. Since α is a cardinal in L, Lemma 7 applies and thus, $0^\sharp$ exists. □

We will need

Remark 8. *In Theorem 6 it is enough to assume that there is a countable ordinal β such that $(*)$ holds for all countable ordinals $\alpha > \beta$.*

Proof. In the proof of Theorem 6 it is sufficient to assume $(*)$ for uncountable ordinals α. Suppose that there is $\beta < \omega_1$ such that $(*)$ holds for all countable $\alpha > \beta$. Let γ be an uncountable ordinal. Fix χ large enough such that

$$(\gamma \text{ is a cardinal})^L \Leftrightarrow (\gamma \text{ is a cardinal})^{L_\chi}.$$

Suppose $L_\gamma[a]$ is admissible. Let $(M, N) \preccurlyeq (L_\chi[a], L_\gamma[a])$ be such that M is countable. Then there is a countable ordinal δ such that $M \cong L_\delta[a]$. Let $\pi : M \to L_\delta[a]$ be the Mostowski collapse. $\pi[N]$ is transitive and thus, there is an ordinal α with $\pi[N] \cong L_\alpha[a]$. Clearly, α is countable. However, by passing to a larger set M if neccessary, we may assume $\alpha > \beta$. Since $L_\alpha[a]$ is admissible, α is a cardinal in L by our assumption. Therefore, we have

$$L_\delta[a] \models \alpha \text{ is a cardinal in } L,$$

and thus,

$$L_\chi[a] \models \gamma \text{ is a cardinal in } L.$$

From the choice of χ it follows that γ is a cardinal in L. This shows that $(*)$ in Theorem 6 holds for all uncountable ordinals γ. □

2. The partial orderings Q_α

Definition 9. In the following a *tree* will be a non-empty subtree of ${}^{<\omega}\omega$. For a tree T let the *height-function* $H_T : T \to \mathrm{ON} \cup \{\infty\}$ be defined as follows:

If $\tau \in T$ is contained in an infinite branch of T, let $H_T(\tau) := \infty$. Otherwise let

$$H_T(\tau) := \sup\{H_T(\eta) + 1 : \eta \in T \text{ is a non-trivial extension of } \tau\}.$$

(As it turns out, H_T is well defined.) □

As usual, we identify trees with subsets of ω. Sometimes it may happen that we use some notions or theorems that have been defined, respectively proved for subsets of ω for elements of ${}^\omega\omega$ instead. (In descriptive set theory we usually have $\mathbb{R} = {}^\omega\omega = \mathcal{P}(\omega) = \mathcal{P}({}^{<\omega}\omega)$ anyway.)

Note: For all ordinals β we have

$$\beta \in H_T[T] \Rightarrow \beta < \omega_1(T).$$

Definition 10. For an ordinal α let Q_α be the set of those pairs (t,h) where t is a finite tree and $h : t \to \omega \cdot \alpha \cup \{\infty\}$ is a function such that $h(\emptyset) = \infty$ and $h(\eta) < h(\tau)$ for all $\eta, \tau \in t$ with $\tau \subseteq \eta$, $\tau \neq \eta$, and $h(\tau) \neq \infty$. For $(t,h),(t',h') \in Q_\alpha$ let $(t,h) \leq_\alpha (t',h')$ if and only if $t' \subseteq t$ and $h' \subseteq h$. (Harrington follows the Jerusalem way and defines $\leq_\alpha$ just the other way round. However, for us $p \leq_\alpha q$ means that p is stronger than q.) □

We will need a simple forcing language, which however allows infinite conjunctions. In order to classify the sentences of this language according to their complexity, we introduce the *rank* $\mathrm{rnk}(\phi)$ of a sentence ϕ.

Definition 11. (i) For all $\eta \in {}^{<\omega}\omega$, $\eta \in \dot{T}$ is a sentence of rank 1.

(ii) If S is a set of sentences, then $\bigwedge S$ is a sentence of rank $\sup\{\mathrm{rnk}(\phi)+1 : \phi \in S\}$.

(iii) For a sentence ϕ let $\mathrm{rnk}(\neg\phi) := \mathrm{rnk}(\phi) + 1$.

For a sentence ϕ and a tree T, $\phi(T)$ has the obvious meaning. □

Definition 12. The relation $\Vdash_\alpha$ between conditions in Q_α and sentences of the forcing language is defined inductively:

(i) $(t,h) \Vdash_\alpha \eta \in \dot{T}$ if and only if either $\eta \in t$ or there is $\tau \in t$ such that $\tau \subseteq \eta$, $|\eta \setminus \tau| = 1$, and $h(\tau) \neq 0$.

(ii) $p \Vdash_\alpha \bigwedge S$ if and only if $p \Vdash_\alpha \phi$ for all $\phi \in S$.

(iii) $p \Vdash_\alpha \neg\phi$ if and only if $q \not\Vdash_\alpha \phi$ for all $q \leq_\alpha p$. □

Definition 13. Let T be a tree and M a transitive set.

(i) T *extends* a condition $(t,h) \in Q_\alpha$ if and only if $t \subseteq T$ and $h(\tau) = H_T(\tau)$ for all $\tau \in t$ with $h(t) \neq \infty$ or $H_T(\tau) < \omega \cdot \alpha$.

(ii) T is generic over M if and only if for all sentences $\phi \in M$ of rank α there is $p \in Q_\alpha$ such that T extends p and $p \Vdash_\alpha \phi$. □

As one might exspect, the following is true: For all $p \in Q_\alpha$, $p \Vdash_\alpha \phi$ if and only if $\phi(T)$ holds for all sufficiently Q_α-generic trees T extending p. Here "sufficiently Q_α-generic" means that there is a filter in Q_α which intersects enough dense subsets of Q_α such that T extends all conditions in the filter.

We also have

Lemma 14. *a) For all countable transitive sets M there is a tree T generic over M.*

b) If M is transitive, sufficiently closed, $\alpha \in M$ is an ordinal, and T is generic over B, then $\alpha \in H_T[T]$.

Proof. a) We may assume that M is admissible. In order to be generic over M, it is sufficient that for all $\alpha \in M$, T extends all conditions in a filter $G_\alpha \subseteq Q_\alpha$ which intersects all dense subsets of Q_α that are contained in M. Since M is countable, there are only countably many dense sets to consider and thus, we can build T by induction in ω steps.

b) By induction, for all countable ordinals α and all $\eta \in {}^{<\omega}\omega$ we can build a sentence $\phi_{\eta,\alpha}$ of rank $\leq \alpha+1$ saying $\eta \in \dot{T}$ and $H_{\dot{T}}(\eta) = \alpha$. If M is sufficiently closed, for $\alpha \in M$ we have $\phi_{\eta,\alpha} \in M$. It follows that for $\alpha \in M$ there is a sentence ϕ_α of rank $\leq \alpha + 1$ in M saying $\alpha \in H_{\dot{T}}[\dot{T}]$. Now if T is generic over M and $\alpha \notin H_T[T]$, then there is $p \in Q_{\alpha+1}$ forcing $\neg\phi_\alpha$. But clearly, p has an extension (t,h) with $\alpha \in h[t]$ and (t,h) forces $\alpha \in H_{\dot{T}}[\dot{T}]$. A contradiction. □

Note that using sentences of the form $\phi_{\eta,\alpha}$ as in the proof of part b) of Lemma 14, for each condition $p \in Q_\beta$ we can build a sentence $\phi_p(\dot{T})$ saying that T extends p.

The definitions and observations concerning the Q_α's are due to Steel [10].

3. The Σ^1_1-set A

Definition 15. Let M and N be sets with $M \subseteq N$ and let R and S be binary relations on M, respectively N. (N, S) is an *end extension* of (M, R) if $S \restriction M = R$ and for all $m \in M$ and $n \in N$, $(n,m) \in S \Rightarrow n \in M$. □

Definition 16. Let $A \subseteq {}^\omega\omega$ be the following set:

$a \in A$ if and only if there is a binary relation R on ω which is recursive in a such that (ω, R) is isomorphic to an end-extension of $(L_{\omega_1(a)}, \in)$. □

Lemma 17. *A is Σ^1_1.*

Proof. The statement

$$R \subseteq \omega \times \omega \text{ in recursive in } a$$

is Δ^1_1 since it just says that there is a function recursive in a which is the characteristic function of R. A function recursive in a is coded by a natural number.

(ω, R) is isomorphic to an end extension of $L_{\omega_1(a)}$ if and only if (ω, R) has an end extension (M, S) with the following properties:

(i) a is an element of (M, S) in the sense that there is $a^{(M,S)} \in M$ such that $a = \{n \in \omega : (n^{(M,S)}, a^{(M,S)}) \in S\}$ where $n^{(M,S)}$ is the element of M which has the same definition in (M, S) as n has in the universe.

(ii) In M there is an ordinal α such that $(L_\alpha[a^{(M,S)}])^{(M,S)}$ is a model of KP and $L_\alpha^{(M,S)}$ is already included in ω (where ω should be viewed as the underlying set of the structure (ω, R)).

Since it is sufficient to consider countable structures (M, S) here, the quantifier "there is a structure (M, S)" can be viewed as a quantifier ranging over reals. Therefore the statement

$$(\omega, R) \text{ is isomorphic to an end extension of } L_{\omega_1(a)}$$

is Σ^1_1. It follows that A is analytic. $\square$

Clearly, A is closed under Turing equivalence. Surprisingly enough, A is non-empty. We even have

Lemma 18. *For all $a \in {}^\omega\omega$ there is $b \in {}^\omega\omega$ such that $\langle a, b\rangle \in A$ where $\langle a, b\rangle$ denotes the* Turing join *of a and b, i.e., $\langle a, b\rangle$ is an element of ${}^\omega\omega$ whose Turing degree is the least upper bound of the Turing degrees of a and b.* $\square$

The proof of this lemma needs two theorems which can be found in [5].

Theorem 19 (Keisler, Morley). *Let M be a countable transitive model of a sufficiently large fragment of ZF + $V = L$. Then there is an elementary end extension N of M such that there is a sequence $(d_i)_{i\in\omega}$ of indiscernibles in N such that the following hold:*

(i) *Each d_i is an ordinal in N.*

(ii) *For $i < j$ we have $N \models d_i \in d_j$.* $\square$

Remark 20. *For $a \subseteq \omega$ this theorem also holds for models of a sufficiently large fragment of ZF + $V = L[a]$.* $\square$

Theorem 21 (Ehrenfeucht, Mostowski). *Let N be a model of a countable fragment L_N of $L_{\omega_1,\omega}$ with Skolem functions and let $(X, <)$ be a set of indiscernibles in N. Then for each infinite linearly ordered set $(Y, <)$ there is a model N' of L_N such that $(Y, <)$ is a set of indiscernibles in N' and finite ascending sequences in X and Y realize the same types.* $\square$

Proof of Lemma 18. Let $\beta < \omega_1$ be such that $L_\beta[a]$ is a model of a sufficiently large fragment of ZF + $V = L[a]$. In particular, let β be a-admissible. By Theorem 19, or rather the remark following it, there is a countable elementary end extension N of $L_\beta[a]$ such that in N there is a sequence $d_0 < d_1 < \dots$ of indiscernibles which are ordinals.

Using Theorem 21, we can get an elementary end extension M of $L_\beta[a]$ with a set Y of indiscernibles which are ordinals such that Y has the ordertype of the rationals. Here we use a fragment L_N of $L_{\omega_1,\omega}$ which codes $L_\beta[a]$ in the sense that every model of L_N is an elementary end extension of $L_\beta[a]$ (up to isomorphism). We may assume that the wellfounded core of M is a transitive set.

In M there is a nonstandard ordinal α such that $M \models |\alpha| = \aleph_0$. Moreover, in M there is a relation $b \subseteq \omega \times \omega$ such that $(\omega, b) \cong L_\alpha^M$. Clearly, a and b are elements of the wellfounded core of M. Since M is a model of KP, the wellfounded core of M is admissible. Therefore, $\omega_1(a,b) \subseteq M$. It follows that for all $\xi < \omega_1(a,b)$, L_α is an end extension of L_ξ and thus, L_α is an end extension of $L_{\omega_1(a,b)}$. By the definition of A, $\langle a, b\rangle \in A$. □

Using the remark derived from Theorem 6 we can prove

Lemma 22. *Let $a \subseteq \omega$ and $\rho < \omega_1$. Suppose for all $\xi < \omega_1$ there is a tree T generic over $L_\xi[a]$ and $b \subseteq \omega$ such that $b \in L_\rho[T, a]$, $b \in A$, and $\omega_1(b) \geq \xi$. Then $0^\sharp \in L[a]$.*

Proof. Let α be a countable a-admissible ordinal such that $\alpha > \rho$. By the remark following Theorem 6, it suffices to show that α is a cardinal in L. For this it is sufficient to show that for all $\kappa < \alpha$, $(\kappa^+)^L \leq \alpha$. Since every ordinal $< (\kappa^+)^L$ is coded by a constructible subset X of κ and since every ordinal coded by an element of an admissible set is an element of that admissible set as well, it is even sufficient to prove that for all constructible $X \subseteq \kappa$, $X \in L_\alpha[a]$.

Let δ be a countable ordinal such that $X \in L_\delta$. Let $\xi < \omega_1$ be sufficiently large. In particular, let ξ be such that $\xi > \delta, \alpha$. Let T be generic over $L_\xi[a]$ such that there is $b \in L_\rho[a, T]$ with $b \in A$ and $\omega_1(b) \geq \xi$.

Claim: $X \in L_{\kappa \cdot 3}[b]$.

Proof of the claim: Since $\xi < \omega_1(b)$, there is a binary relation R on ω recursive in b such that R is a wellordering of ω of order type $\geq \xi$. (See [1] for this.) (ω, R) is an element of $L_{\omega+7}[b]$. It follows that in $L_{\omega+8}[b]$ there is a binary relation S on ω such that $N = (\omega, S)$ is isomorphic to an end extension of L_ξ since S can be defined using recursion along the wellfounded relation R. In N there are elements κ' and X' corresponding to

κ, respectively X in L_ξ. The isomorphism between κ' and κ exists in $L_{\kappa\cdot 3}[b]$ and maps X' onto X. Thus, $X \in L_{\kappa\cdot 3}[b]$. This concludes the proof of the claim.

Since $\alpha > \rho, \kappa \cdot 3$ and $b \in L_\rho[a,T]$, we have $X \in L_\alpha[a,T]$. Thus, X is definable over $L_\alpha[a]$ from an initial segment of T. Therefore, there are $\beta < \alpha$ and a Q_β-name $\sigma \in L_\alpha[a]$ such that $X = \sigma(T)$. Since we can use infinite conjunctions in our forcing language, we can write down a sentence ϕ expressing $\sigma(T) = X$, namely

$$\left(\bigwedge_{i\in X} i \in \sigma(\dot{T})\right) \wedge \left(\bigwedge_{i\in\kappa\setminus X} i \notin \sigma(\dot{T})\right),$$

where the statements "$i \in \sigma(\dot{T})$" are sentences of rank $< \alpha$, which can be constructed by looking at σ. Clearly, $\phi \in L_\xi[a]$. We have $\gamma := \mathrm{rnk}(\phi) < \alpha$. Therefore, there is $p \in Q_\gamma$ with $p \Vdash_\gamma \phi(T)$. But now $X = \{i \in \kappa : \exists q \leq_\gamma p(q \Vdash_\gamma i \in \sigma(\dot{T}))\}$. Since α is a limit ordinal, $p \in L_\alpha[a]$ and thus, $X \in L_\alpha[a]$. $\square$

4. Showdown

Definition 23. A subset B of ${}^\omega\omega$ is a *cone of Turing degrees* if there is $a \subseteq \omega$ such that $b \subseteq \omega$ is an element of B if and only if b is recursive in a. In this case a is called a *base* of B. $\square$

In [7] Martin showed

Theorem 24. *For every determined set $B \subseteq {}^\omega\omega$ which is closed under Turing equivalence either B or ${}^\omega\omega \setminus B$ includes a cone of Turing degrees.*

Proof. Suppose $B \subseteq {}^\omega\omega$ is determined and closed under Turing equivalence. Without loss of generality, we may assume that the first player has a winning strategy σ for $G(B)$. σ is a function from ${}^{<\omega}\omega$ to ω. Thus, via the usual identification, we can regard σ as an element of ${}^\omega\omega$. Let $b \in {}^\omega\omega$ be recursive in σ. Let $c \in {}^\omega\omega$ be the play where the first player plays according to σ and the second player plays the Turing join of b and σ. Then $c \in B$ since σ is a winning strategy for the first player and c is Turing equivalent to b since σ is recursive in b and c is recursive in $\langle b, \sigma\rangle$. Since B is closed under Turing equivalence, we have $b \in B$. $\square$

We are now in the position to prove Theorem 1.

Proof of Theorem 1. By Lemma 18, ${}^\omega\omega \setminus A$ does not include a cone of Turing degrees. If all analytic sets are determined, this means that A includes a cone of Turing degrees. We are done if we can show the following

Claim: A base a of a cone of Turing degrees included in A satisfies the conditions in Lemma 22.

Let a be a base of a cone of Turing degrees included in A and let $\rho < \omega$ be reasonably big. For $\xi < \omega_1$ let T be generic over $L_\xi[a]$ and let $b := \langle a, T\rangle$. T exists by Lemma 14. Now $b \in A$, $b \in L_\rho[a, T]$, and $\omega_1(b) \leq \xi$. This finishes the proof of the claim and thus, of Theorem 1. $\square$

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