A HYPERSPACE COMPLETION FOR SEMIUNIFORM CONVERGENCE SPACES AND RELATED HYPERSPACE STRUCTURES

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Dedicated to my friend and colleague Horst Herrlich on the occasion of his 65th birthday and his retirement

Abstract. Since hyperspaces of complete (separated) uniform spaces are not complete in general, it is highly remarkable that in the more general context of semiuniform convergence spaces even a hyperspace completion exists which preserves several invariants, e.g. precompactness (and thus compactness), connectedness (and uniform connectedness), the property of being a filter space (or a semiuniform space), etc. This completion is used to characterize the subspaces of the compact spaces in the realm of semiuniform convergence spaces axiomatically. The complete hyperspace structure is coarser than the usual uniform hyperspace structure in case uniform spaces are considered.

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0. Introduction

If $\mathbf{X}$ is a space (e.g. a metric space, a topological space, a uniform space or a semiuniform convergence space), then a hyperspace of $\mathbf{X}$, denoted by $H(\mathbf{X})$, is a space whose points are suitable subsets of $\mathbf{X}$ and in which $\mathbf{X}$ can be embedded (possibly under additional assumptions). The embedding of $\mathbf{X}$ into $H(\mathbf{X})$ is said to preserve (resp. reflect) some property P if $H(\mathbf{X})$ (resp. $\mathbf{X}$) has P whenever $\mathbf{X}$ (resp. $H(\mathbf{X})$) has P. Hyperspaces of metric spaces have been studied first by F. Hausdorff [3; pp. 290–294] whose distance on the set of all non-empty, closed and bounded subsets of a metric space is known

today as Hausdorff metric. Later on, L. Vietoris [8] defined a topology on the set of all non-empty closed subsets of a topological space $\mathbf{X}$, which is called now the Vietoris topology. In this way one obtains a topological hyperspace provided that $\mathbf{X}$ is a topological T_1-space. It has been studied extensively by E. Michael [5]. Hyperspaces of (separated) uniform spaces have been considered first by N. Bourbaki [1; p.97, ex.7)] and later on partially by E. Michael [5] and in some detail by J. Isbell [4]. Though the hyperspace of a complete metric space is a complete metric space, a corresponding result for separated uniform spaces is not true.
In this article we will prove that the situation becomes much better, when we study semiuniform convergence spaces, which include uniform spaces as well as (symmetric) topological spaces and have many convenient properties (cf. [6]). In particular, if $\mathcal{A}$ denotes an arbitrary set of non-empty subsets of a semiuniform convergence space $\mathbf{X}$ containing all singletons, there is a coarsest semiuniform convergence structure $\mathcal{J}_{\mathcal{A}}^f$ on $\mathcal{A}$ such that $\mathbf{X}$ is a subspace of $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$. It turns out that $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is already a completion of $\mathbf{X}$, whenever $\mathcal{A}$ contains a subset of $\mathbf{X}$ with at least two points.
This completion generalizes the construction of one-extensions (cf. [6]) for semiuniform convergence spaces with more than one point.
The embedding from $\mathbf{X}$ into $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ preserves and reflects the following invariants for semiuniform convergence spaces:

1. precompact
2. semiuniform limit space
3. diagonal
4. semi-pseudouniform
5. semiuniform

Furthermore, it preserves compactness and connectedness as well as uniform connectedness. Last not least, the hyperspace completion $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is used in order to characterize the subspaces of compact spaces in the realm of semiuniform convergence spaces.
Finally, the hyperspace for diagonal uniform limit spaces (= Cook Fischer spaces), introduced by R. J. Gazik [2], is generalized to diagonal semiuniform limit spaces. The embedding from a diagonal semiuniform limit space $\mathbf{X}$ into this hyperspace preserves and reflects the following properties:

1. Cook Fischer space
2. semiuniform
3. uniform

For uniform spaces the original construction mentioned above is reobtained.

The terminology of this paper corresponds to [6].

Notation. Unless otherwise stated, in the following, let $\mathcal{A}$ be a set of non-empty subsets of a semiuniform convergence space $(X, \mathcal{J}_X)$ such that $\{x\} \in \mathcal{A}$ for each $x \in X$. If an injective map $i : X \to \mathcal{A}$ is defined by $i(x) = \{x\}$ for each $x \in X$, we put $i[X] = X'$ and assume without loss of generality that $X = X'$, i.e. i is an inclusion map.

1. Hyperspace structures for all semiuniform convergence spaces

1.1 Proposition. *If $\mathcal{J}^s_{\mathcal{A}} = \{\mathcal{H} \in F(\mathcal{A} \times \mathcal{A})$: there is some $\mathcal{F} \in \mathcal{J}_X$ with $(i \times i)(\mathcal{F}) \subset \mathcal{H}$ or there is an ultrafilter $\mathcal{U}$ on $\mathcal{A}$ having no trace on X such that $\mathcal{U} \times \mathcal{U} \subset \mathcal{H}\}$, then $(\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$ is a hyperspace of $(X, \mathcal{J}_X)$ such that the inclusion map $i : (X, \mathcal{J}_X) \to (\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$ preserves (and reflects) precompactness.*

Proof. 1) $(\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$ is a semiuniform convergence space:
$UC1$) For each $x \in X, \dot{i(x)} \times \dot{i(x)} = i(\dot{x}) \times i(\dot{x}) = i \times i(\dot{x} \times \dot{x}) \in \mathcal{J}^s_{\mathcal{A}}$, and for each $A \in \mathcal{A} \setminus X, \dot{A} \times \dot{A} \in \mathcal{J}^s_{\mathcal{A}}$, since $\dot{A}$ is an ultrafilter having no trace on X.
$UC2$) is fulfilled by definition of $\mathcal{J}^s_{\mathcal{A}}$.
$UC3$) Let $\mathcal{H} \in \mathcal{J}^s_{\mathcal{A}}$. Then 1. $\mathcal{H} \supset i \times i(\mathcal{F})$ for some $\mathcal{F} \in \mathcal{J}_X$ or 2. $\mathcal{H} \supset \mathcal{U} \times \mathcal{U}$ for some ultrafilter $\mathcal{U}$ on $\mathcal{A}$ having no trace on X. In the first case,
$\mathcal{H}^{-1} \supset ((i \times i)(\mathcal{F}))^{-1} = (i \times i)(\mathcal{F}^{-1})$, which implies $\mathcal{H}^{-1} \in \mathcal{J}^s_{\mathcal{A}}$, since $\mathcal{F}^{-1} \in \mathcal{J}_X$. In the second case, $\mathcal{H}^{-1} \supset (\mathcal{U} \times \mathcal{U})^{-1} = \mathcal{U} \times \mathcal{U}$, i.e. $\mathcal{H}^{-1} \in \mathcal{J}^s_{\mathcal{A}}$.
2) $(X, \mathcal{J}_X)$ is a subspace of $(\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$, i.e. $\mathcal{J}_X = (\mathcal{J}^s_{\mathcal{A}})_X = \{\mathcal{F} \in F(X \times X) : (i \times i)(\mathcal{F}) \in \mathcal{J}^s_{\mathcal{A}}\}$:
a) $\mathcal{J}_X \subset (\mathcal{J}^s_{\mathcal{A}})_X$ follows directly from the definition of $\mathcal{J}^s_{\mathcal{A}}$.
b) Let $\mathcal{F} \in \mathcal{J}^s_{\mathcal{A}})_X$. Then $(i \times i)(\mathcal{F}) \in \mathcal{J}^s_{\mathcal{A}}$, i.e.
   1. $(i \times i)(\mathcal{F}) \supset (i \times i)(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{J}_X$
or 2. $(i \times i)(\mathcal{F}) \supset \mathcal{U} \times \mathcal{U}$ for some ultrafilter $\mathcal{U}$ on $\mathcal{A}$ having no trace on X .
In the first case, $(i \times i)^{-1}(i \times i)(\mathcal{F}) = \mathcal{F} \supset (i \times i)^{-1}(i \times i)(\mathcal{G}) = \mathcal{G}$, which implies $\mathcal{F} \in \mathcal{J}_X$. The second case cannot occur, since otherwise $\mathcal{U}$ would have a trace on X.
3) $(X, \mathcal{J}_X)$ is precompact iff $(\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$ ist precompact:
"$\Leftarrow$". Since precompactness is hereditary, this implication follows from 2).

"$\Rightarrow$". Let $\mathcal{U}$ be an ultrafilter on $\mathcal{A}$. Then 1. $i^{-1}(\mathcal{U})$ does not exist or 2. $\mathcal{U}$ has a trace on X. In the first case, $\mathcal{U} \times \mathcal{U} \in \mathcal{J}^s_{\mathcal{A}}$ by definition, i.e. $\mathcal{U}$ is a Cauchy filter. In the second case, $i^{-1}(\mathcal{U})$ is an ultrafilter because $\mathcal{U}$ is an ultrafilter. Furthermore, $X \in \mathcal{U}$(otherwise $\mathcal{A} \setminus X \in \mathcal{U}$ and $\mathcal{U}$ would not have a trace on X). Since $(X, \mathcal{J}_X)$ is precompact, $i^{-1}(\mathcal{U})$ is a Cauchy filter, i.e. $i^{-1}(\mathcal{U}) \times i^{-1}(\mathcal{U}) = (i \times i)^{-1}(\mathcal{U} \times \mathcal{U})$ belongs to $\mathcal{J}_X$, and thus (note $X \times X \in \mathcal{U} \times \mathcal{U}$), $(i \times i)(i \times i)^{-1}(\mathcal{U} \times \mathcal{U}) = \mathcal{U} \times \mathcal{U}$ belongs to $\mathcal{J}^s_{\mathcal{A}}$ by definition, i.e. $\mathcal{U}$ is a Cauchy filter.

1.2 Definition. $(\mathcal{A}, \mathcal{J}^s_{\mathcal{A}})$ is called the *simple hyperspace* of $(X, \mathcal{J}_X)$.

1.3 Remark. Let $(\mathcal{J}^j_{\mathcal{A}})_{j \in J}$ be the family of all semiuniform convergence structures on $\mathcal{A}$ inducing $\mathcal{J}_X$, i.e. $(\mathcal{J}^j_{\mathcal{A}})_X = \mathcal{J}_X$ for each $j \in J$. Then the final **SUConv**-structure $\mathcal{J}^f_{\mathcal{A}}$ on $\mathcal{A}$ w.r.t. $(1^j_{\mathcal{A}} : (\mathcal{A}, \mathcal{J}^j_{\mathcal{A}}) \to \mathcal{A})_{j \in J}$, where $1^j_{\mathcal{A}} = 1_{\mathcal{A}}$ for each $j \in J$, i.e. $\sup_{j \in \mathcal{J}} \mathcal{J}^j_{\mathcal{A}}$ w.r.t. $\leq$, is the finest semiuniform convergence structure on $\mathcal{A}$ which is coarser than each $\mathcal{J}^j_{\mathcal{A}}$. Obviously, $\mathcal{J}^f_{\mathcal{A}} = \bigcup\limits_{j \in \mathcal{J}} \mathcal{J}^j_{\mathcal{A}}$. $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is called the *final hyperspace* of $(X, \mathcal{J}_X)$.

1.4 Proposition. *Let $\mathcal{J}_{\mathcal{A}}$ be a semiuniform convergence structure on $\mathcal{A}$. Then the following are equivalent:*
(1) *$\mathcal{J}_{\mathcal{A}}$ is the final hyperspace structure $\mathcal{J}^f_{\mathcal{A}}$ on $\mathcal{A}$.*
(2) *$\mathcal{J}_{\mathcal{A}}$ is the coarsest semiuniform convergence structure on $\mathcal{A}$ such that $(X, \mathcal{J}_X)$ is a subspace of $(\mathcal{A}, \mathcal{J}_{\mathcal{A}})$.*
(3) *$\mathcal{J}_{\mathcal{A}} = \{\mathcal{H} \in F(\mathcal{A} \times \mathcal{A}) : (i \times i)^{-1}(\mathcal{H})$ exists and belongs to $\mathcal{J}_X$ or $(i \times i)^{-1}(\mathcal{H})$ does not exist.$\}$*

Proof. "(1)$\Rightarrow$ (2)". 1. $(X, \mathcal{J}_X)$ is a subspace of $(\mathcal{A}, \mathcal{J}_{\mathcal{A}})$, i.e. $\mathcal{J}_X = (\mathcal{J}_{\mathcal{A}})_X$: If $\mathcal{F} \in \mathcal{J}_X$, then $(i \times i)(\mathcal{F}) \in \mathcal{J}^s_{\mathcal{A}} \subset \mathcal{J}^f_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$, i.e. $\mathcal{F} \in (\mathcal{J}_{\mathcal{A}})_X$. Conversely, let $\mathcal{F} \in (\mathcal{J}_{\mathcal{A}})_X$. Then $(i \times i)(\mathcal{F}) \in \mathcal{J}^f_{\mathcal{A}} = \bigcup\limits_{j \in J} \mathcal{J}^j_{\mathcal{A}}$. Hence, there is some $j \in J$ such that $(i \times i)(\mathcal{F}) \in \mathcal{J}^j_{\mathcal{A}}$, i.e. $\mathcal{F} \in (\mathcal{J}^j_{\mathcal{A}})_X = \mathcal{J}_X$.
2. Let $\mathcal{J}'_{\mathcal{A}}$ be a semiuniform convergence structure on $\mathcal{A}$ inducing $\mathcal{J}_X$. Then there is some $j \in J$ with $\mathcal{J}'_{\mathcal{A}} = \mathcal{J}^j_{\mathcal{A}} \subset \mathcal{J}^f_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$.
"(2) $\Rightarrow$ (3)". A semiuniform convergence structure $\mathcal{J}'_{\mathcal{A}}$ on $\mathcal{A}$ is defined by $\mathcal{H} \in \mathcal{J}'_{\mathcal{A}}$ iff $(i \times i)^{-1}(\mathcal{H})$ exists and belongs to $\mathcal{J}_X$ or $(i \times i)^{-1}(\mathcal{H})$ does not exist. Then
1. $(X, \mathcal{J}_X)$ is a subspace of $(\mathcal{A}, \mathcal{J}'_{\mathcal{A}})$:
a) If $\mathcal{F} \in \mathcal{J}_X$, then the filter $(i \times i)(\mathcal{F})$ on $\mathcal{A} \times \mathcal{A}$ has the filter $\mathcal{F}$ as a

trace on $X \times X$, i.e. $(i \times i)(\mathcal{F}) \in \mathcal{J}'_{\mathcal{A}}$ or equivalently, $\mathcal{F} \in (\mathcal{J}'_{\mathcal{A}})_X$.

b) If $\mathcal{F} \in (\mathcal{J}'_{\mathcal{A}})_X$, i.e. $(i \times i)(\mathcal{F}) \in \mathcal{J}'_{\mathcal{A}}$, then $(i \times i)^{-1}(i \times i)(\mathcal{F}) = \mathcal{F}$ exists and belongs to $\mathcal{J}_X$.

2. Let $\hat{\mathcal{J}}_{\mathcal{A}}$ be a semiuniform convergence structure on $\mathcal{A}$ inducing $\mathcal{J}_X$, and let $\mathcal{F} \in \hat{\mathcal{J}}_{\mathcal{A}}$. Then 1° $(i \times i)^{-1}(\mathcal{F})$ does not exist, i.e. $\mathcal{F} \in \mathcal{J}'_{\mathcal{A}}$, or 2° $(i \times i)^{-1}(\mathcal{F})$ exists, which implies that $(i \times i)(i \times i)^{-1}(\mathcal{F}) \supset \mathcal{F}$ belongs to $\hat{\mathcal{J}}_{\mathcal{A}}$, i.e. $(i \times i)^{-1}(\mathcal{F}) \in (\hat{\mathcal{J}}_{\mathcal{A}})_X = \mathcal{J}_X$ or equivalently, $\mathcal{F} \in \mathcal{J}'_{\mathcal{A}}$. Thus, $\hat{\mathcal{J}}_{\mathcal{A}} \subset \mathcal{J}'_{\mathcal{A}}$.

It follows from 1. and 2. that $\mathcal{J}'_{\mathcal{A}}$ is the coarsest semiuniform convergence structure on $\mathcal{A}$ inducing $\mathcal{J}_X$. By assumption, $\mathcal{J}'_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$.

"(3) $\Rightarrow$ (1)". $\mathcal{J}'_{\mathcal{A}}$ as defined under "(2) $\Rightarrow$ (3)" is a semiuniform convergence structure on $\mathcal{A}$ inducing $\mathcal{J}_X$. Thus, $\mathcal{J}'_{\mathcal{A}} \subset \bigcup\limits_{j \in J} \mathcal{J}^j_{\mathcal{A}} = \mathcal{J}^f_{\mathcal{A}}$ (cf. 1.3). It has been proved under "(1) $\Rightarrow$ (2)" that $\mathcal{J}^f_{\mathcal{A}}$ is also a semiuniform convergence structure inducing $\mathcal{J}_X$. Consequently, by 2. under "(2) $\Rightarrow$ (3)", $\mathcal{J}^f_{\mathcal{A}} \subset \mathcal{J}'_{\mathcal{A}}$. Hence, $\mathcal{J}_{\mathcal{A}} = \mathcal{J}'_{\mathcal{A}}$ is the final hyperspace structure $\mathcal{J}^f_{\mathcal{A}}$.

1.5 Corollary. $\mathcal{J}^s_{\mathcal{A}} \leq \mathcal{J}^f_{\mathcal{A}}$, *i.e.* $1_X : (X, \mathcal{J}^s_{\mathcal{A}}) \to (X, \mathcal{J}^f_{\mathcal{A}})$ *is uniformly continuous.*

1.6 Corollary. *A semiuniform convergence space is precompact iff its final hyperspace is precompact.*

Proof. "$\Rightarrow$". This follows immediately from 1.1. and 1.5.
"$\Leftarrow$". Precompactness is hereditary.

1.7 Remark. A filter $\mathcal{H}$ on $\mathcal{A}$ converges to $A \in \mathcal{A}$ (w.r.t. $q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}$) iff $(\mathcal{H} \cap \dot{A}) \times (\mathcal{H} \cap \dot{A}) \in \mathcal{J}^f_{\mathcal{A}}$. Thus, using 1.4.(3),

1. $\mathcal{H} \xrightarrow{q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}} A \in \mathcal{A} \setminus X$ iff $\mathcal{H} \in \gamma_{\mathcal{J}^f_{\mathcal{A}}}$, i.e. $\mathcal{H}$ is a $\mathcal{J}^f_{\mathcal{A}}$–Cauchy filter,
2. $\mathcal{H} \xrightarrow{q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}} x \in X$ iff $i^{-1}(\mathcal{H} \cap \dot{x}) \in \gamma_{\mathcal{J}_X}$.

This means that for each $\mathcal{H} \in F(\mathcal{A})$, we obtain:
If $i^{-1}(\mathcal{H})$ exists, then

a) for each $x \in X, \mathcal{H} \xrightarrow{q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}} x$ iff $i^{-1}(\mathcal{H}) \xrightarrow{q_{\gamma_{\mathcal{J}_X}}} x$

b) for each $A \in \mathcal{A} \setminus X, \mathcal{H} \xrightarrow{q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}} A$ iff $i^{-1}(\mathcal{H}) \in \gamma_{\mathcal{J}_X}$ (i.e. $\mathcal{H} \in \gamma_{\mathcal{J}^f_{\mathcal{A}}}$),

and if $i^{-1}(\mathcal{H})$ does not exist, then for each $A \in \mathcal{A}, \mathcal{H} \xrightarrow{q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}}} A$.
In particular, a filter $\mathcal{H}$ on $\mathcal{A}$ converging to some $x \in X$, converges to each $A \in \mathcal{A} \setminus X$.

1.8 Theorem. *For each semiuniform convergence space $(X, \mathcal{J}_X)$, the final hyperspace $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is complete whenever $\mathcal{A} \setminus X$ is non-empty, and it contains X as a dense subset, i.e. it is a completion of $(X, \mathcal{J}_X)$.*

Proof. Each $\mathcal{J}^f_{\mathcal{A}}$–Cauchy filter $\mathcal{H}$ on $\mathcal{A}$ converges to each $A \in \mathcal{A} \setminus X$ (cf. 1.7). Furthermore, each $A \in \mathcal{A}$ is an adherent point of X: For each $x \in X, i(\dot{x})$ converges to x and contains X, i.e. x is an adherent point of X, and it converges to each $A \in \mathcal{A} \setminus X$, i.e. each $A \in \mathcal{A} \setminus X$ is an adherent point of X.

1.9 Corollary. *Let (X, γ) be a filter space, $(X, \mathcal{J}_\gamma)$ its corresponding semiuniform convergence space, and $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ the final hyperspace of $(X, \mathcal{J}_\gamma)$, where $\mathcal{A} \setminus X$ is non-empty. Then the underlying filter space $(\mathcal{A}, \gamma_{\mathcal{J}^f_{\mathcal{A}}})$ is complete (i.e. a symmetric Kent convergence space) and contains (X, γ) as a (dense) subspace. In particular, $\gamma_{\mathcal{J}^f_{\mathcal{A}}}$ is the coarsest filter space structure on $\mathcal{A}$ inducing γ .*

Proof. Since $(X, \mathcal{J}_\gamma)$ is a subspace of $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}}), (X, \gamma_{\mathcal{J}_\gamma}) = (X, \gamma)$ is a subspace of $(\mathcal{A}, \gamma_{\mathcal{J}^f_{\mathcal{A}}})$ (cf. [6; 2.3.3.17]). Obviously, $\gamma_{\mathcal{J}^f_{\mathcal{A}}} = \{\mathcal{F} \in F(\mathcal{A}) : i^{-1}(\mathcal{F})$ does not exist or $(i^{-1}(\mathcal{F})$ exists and belongs to $\gamma)\}$, and by 1.8, $(\mathcal{A}, \gamma_{\mathcal{J}^f_{\mathcal{A}}})$ is complete and X is dense in $(\mathcal{A}, \gamma_{\mathcal{J}^f_{\mathcal{A}}})$, i.e. in $(\mathcal{A}, q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}})$. If γ' is a filter space structure on $\mathcal{A}$ inducing γ and $\mathcal{F} \in \gamma'$ then 1° $i^{-1}(\mathcal{F})$ does not exist, i.e. $\mathcal{F} \in \gamma_{\mathcal{J}^f_{\mathcal{A}}}$, or 2° $i^{-1}(\mathcal{F})$ exists, where $i(i^{-1}(\mathcal{F})) \supset \mathcal{F}$ belongs to γ', which implies $i^{-1}(\mathcal{F}) \in \gamma$, since γ is induced by γ', and consequently, $\mathcal{F} \in \gamma_{\mathcal{J}^f_{\mathcal{A}}}$. Thus, $\gamma_{\mathcal{J}^f_{\mathcal{A}}}$ is the coarsest filter space structure on $\mathcal{A}$ inducing γ.

1.10 Corollary. *Let (X, q) be a symmetric Kent convergence space, (X, γ_q) its corresponding filter space and $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ the final hyperspace of $(X, \mathcal{J}_{\gamma_q})$. Then the underlying (symmetric) Kent convergence space $(\mathcal{A}, q_{\gamma_{\mathcal{J}^f_{\mathcal{A}}}})$ of $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ contains (X, q) as a (dense) subspace.*

Proof. Since $(X, \mathcal{J}_{\gamma_q})$ is a dense subspace of $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$, (X, q) is a dense subspace of $(\mathcal{A}, q_{\gamma_{\mathcal{J}_{\mathcal{A}}^f}})$.

1.11 Remarks. 1) If $\mathcal{A}$ contains all singletons and exactly one A with cardinality at least 2, then $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is the *one-point extension of the semiuniform convergence space* $(X, \mathcal{J}_X)$ (cf. [6; 3.2.4]), and it is called *the one-point completion of* $(X, \mathcal{J}_X)$.
2) Since subspaces of convergence spaces (=symmetric Kent convergence spaces) are filter spaces, it follows from 1.9 that *the filter spaces are exactly the* [*dense*] *subspaces* (*formed in the construct* **SUConv** *of semiuniform convergence spaces*) *of the convergence spaces* – a result which is already known (cf. [6; 4.4.7]).

1.12 Theorem. *Let* $(X, \mathcal{J}_X)$ *be a semiuniform convergence space. Then the following are equivalent:*
(1) $(X, \mathcal{J}_X)$ *is precompact.*
(2) $(X, \mathcal{J}_X)$ *is a dense subspace* (*in* **SUConv**) *of a compact semiuniform convergence space.*
(3) $(X, \mathcal{J}_X)$ *is a subspace* (*in* **SUConv**) *of a compact semiuniform convergence space.*

Proof. "(1) $\Rightarrow$ (2)". If $(X, \mathcal{J}_X)$ is a semiuniform space, then 1. $|X| \leq 1$ or 2. $|X| > 1$. In case 1, the assertion is obvious, since $(X, \mathcal{J}_X)$ is already compact. In case 2, let $\mathcal{A}$ contain all singletons and a subset A of X with at least two points:
Then $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is a completion of $(X, \mathcal{J}_X)$, which is precompact by 1.6. Consequently, $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is a compact semiuniform convergence space containing $(X, \mathcal{J}_X)$ as a dense subspace.
"(2) $\Rightarrow$ (3)". This implication is obvious.
"(3) $\Rightarrow$ (1)". Subspaces of compact semiuniform convergence spaces are precompact, since "compact" implies "precompact", and precompactness is hereditary.

1.13 Proposition. *If* $(X, \mathcal{J}_X)$ *is a compact semiuniform convergence space, then its final hyperspace* $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ *is compact.*

Proof. Let $|X| \leq 1$. Then the assertion is obvious. If $|X| > 1$, then the assertion follows from 1.6 and 1.8 provided that $\mathcal{A} \setminus X \neq \emptyset$. In case

$\mathcal{A} \setminus X = \emptyset$ and $|X| > 1, (\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f) \cong (X, \mathcal{J}_X)$ is compact.

1.14 Proposition. *Let $\mathcal{E}$ be a class of semiuniform convergence spaces consisting of T_1-spaces. If $(X, \mathcal{J}_X)$ is an $\mathcal{E}-$connected semiuniform convergence space, then its final hyperspace $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$ is $\mathcal{E}-$connected.*

Proof. Since X is dense in $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^f)$, the result follows from [6; 5.1.20].

1.15 Remark. Let $(X, \mathcal{V})$ be a separated uniform space in the sense of A. Weil [9]. In case $\mathcal{A}$ is the set of all non-empty closed subsets, then for each $V \in \mathcal{V}$, let

$$H(V) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \subset V[B] \text{ and} B \subset V[A]\}.$$

Obviously, $\{H(V) : V \in \mathcal{V}\}$ is a base for a uniformity $H(\mathcal{V})$ on $\mathcal{A}$, which has been introduced by N. Bourbaki [1; p. 97].
The following are valid:
(1) $(X, \mathcal{V})$ is a subspace of $(\mathcal{A}, H(\mathcal{V}))$.
(2) $(\mathcal{A}, H(\mathcal{V}))$ is separated (use that for each $A \in \mathcal{A}, A = \bigcap_{V \in \mathcal{V}} V[A]$).
$(\mathcal{A}, H(\mathcal{V}))$ is called the *uniform hyperspace*. In contrast to the situation for the final hyperspace of a semiuniform convergence space, a) X is not dense in $(\mathcal{A}, H(\mathcal{V}))$ and b) $(\mathcal{A}, H(\mathcal{V}))$ is not complete, in general:

a) Let $(X, \mathcal{V})$ be the usual uniform space $\mathbb{R}_u$ of real numbers and A the closed unit interval $[0, 1]$. Then $A \in \mathcal{A}$ is not an adherent point of X: Choose $V = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < \frac{1}{4}\right\}$. Then $H(V)(A) = \{B \in \mathcal{A} : (A, B) \in H(V)\}$ is a neighborhood of A which does not contain any point of X , since there is no $x \in \mathbb{R}$ such that $A \subset V(x) = \{y \in X : (x, y) \in V\}$.
b) Even if $(X, \mathcal{V})$ is complete, $(\mathcal{A}, H(\mathcal{V}))$ need not be complete (cf. [10; 39.D.2.] or [4; II.50]).

On the other hand, the uniform hyperspace of a (completely) metrizable semiuniform convergence space [= (completely) metrizable uniform space] is (completely) metrizable (cf. [10; 36.E.2. and 39.D.1.], where the metrizability is realized by means of the Hausdorff metric.
J. Isbell [4] uses (separated) uniform spaces in the sense of W. J. Tukey [7] and defines on the set $\mathcal{A}$ of non-empty closed subsets of a separated uniform space (X, μ) (described by means of uniform covers) a uniformity as follows:

Put $H(\mathcal{U}) = \{(A,B) \in \mathcal{A} \times \mathcal{A} : A \subset St(B,\mathcal{U}) \text{ and } B \subset St(A,\mathcal{U})\}$ for each $\mathcal{U} \in \mu$. Then $\{H(\mathcal{U}) : \mathcal{U} \in \mu\}$ is a base for a separated uniformity in the sense of Weil. If $(X, \mathcal{V})$ is a separated uniform space in the sense of Weil and $(X, \mu_{\mathcal{V}})$ its corresponding uniform space in the sense of Tukey, then it is easily checked that $H(\mathcal{V})$ has the base $\{H(\mathcal{U}) : \mathcal{U} \in \mu_{\mathcal{V}}\}$, i.e. Bourbaki's and Isbell's definition of the uniform hyperspace coincide.
If $(X, \mathcal{V})$ is a separated uniform space and $(X, [\mathcal{V}])$ its corresponding semiuniform convergence space, i.e. $[\mathcal{V}] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{V}\}$, then $(\mathcal{A}, [H(\mathcal{V})])$ is a separated semiuniform convergence space containing $(X, [\mathcal{V}])$ as a subspace. Therefore, $[H(\mathcal{V})]$ is finer than $\mathcal{J}^f_{\mathcal{A}}$, i.e. $[H(\mathcal{V})] \subset \mathcal{J}^f_{\mathcal{A}}$.

2. Hyperspace structures for semiuniform limit spaces

2.1 Proposition. *A semiuniform convergence space $(X, \mathcal{J}_X)$ is a semiuniform limit space iff $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is a semiuniform limit space.*

Proof. "$\Leftarrow$". Since the construct **SULim** of semiuniform limit spaces is bireflective in **SUConv** (cf. [6; 2.3.2.3]), this implication is obvious.
"$\Rightarrow$". Let $\mathcal{F}, \mathcal{G} \in \mathcal{J}^f_{\mathcal{A}}$. Then 1. $(i \times i)^{-1}(\mathcal{F})$ exists and belongs to $\mathcal{J}_X$ or 2. $(i \times i)^{-1}(\mathcal{F})$ does not exist, and 3. $(i \times i)^{-1}(\mathcal{G})$ exists and belongs to $\mathcal{J}_X$ or 4. $(i \times i)^{-1}(\mathcal{G})$ does not exist. If "1. and 3." is valid, $(i \times i)^{-1}(\mathcal{F} \cap \mathcal{G}) = (i \times i)^{-1}(\mathcal{F}) \cap (i \times i)^{-1}(\mathcal{G})$ belongs to $\mathcal{J}_X$ by assumption, i.e. $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}^f_{\mathcal{A}}$. In case "1. and 4." is true, $(i \times i)^{-1}(\mathcal{F} \cap \mathcal{G}) = (i \times i)^{-1}(\mathcal{F}) \in \mathcal{J}_X$, i.e. $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}^f_{\mathcal{A}}$. Similarly, one concludes in case "2. and 3." is true. If "2. and 4." is satisfied, $(i \times i)^{-1}(\mathcal{F} \cap \mathcal{G})$ does not exist, i.e. $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}^f_{\mathcal{A}}$.

2.2 Corollary. *Let $(X, \mathcal{J}_X)$ be a semiuniform limit space. Then $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is the coarsest semiuniform limit space structure on $\mathcal{A}$ inducing $\mathcal{J}_X$.*

2.3 Proposition. *A semiuniform convergence space $(X, \mathcal{J}_X)$ is semi-pseudouniform iff $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is semi-pseudouniform.*

Proof. "$\Leftarrow$". By [6; 6.3.2.3) b)], **SPsU** is bireflective in **SUConv**.
"$\Rightarrow$". If $\mathcal{H}$ is a filter on $\mathcal{A} \times \mathcal{A}$ such that $\mathcal{H} \notin \mathcal{J}^f_{\mathcal{A}}$, then $(i \times i)^{-1}(\mathcal{H})$ exists and does not belong to $\mathcal{J}_X$. This implies that there is an ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \supset (i \times i)^{-1}(\mathcal{H})$ and $\mathcal{U} \notin \mathcal{J}_X$. Consequently, $i \times i\ (\mathcal{U}) \supset \mathcal{H}$ is an ultrafilter which does not belong to $\mathcal{J}^f_{\mathcal{A}}$, since $(i \times i)^{-1}(i \times i)(\mathcal{U}) = \mathcal{U} \notin \mathcal{J}_X$.

Thus, the assertion is proved.

2.4 Remark. The inclusion map $i : (X, \mathcal{J}_X) \to (\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ does not preserve the properties "uniform limit space" and "uniform space", e.g. if $\mathcal{C}$ denotes one of the constructs **ULim** or **Unif**, there is no coarsest $\mathcal{C}-$structure ξ on $\{0,1,2\}$ such that $(\{0,1,2\},\xi)$ contains the discrete uniform space D_2 with underlying set $\{0,1\}$ as a subspace (cf. [6; 3.2.7.2]).

2.5 Proposition. *A semiuniform convergence space* $(X, \mathcal{J}_X)$ *is semiuniform (=principal semiuniform limit space) iff* $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ *is semiuniform.*

Proof. "$\Leftarrow$". The construct **SUnif** of semiuniform spaces is bireflective in **SUConv** (cf. [6; 6.3.2]).
"$\Rightarrow$". By 2.1, $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is a semiuniform limit space. In order to prove that $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is principal, it must be verified that $\bigcap\limits_{\mathcal{H}\in\mathcal{J}^f_{\mathcal{A}}} \mathcal{H} \in \mathcal{J}^f_{\mathcal{A}}$: Since $\mathcal{J}_X = [\mathcal{G}]$, i.e. there is some $\mathcal{G} \in \mathcal{J}_X$ such that each $\mathcal{F} \in \mathcal{J}_X$ contains $\mathcal{G}$, and the trace of $\bigcap\limits_{\mathcal{H}\in\mathcal{J}^f_{\mathcal{A}}} \mathcal{H}$ on $X \times X$ exists, one obtains $(i\times i)^{-1}(\bigcap\limits_{\mathcal{H}\in\mathcal{J}^f_{\mathcal{A}}} \mathcal{H}) = (i\times i)^{-1}(\bigcap\{\mathcal{H}' \in \mathcal{J}^f_{\mathcal{A}} : (i\times i)^{-1}(\mathcal{H}')$ exists and belongs to $\mathcal{J}_X\} = \bigcap\{(i\times i)^{-1}(\mathcal{H}') : \mathcal{H}' \in \mathcal{J}^f_{\mathcal{A}}$ and $(i\times i)^{-1}(\mathcal{H}') \in \mathcal{J}_X\} \supset \mathcal{G}$, i.e. $(i\times i)^{-1}(\bigcap\limits_{\mathcal{H}\in\mathcal{J}^f_{\mathcal{A}}} \mathcal{H}) \in \mathcal{J}_X$ which implies

$$\bigcap_{\mathcal{H}\in\mathcal{J}^f_{\mathcal{A}}} \mathcal{H} \in \mathcal{J}^f_{\mathcal{A}}.$$

2.6 Proposition. *A semiuniform convergence space* $(X, \mathcal{J}_X)$ *is diagonal iff* $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ *is diagonal.*

Proof. "$\Leftarrow$". Since, obviously, the construct $\Delta-$**SUConv** of diagonal semiuniform convergence spaces is bireflective in **SUConv**, this implication is true.
"$\Rightarrow$". Since $(i\times i)^{-1}((\Delta_{\mathcal{A}})) = (\Delta_X) \in \mathcal{J}_X, (\Delta_{\mathcal{A}}) \in \mathcal{J}^f_{\mathcal{A}}$.

2.7 Remark. It follows from the above results that for each diagonal semiuniform limit space $(X, \mathcal{J}_X)$ the hyperspace $(\mathcal{A}, \mathcal{J}^f_{\mathcal{A}})$ is a diagonal semiuniform limit space. In order to obtain a hyperspace $(\mathcal{A}, \mathcal{J}^*_{\mathcal{A}})$ of $(X, \mathcal{J}_X)$ such that the inclusion map from $(X, \mathcal{J}_X)$ into $(\mathcal{A}, \mathcal{J}^*_{\mathcal{A}})$ preserves the properties "diagonal uniform limit space" and "uniform" some further definitions are necessary in the following.

2.8 Definition. 1) Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space. A non-empty set $\mathcal{B} \subset \mathcal{J}_X$ is called a *base* for $\mathcal{J}_X$ provided that each $\mathcal{F} \in \mathcal{J}_X$ contains some $\mathcal{H} \in \mathcal{B}$.
2) Let $(X, \mathcal{J}_X)$ be a diagonal semiuniform limit space. Then
$\mathcal{B}_c = \{\mathcal{F} \cap \mathcal{F}^{-1} \cap (\Delta_X) : \mathcal{F} \in \mathcal{J}_X\}$ is called the *canonical base* for $\mathcal{J}_X$.

2.9 Remarks. 1) Each filter $\mathcal{H} \in \mathcal{B}_c$ has a base consisting of symmetric elements containing the diagonal. In particular, each $\mathcal{H} \in \mathcal{B}_c$ contains the diagonal. Obviously, $\mathcal{B}_c = \{\mathcal{H} \in \mathcal{J}_X : \mathcal{H} = \mathcal{H}^{-1}$ and $\mathcal{H} \subset (\Delta_X)\}$.
2) If $(X, \mathcal{J}_X)$ is a diagonal semiuniform limit space, then for each $\mathcal{H} \in \mathcal{B}_c$, a filter $T(\mathcal{H})$ on $\mathcal{A} \times \mathcal{A}$ is defined which is generated by the filter base $\{T(\mathcal{H}, H) : H \in \mathcal{H}\}$ where $T(\mathcal{H}, H) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : A \subset H[B]$ and $B \subset H[A]\}$.

2.10 Theorem. *Let $(X, \mathcal{J}_X)$ be a diagonal semiuniform limit space. Then $\mathcal{J}_\mathcal{A}{}^* = \{\mathcal{K} \in F(\mathcal{A} \times \mathcal{A}) : \mathcal{K} \supset T(\mathcal{H})$ for some $\mathcal{H} \in \mathcal{B}_c\}$ is a diagonal semiuniform limit space structure on $\mathcal{A}$ inducing $\mathcal{J}_X$.*

Proof. 1) $\{T(\mathcal{H}, H) : H \in \mathcal{H}\}$ is a filter base for each $\mathcal{H} \in \mathcal{B}_c$:
a. $T(\mathcal{H}, H) \neq \phi$, since $\Delta_\mathcal{A} \subset T(\mathcal{H}, H)$.
b. If $H_1, H_2 \in \mathcal{H}$, then $T(\mathcal{H}, H_1 \cap H_2) \subset T(\mathcal{H}, H_1) \cap T(\mathcal{H}, H_2)$.
2) a) $(\Delta_\mathcal{A}) \in \mathcal{J}_\mathcal{A}{}^*$ follows from $T((\Delta_X)) \subset (\Delta_\mathcal{A})$.
 b) $\mathcal{J}_\mathcal{A}{}^*$ is closed under formation of superfilters.
 c) $\mathcal{K} \in \mathcal{J}_\mathcal{A}^f$ implies that there is some $\mathcal{H} \in \mathcal{B}_c$ such that $\mathcal{K} \supset T(\mathcal{H})$.
 Consequently, $\mathcal{K}^{-1} \supset (T(\mathcal{H}))^{-1} = T(\mathcal{H})$ (note: $T(\mathcal{H}, H) = (T(\mathcal{H}, H))^{-1}$ for each $H \in \mathcal{H}$), i.e. $\mathcal{K}^{-1} \in \mathcal{J}_\mathcal{A}{}^*$.
 d) Let $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{J}_\mathcal{A}{}^*$. Then $\mathcal{K}_1 \supset T(\mathcal{H}_1)$ and $\mathcal{K}_2 \supset T(\mathcal{H}_2)$ with $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{B}_c$. Hence, $T(\mathcal{H}_1 \cap \mathcal{H}_2) \subset T(\mathcal{H}_1) \cap T(\mathcal{H}_2) \subset \mathcal{K}_1 \cap \mathcal{K}_2$ which implies $\mathcal{K}_1 \cap \mathcal{K}_2 \in \mathcal{J}_\mathcal{A}{}^*$, since $\mathcal{H}_1 \cap \mathcal{H}_2 \in \mathcal{B}_c$.
3) a) For each $\mathcal{H} \in \mathcal{B}_c$, $(i \times i)^{-1}(T(\mathcal{H})) = \mathcal{H}$:
 By 2.9.1), $\mathcal{H}$ has a base $\mathcal{B}_\mathcal{H}$ consisting of symmetric elements of $\mathcal{H}$ (which contain the diagonal). Then $\{T(\mathcal{H}, B_\mathcal{H}) : B_\mathcal{H} \in \mathcal{B}_\mathcal{H}\}$ is a base of $T(\mathcal{H})$. Furthermore, $T(\mathcal{H}, B_\mathcal{H}) \cap (X \times X) = B_\mathcal{H}$, since $B_\mathcal{H}$ is symmetric. Thus, $(i \times i)^{-1}(T(\mathcal{H}))$ and $\mathcal{H}$ have the same base $\mathcal{B}_\mathcal{H}$, i.e. a) is proved.

b) $\mathcal{J}_{\mathcal{A}}{}^*$ induces $\mathcal{J}_X$, i.e. $(X, \mathcal{J}_X)$ is a subspace (in **SUConv**) of $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$:

α) Let $\mathcal{F} \in \mathcal{J}_X$. Then there is some $\mathcal{H} \in \mathcal{B}_c$ such that $\mathcal{F} \supset \mathcal{H}$. Using a), $(i\times i)(\mathcal{F}) \supset (i\times i)(\mathcal{H}) = (i\times i)((i\times i)^{-1}(T(\mathcal{H}))) \supset T(\mathcal{H})$, i.e. $(i\times i)(\mathcal{F}) \in \mathcal{J}_{\mathcal{A}}{}^*$.

β) Let $\mathcal{F} \in F(X\times X)$ such that $(i\times i)(\mathcal{F}) \in \mathcal{J}_{\mathcal{A}}{}^*$, i.e. there is some $\mathcal{H} \in \mathcal{B}_c$ with $(i\times i)(\mathcal{F}) \supset T(\mathcal{H})$. Hence, $\mathcal{F} = (i\times i)^{-1}((i\times i)(\mathcal{F})) \supset (i\times i)^{-1}(T(\mathcal{H})) = \mathcal{H}$, which implies $\mathcal{F} \in \mathcal{J}_X$.

2.11 Corollary. *Let $(X, \mathcal{J}_X)$ be a diagonal semiuniform limit space. Then $\mathcal{J}_{\mathcal{A}}{}^* \subset \mathcal{J}_{\mathcal{A}}^f$, i.e. $\mathcal{J}_{\mathcal{A}}{}^*$ is finer than $\mathcal{J}_{\mathcal{A}}^f$.*

2.12 Corollary. *A semiuniform convergence space $(X, \mathcal{J}_X)$ is a Cook-Fischer space (=diagonal uniform limit space) iff $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$ is a Cook-Fischer space.*

Proof. "$\Leftarrow$". Since the property "Cook-Fischer space" is hereditary, this implication is obvious.
"$\Rightarrow$". Let $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{J}_{\mathcal{A}}{}^*$ such that $\mathcal{K}_1 \circ \mathcal{K}_2$ exists. Then there are $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{B}_c$ with $T(\mathcal{H}_i) \subset \mathcal{K}_i$ for each $i \in \{1,2\}$. In particular, $T(\mathcal{H}_1)\circ T(\mathcal{H}_2) \subset \mathcal{K}_1 \circ \mathcal{K}_2$, and because of $T((\mathcal{H}_2\circ\mathcal{H}_1)\cap(\mathcal{H}_1\circ\mathcal{H}_2)) \subset T(\mathcal{H}_1)\circ T(\mathcal{H}_2)$ the assertion follows since $(\mathcal{H}_2\circ\mathcal{H}_1)\cap(\mathcal{H}_1\circ\mathcal{H}_2) \in \mathcal{B}_c$ (note that the composition of elements of $\mathcal{B}_c$ always exists).

2.13. Corollary. *A semiuniform convergence space $(X, \mathcal{J}_X)$ is semiuniform (uniform) iff $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$ is semiuniform (uniform).*

Proof. "$\Leftarrow$". The property "semiuniform" ("uniform") is hereditary.
"$\Rightarrow$". a) If $(X, \mathcal{J}_X)$ is semiuniform, i.e. a principal semiuniform limit space, then $(X, \mathcal{J}_X)$ is diagonal. In order to prove that $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$ is a principal (diagonal) semiuniform limit space, it suffices, by 2.10., to verify that $\mathcal{J}_{\mathcal{A}}{}^*$ is generated by a single element of $\mathcal{J}_{\mathcal{A}}{}^*$. By assumption, there is some $\mathcal{H} \in \mathcal{J}_X$ such that $\mathcal{J}_X = [\mathcal{H}]$. Since $\mathcal{H} = \bigcap_{\mathcal{F}\in\mathcal{J}_X} \mathcal{F} \in \mathcal{J}_X$ is a semiuniformity, $\mathcal{H} \in \mathcal{B}_c$, and $[T(\mathcal{H})] = \mathcal{J}_{\mathcal{A}}{}^*$ (namely, if $\mathcal{K} \in \mathcal{J}_{\mathcal{A}}{}^*$, there is some $\mathcal{H}' \in \mathcal{B}_c \subset \mathcal{J}_X$ with $\mathcal{K} \supset T(\mathcal{H}')$; hence, $\mathcal{H} \subset \mathcal{H}'$ and $T(\mathcal{H}) \subset T(\mathcal{H}')$, i.e. $\mathcal{K} \in [T(\mathcal{H})]$).
b) If $(X, \mathcal{J}_X)$ is uniform, i.e. a principal uniform limit space, then $(X, \mathcal{J}_X)$ is diagonal, and by 2.12, $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$ is a diagonal uniform limit space. Thus by a), $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}{}^*)$ is principal, i.e. a uniform space.

2.14 Remark. If $(X, \mathcal{J}_X)$ is a separated Cook-Fischer space and $\mathcal{A}$ is the set of all non-empty compact subsets of $(X, \mathcal{J}_X)$, then $(\mathcal{A}, \mathcal{J}_{\mathcal{A}}^*)$ is the hyperspace constructed by R. J. Gazik [2]. In case $(X, \mathcal{J}_X)$ is a separated uniform space, i.e. $\mathcal{J}_X = [\mathcal{V}]$ where $\mathcal{V}$ is a separated uniformity on X, and $\mathcal{A}$ is the set of all non-empty closed subsets of $(X, \mathcal{J}_X)$, then $\mathcal{J}_{\mathcal{A}}^* = [T(\mathcal{V})]$, and $(\mathcal{A}, T(\mathcal{V}))$ is Bourbaki's uniform hyperspace, i.e. $T(\mathcal{V}) = H(\mathcal{V})$ (cf. 1.15).

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