A STONE-WEIERSTRASS TYPE THEOREM FOR SEMIUNIFORM CONVERGENCE SPACES

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Abstract. A Stone-Weierstraß type theorem for semiuniform convergence spaces is proved. It implies the classical Stone-Weierstraß theorem as well as a Stone-Weierstraß type theorem for filter spaces due to Bentley, Hušek and Lowen-Colebunders [1].

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0. Introduction

In 1885 K. Weierstraß [8] proved his approximation theorem. M. H. Stone [7] proved the so-called Stone-Weierstraß theorem in 1937. A reformulation of the latter one can be found in Gillman-Jerison's book [4]. Furthermore, a Stone-Weierstraß type theorem for proximity spaces is proved in Čech's book [2], where a proximity space there is more general than an Efremovič proximity space (cf.[3]). It is well-known that the construct of Efremovič proximity spaces (and proximally continuous maps) is concretely isomorphic to the construct of precompact (=totally bounded) uniform spaces (and uniformly continuous maps). In 2000, H. L. Bentley, M. Hušek and E. Lowen-Colebunders [1] established a Stone-Weierstraß type theorem for an unstructured set using the above mentioned Stone-Weierstraß theorem for Efremovič proximity spaces. Indeed, they showed that both theorems are equivalent.

Here their Stone-Weierstraß type theorem for an unstructured set is used in order to derive a Stone-Weierstraß type theorem for semiuniform convergence spaces. Semiuniform convergence spaces play an essential role in Convenient Topology since they form a strong topological universe in which topological and uniform concepts are available. Additionally, the construct **Fil** of filter spaces (and Cauchy continuous maps) can be nicely embedded into the construct **SUConv** of semiuniform convergence spaces (and uniformly continuous maps) (cf. [6] for more detailed information).

From the Stone-Weierstraß type theorem for semiuniform convergence spaces a corresponding theorem for filter spaces due to Bentley/Hušek/Lowen-Colebunders [1] can be derived as well as the classical Stone-Weierstraß theorem (as formulated by Gillman-Jerison).

Finally, concerning Stone-Weierstraß type theorems it turns out that the semiuniform convergence space case, the filter space case and the unstructured set case are equivalent.
The terminology of this paper corresponds to [6].

1. Preliminaries

For each set X, let $F(X)$ be the set of all real-valued functions on X endowed with the uniformity of uniform convergence. In the following all subsets of $F(X)$ are assumed to be endowed with the subspace uniformity of this uniformity. In particular the subspace $F^*(X)$ of all bounded real-valued functions on X is metrizable by means of the metric d defined by

$$d(f,g) = \sup\{|\, f(x) - g(x)\,| : x \in X\}.$$

(Let $\mathbb{R}_u$ be the usual uniform space of real numbers, i.e. $\{V_\varepsilon : \varepsilon > 0\}$ is a base for its uniformity $\mathcal{U}$, where $V_\varepsilon = \{(x,y) : |\, x-y\,| < \varepsilon\}$. Then $\{W(V_\varepsilon) : \varepsilon > 0\}$ is a base for the uniformity of uniform convergence on $F(X)$, where $W(V_\varepsilon) = \{(f,g) : (f(x), g(x)) \in V_\varepsilon\}$, and $\mathcal{B} = \{W(V_\varepsilon) \cap (F^*(X) \times F^*(X)) : \varepsilon > 0\}$ is a base for the uniformity of $F^*(X)$. Thus,
(1) $\mathcal{B}' = \{\{(f,g) \in F^*(X) \times F^*(X) : |\, f(x) - g(x)\,| \leq \varepsilon \text{ for each } x \in X\} : \varepsilon > 0\}$
is also a base for this uniformity, whereas
(2) $\mathcal{B}'' = \{\{(f,g) \in F^*(X) \times F^*(X) : d(f,g) \leq \varepsilon\} : \varepsilon > 0\}$
is a base for the uniformity induced by d. Since for each $(f,g) \in F^*(X) \times F^*(X)$,

$$d(f,g) \leq \varepsilon \text{ iff } |\, f(x) - g(x)\,| \leq \varepsilon \text{ for each } x \in X,$$

it follows from (1) and (2) that the uniformity of $F^*(X)$ coincides with the uniformity induced by d.).
$F(X)$ may also be regarded as an algebra over the field $\mathbb{R}$ of real numbers containing a unit element different from zero, namely the constant function $\overline{1} : X \to \mathbb{R}$, defined by $\overline{1}(x) = 1$ for each $x \in X$. In the following subalgebras of $F(X)$ are also assumed to contain $\overline{1}$ (and thus all constant functions), e.g. $F^*(X)$ is a subalgebra of $F(X)$.

H. L. Bentley, M. Hušek and E. Lowen-Colebunders [1] have proved the following Stone-Weierstraß type theorem for an unstructured set:

1.1 Theorem. *Let X be a set, let $\mathcal{B}$ be a subalgebra of $F^*(X)$, and let $f \in F^*(X)$. Then f belongs to the closure $\mathrm{cl}_{F^*(X)}\mathcal{B}$ of $\mathcal{B}$ in $F^*(X)$ iff whenever $\mathcal{F}$ is a filter on X such that $g(\mathcal{F})$ converges for every $g \in \mathcal{B}$ then $f(\mathcal{F})$ converges too.*

In the realm of semiuniform convergence spaces we do not make a notational distinction between the usual uniform space $\mathbb{R}_u$ of real numbers and its corresponding semiuniform convergence space whose uniform filters are exactly those filters on $\mathbb{R} \times \mathbb{R}$ containing the usual uniformity $\mathcal{U}$ of $\mathbb{R}$.

1.2 Proposition. *Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space and let $f \in F(X)$. Then the following are equivalent:*
(1) $f : (X, \mathcal{J}_X) \to \mathbb{R}_u$ is uniformly continuous.
(2) For each $\mathcal{F} \in \mathcal{J}_X$ the following is satisfied: For each $\varepsilon > 0$ there is some $F \in \mathcal{F}$ such that $|f(x) - f(y)| < \varepsilon$ for all $(x, y) \in F$, i.e. $f \times f[F] \subset V_\varepsilon$.
(3) $(f \times f)^{-1}[U] \in \bigcap_{\mathcal{F} \in \mathcal{J}_X} \mathcal{F}$ for each $U \in \mathcal{U}$.

1.3 Lemma. *Let $\mathbf{X} = (X, \mathcal{J}_X)$ be a semiuniform convergence space and let $U(\mathbf{X})$ be the set of all uniformly continuous maps from $\mathbf{X}$ into $\mathbb{R}_u$. Then $U(\mathbf{X})$ is closed in $F(X)$.*

Proof. In order to prove that $F(X) \setminus U(\mathbf{X})$ is open, let $f \in F(X) \setminus U(\mathbf{X})$. Thus, there is some $U \in \mathcal{U}$ such that $(f \times f)^{-1}[U] \notin \bigcap_{\mathcal{F} \in \mathcal{J}_X} \mathcal{F}$. Furthermore, there is some symmetric $V \in \mathcal{U}$ with $V^3 \subset U$. If $g \in W(V)(f)$, i.e. $(f(x), g(x)) \in V$ for each $x \in X$, then $(g \times g)^{-1}[V] \subset (f \times f)^{-1}[U]$, and consequently, $(g \times g)^{-1}[V] \notin \bigcap_{\mathcal{F} \in \mathcal{J}_X} \mathcal{F}$, i.e. g is not uniformly continuous. Hence, $W(V)(f) \subset F(X) \setminus U(\mathbf{X})$.

1.4 Corollary. *The set $U^*(\mathbf{X})$ of all bounded uniformly continuous maps from a semiuniform convergence space $\mathbf{X} = (X, \mathcal{J}_X)$ into $\mathbb{R}_u$ may be regarded as a subalgebra of $F^*(X)$, which is closed in the subspace $F^*(X)$ of $F(X)$.*
Proof. Let $f, g : \mathbf{X} \to \mathbb{R}_u$ be bounded uniformly continuous maps from a

semiuniform uniform convergence space $\mathbf{X}$ into $\mathbb{R}_u$ and let $\lambda \in \mathbb{R}$. Using 1.2.(2), $f+g, f \cdot g, \lambda f$ and $\overline{1}$ belong to $U^*(\mathbf{X})$, i.e. $U^*(\mathbf{X})$ is a subalgebra of $F^*(X)$. By the preceeding lemma, $cl_{F(X)}U(\mathbf{X}) = U(\mathbf{X})$. Furthermore,

$$U^*(\mathbf{X}) \subset cl_{F^*(X)}U^*(\mathbf{X}) = (cl_{F(X)}U^*(\mathbf{X})) \cap F^*(X)$$
$$\subset U(\mathbf{X}) \cap F^*(X) = U^*(\mathbf{X}),$$

since $cl_{F(X)}U^*(\mathbf{X}) \subset cl_{F(X)}U(\mathbf{X}) = U(\mathbf{X})$. Thus, $U^*(\mathbf{X}) = cl_{F^*(x)}U^*(\mathbf{X})$.

2. The Main Result

2.1 Theorem. *Let $\mathbf{X} = (X, \mathcal{J}_X)$ be a semiuniform convergence space, let $\mathcal{B}$ be a subalgebra of $U^*(\mathbf{X})$, and let $f \in U^*(\mathbf{X})$. Then $f \in cl_{U^*(\mathbf{X})}\mathcal{B}$ iff the following is satisfied: Whenever $\mathcal{F}$ is a filter on X such that $g(\mathcal{F})$ converges for each $g \in \mathcal{F}$, then $f(\mathcal{F})$ converges too.*

Proof. $cl_{U^*(\mathbf{X})}\mathcal{B} = (cl_{F^*(X)}\mathcal{F}) \cap U^*(\mathbf{X}) = cl_{F^*(X)}\mathcal{B}$ since $cl_{F^*(X)}\mathcal{B} \subset cl_{F^*(X)}U^*(\mathbf{X}) = U^*(\mathbf{X})$ (cf. 1.4). By 1.1, $f \in cl_{F^*(X)}\mathcal{B}$ iff the condition in 2.1 is fulfilled.

2.2 Corollary. *Let $\mathbf{X} = (X, \mathcal{J}_X)$ be a semiuniform convergence space, and let $\mathcal{B}$ be a subalgebra of $\mathcal{U}^*(\mathbf{X})$ such that $\mathcal{J}_X$ is the initial* **SUConv***-structure w.r.t. $(g)_{g\in\mathcal{B}}$. Then $\mathcal{B}$ is dense in $U^*(\mathbf{X})$.*

Proof. Since $(g : (X, \mathcal{J}_X) \to (\mathbb{R}, [\mathcal{U}]^g))_{g\in\mathcal{B}}$ with $[\mathcal{U}]^g = [\mathcal{U}]$ for each $g \in \mathcal{B}$ is an initial source in **SUConv**, $(g : (X, \gamma_{\mathcal{J}_X}) \to (\mathbb{R}, \gamma_{[\mathcal{U}]^g}))_{g\in\mathcal{B}}$ is an initial source in **Fil** (cf. [6; 2.3.3.17]) where $\gamma_{[\mathcal{U}]^g}$ is the set of all Cauchy filter in $\mathbb{R}_u$, i.e. the set of all convergent filters in the usual topological space $\mathbb{R}_t$ (in other words: $(\mathbb{R}, \gamma_{[\mathcal{U}]^g})$ is the space $\mathbb{R}_t$ regarded as a filter space.). Thus, the condition in 2.1 means that $f : (X, \gamma_{\mathcal{J}_X}) \to \mathbb{R}_t$ is Cauchy continuous. Since this is true for each $f \in U^*(\mathbf{X})$, $f \in cl_{U^*(\mathbf{X})}\mathcal{B}$.

2.3 Remarks. 1) The semiuniform convergence $\mathbf{X}$ in 2.2 is a uniform space, since **Unif** is bireflective in **SUConv**.
2) Let A be a compact subset of the usual topological space $\mathbb{R}_t$ of real numbers, e.g. $A = [0, 1]$ (=closed unit interval). Since there is a unique uniformity on A which induces the topology of A (i.e. the topology induced by the Euclidean metric on $\mathbb{R}$), A may be regarded as a (uniform) subspace of $\mathbb{R}_u$ denoted

by A_u. In particular, the inclusion map $i : A_u \to \mathbb{R}_u$ is uniformly continuous, and the uniformity of A_u is the coarsest uniformity such that i is uniformly continuous, i.e. the initial uniformity w.r.t. $i : A \to \mathbb{R}_u$. If $\mathbb{R}_u$ and A_u are considered to be semiuniform convergence spaces, the semiuniform convergence structure of A_u is the initial **SUConv**-structure w.r.t. $i : A \to \mathbb{R}_u$ (**Unif** is bireflectively embedded in **SUConv**).
The smallest subalgebra $\mathcal{B}$ of $U^*(A_u)$ containing $i : A_u \to \mathbb{R}_u$ and $\overline{1} : A_u \to \mathbb{R}_u$ is the algebra of all real-valued polynomial functions on A_u, and it generates A_u initially, i.e. the **SUConv**-structure of A_u is initial w.r.t. $\mathcal{B}$ (since it is already initial w.r.t. $\{i\} \subset \mathcal{B}$). Let the topological subspace of $\mathbb{R}_t$ determined by A be denoted by A_t, and let $C(A_t)$ be the set of all continuous maps from A_t to $\mathbb{R}_t$, then

$$U^*(A_u) = U(A_u) = C(A_t),$$

since A is compact. By 2.2, for each continuous map $f : A_t \to \mathbb{R}_t$, there is a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomial functions converging uniformly to f. This is *the classical Weierstraß approximation theorem.*

2.4 Proposition. *Let* $\mathbf{X} = (X, \mathcal{J}_X)$ *be a precompact semiuniform convergence space. Then each uniformly continuous map* $f : \mathbf{X} \to \mathbb{R}_u$ *is bounded, i.e.* $\mathbf{U}^*(\mathbf{X}) = U(\mathbf{X})$.

Proof. Since $\mathbf{X}$ is precompact and $f \in U(\mathbf{X}), f[X] \subset \mathbb{R}_u$ is precompact (=totally bounded), i.e. $f[X]$ is bounded.

2.5 Corollary (*Weierstraß theorem for bounded sets*). *Let* $A \subset \mathbb{R}$ *be bounded, and let* $\mathbf{A}$ *be the subspace (in* **SUConv***) of* $\mathbb{R}_u$*. Then the algebra of all real-valued polynomial functions on* $\mathbf{A}$ *is dense in* $U(\mathbf{A})$*, i.e. for each uniformly continuous map* $f : \mathbf{A} \to \mathbb{R}_u$*, there is a sequence* $(P_n)_{n\in\mathbb{N}}$ *of polynomial functions on* $\mathbf{A}$ *converging uniformly to* f*.*

Proof. By assumption, $\mathbf{A}$ is a precompact semiuniform convergence space because $\mathbf{A}$ is metrizable by means of the metric induced by the Euclidean metric on $\mathbb{R}$, and a subset of $\mathbb{R}$ is bounded iff it is totally bounded (=precompact) (cf. e.g. [5; 4.1.12]). By 2.4, $U^*(\mathbf{A}) = U(\mathbf{A})$. Since $\mathbf{A}$ is initially generated by $i : A \to \mathbb{R}_u$, it is also initially generated by $\mathcal{B}$. By 2.2, $\mathcal{B}$ is dense in $U(\mathbf{A})$.

3. Gillman-Jerison's Version of the Stone-Weierstraß Theorem

3.1 Theorem ([4; 16.4]). *Let* $\mathbf{X} = (X, \mathcal{X})$ *be a compact Hausdorff space, and let* $\mathcal{B}$ *be a subalgebra of the algebra* $C(\mathbf{X})$ *of all continuous maps from* $\mathbf{X}$ *into* $\mathbb{R}_t$. *If* $f \in C(\mathbf{X})$, *then* $f \in cl_{C(\mathbf{X})}\mathcal{B}$ *iff the following is satisfied: For each* $S \subset X$ *such that* $g\,|\,S$ *is constant for each* $g \in \mathcal{B}$, $f\,|\,S$ *is constant.*

Proof. Let $\mathcal{V}$ be the unique uniformity inducing $\mathcal{X}$. Since $(X, \mathcal{X})$ is compact, $U^*((X, [\mathcal{V}])) = U((X, [\mathcal{V}])) = C(\mathbf{X})$. Now let us apply 2.1:
"$\Rightarrow$". By assumption, $g\,|\,S$ is constant for each $g \in \mathcal{B}$, i.e. for each $g \in \mathcal{B}$ there is some $x_g \in \mathbb{R}$ such that $g[S] = \{x_g\}$. If (S) denotes the filter generated by S, then $g((S)) = (g[S]) = \dot{x}_g$ converges to x_g for each $g \in \mathcal{B}$ which implies that $f((S))$ converges to some $x \in \mathbb{R}$. Let $s_1, s_2 \in S$. Obviously, $f(\dot{s}_i) \supset f((S))$ converges to x for each $i \in \{1, 2\}$, and by continuity of f, it converges also to $f(s_i)$. Since filter convergence in $\mathbb{R}_t$ is unique, $x = f(s_1) = f(s_2)$, i.e. $f\,|\,S$ is constant.
"$\Leftarrow$". Let $\mathcal{F}$ be a filter on X such that $g(\mathcal{F})$ converges for each $g \in \mathcal{B}$. The set S of all adherence points of $\mathcal{F}$ is non-empty, since $\mathbf{X}$ is compact, and $g\,|\,S$ is constant for each $g \in \mathcal{B}$ because $g(\mathcal{F})$ converges. Consequently, $f\,|\,S$ is constant, i.e. $f[S] = \{x\}$ for some $x \in \mathbb{R}$. In order to prove that $f(\mathcal{F})$ converges to x, let V be an open neighborhood of x. Then $f^{-1}[V] \supset S$ is open in $\mathbf{X}$, i.e. $f^{-1}[V] \in \mathcal{U}(s) = \bigcap\{\mathcal{H} : \mathcal{H}$ is a filter on X converging to s in $\mathbf{X}\}$ for each $s \in S$. Since $\mathbf{X}$ is compact each ultrafilter $\mathcal{U}$ containing $\mathcal{F}$ converges to some $s \in S$. Thus, $f^{-1}[V] \in \bigcap\{\mathcal{U} : \mathcal{U}$ is an ultrafilter on X with $\mathcal{U} \supset \mathcal{F}\} = \mathcal{F}$. Hence, $V \in f(\mathcal{F})$.

3.2 Corollary (*Stone's Theorem* [7]). *Let* $\mathbf{X} = (X, \mathcal{X})$ *be a compact Hausdorff space, and let* $\mathcal{B}$ *be a subalgebra of* $C(\mathbf{X})$. *Then* $\mathcal{B}$ *is dense in* $C(\mathbf{X})$ *iff* $\mathcal{B}$ *separates points of* $\mathbf{X}$.

Proof. If $\mathcal{B}$ is dense in $C(\mathbf{X})$, each $f \in C(\mathbf{X})$ fulfills the condition in the above theorem. Since $\mathbf{X}$ is compact, $C(\mathbf{X})$ separates points of $\mathbf{X}$. Let $x, y \in X$ such that $x \neq y$, and put $S = \{x, y\}$. Then there is some $f \in C(\mathbf{X})$ such that $f\,|\,S$ is non-constant, which implies that there is some $g \in \mathcal{B}$ such that $g\,|\,S$ is non-constant, i.e. $\mathcal{B}$ separates points of $\mathbf{X}$. Conversely, let $\mathcal{B}$ separate points of $\mathbf{X}$. Then any non-empty subset S of X such that $g\,|\,S$ is constant for all $g \in \mathcal{B}$ is a singleton, which implies that for each $f \in C(\mathbf{X}), f\,|\,S$ is constant. By 3.1, $\mathcal{B}$ is dense in $C(\mathbf{X})$.

4. A Stone-Weierstraß Type Theorem for Filter Spaces (and Cauchy

Spaces)

The construct **Fil** of filter spaces (and Cauchy continuous maps) is concretely isomorphic to the construct **Fil-D-SUConv** of **Fil**-determined semiuniform convergence spaces (and uniformly continuous maps) (cf. [6; 2.3.3.5]). In the following $\mathbb{R}_t$ is regarded as a **Fil**-determined semiuniform convergence space (or a filter space), i.e. the Cauchy filters in $\mathbb{R}_t$ are exactly the convergent filters. If $\mathbf{X} = (X, \mathcal{J}_X) \in|$**Fil-D-SUConv**$|$, then the set of all bounded Cauchy continuous maps between $\mathbf{X}$ and $\mathbb{R}_t$ is denoted by $\Gamma^*(\mathbf{X})$. The same notation is used if $\mathbf{X}$ is a filter space. Since $\Gamma^*(\mathbf{X}) = U^*(\mathbf{X})$ (cf. [6; 2.3.3.25.2)]), our main theorem 2.1 can be applied in order to obtain the following theorem.

4.1 Theorem ([1; theorem 3]). *Let* $\mathbf{X}$ *be a filter space* (*or a* **Fil**-*determined semiuniform convergence space*), *let* $\mathcal{B}$ *be a subalgebra of* $\Gamma^*(\mathbf{X})$, *and let* $f \in \Gamma^*(\mathbf{X})$. *Then* $f \in cl_{\Gamma^*(\mathbf{X})}\mathcal{B}$ *iff the following is satisfied: Whenever* $\mathcal{F}$ *is a filter on* X *such that* $g(\mathcal{F})$ *converges for each* $g \in \mathcal{B}$, *then* $f(\mathcal{F})$ *converges.*

Since **Fil-D-SUConv** ($\cong$ **Fil**) is bireflective in **SUConv**, initial structures in **Fil-D-SUConv** (or in **Fil**) are formed as in **SUConv**. Applying corollary 2.2 one obtains the following corollary.

4.2 Corollary ([1; theorem 4]). *Let* $\mathbf{X}$ *be a filter space* (*resp. a* **Fil**-*determined semiuniform convergence space*), *and let* $\mathcal{B}$ *be a subalgebra of* $\Gamma^*(\mathbf{X})$ *initially generating the* **Fil**-*structure* (*resp. the* **Fil-D-SUConv**-*structure*) *of* $\mathbf{X}$. *Then* $\mathcal{B}$ *is dense in* $\Gamma^*(\mathbf{X})$.

4.3 Remarks. 1) Bentley, Hušek and Lowen-Colebunders [1] have observed that 4.2 follows also from 4.1, since the condition characterizing $f \in cl_{\Gamma^*(\mathbf{X})}\mathcal{B}$ in 4.1 means exactly that $f : \mathbf{X} \to \mathbb{R}_t$ is Cauchy continuous provided that $\mathbf{X}$ is initially generated by $\mathcal{B}$.
2) The above results 4.1 and 4.2 can be specialized to Cauchy spaces (concerning 4.2 note that the construct **Chy** of Cauchy spaces [and Cauchy continuous maps] is bireflective in **Fil** which implies that initial structures in **Chy** are formed as in **Fil**) (cf. [1; theorem 6 and theorem 7).

5. A Stone-Weierstraß Theorem for an Unstructured Set

5.1 Theorem (=Theorem 1.1). *Let X be a set, let $\mathcal{B}$ be a subalgebra of $F^*(X)$, and let $f \in F^*(X)$. Then $f \in cl_{F^*(X)}\mathcal{B}$ iff the following is satisfied: Whenever $\mathcal{F}$ is filter on X such that $g(\mathcal{F})$ converges for every $g \in \mathcal{B}$ then $f(\mathcal{F})$ converges too.*

Proof. A) Endow X with the initial **Fil**-structure γ w.r.t. $\mathcal{B}$, where each $g \in \mathcal{B}$ is regarded as a map from X into $\mathbb{R}_t$, i.e. the reals carrying the usual **Fil**-structure (cf. 4.). Then $\mathcal{B}$ is a subalgebra of $\Gamma^*(\mathbf{X})$, where $\mathbf{X} = (X, \gamma)$. By means of 4.2, $\mathcal{B}$ is dense in $\Gamma^*(\mathbf{X})$.
B) Now the above theorem can be proved:
a) "$\Leftarrow$" (indirect). Let $f \in F^*(X)$ such that $f \notin cl_{F^*(X)}\mathcal{B}$. By 1.4, $\Gamma^*(\mathbf{X})$ is closed in $F^*(X)$, which implies $cl_{F^*(X)}\mathcal{B} = cl_{\Gamma^*(\mathbf{X})}\mathcal{B}$. Since, by A), $cl_{\Gamma^*(\mathbf{X})}\mathcal{B} = \Gamma^*(\mathbf{X})$, it follows $cl_{F^*(X)}\mathcal{B} = \Gamma^*(\mathbf{X})$. Thus, $f \notin \Gamma^*(\mathbf{X})$, i.e. $f : \mathbf{X} \to \mathbb{R}_t$ is not Cauchy continuous. Hence, there is a Cauchy filter $\mathcal{F}$ on X such that $f(\mathcal{F})$ does not converge. Since $\mathcal{F}$ is a Cauchy filter, $g(\mathcal{F})$ converges for each $g \in \mathcal{B} \subset \Gamma^*(\mathbf{X})$. Consequently, the condition in 5.1 is not fulfilled.
b) "$\Rightarrow$". Let $f \in cl_{F^*(X)}\mathcal{B} = \Gamma^*(\mathbf{X})$ (cf. a)), and let $\mathcal{F}$ be a filter on X such that $g(\mathcal{F})$ converges for each $g \in \mathcal{B}$. Then $\mathcal{F}$ is a Cauchy filter on $\mathbf{X}$. Since f is Cauchy continuous, $f(\mathcal{F})$ is a Cauchy filter on $\mathbb{R}_t$, i.e. $f(\mathcal{F})$ converges.

6. Final Remark

In this paper the following implications have been proved:

$$\begin{tikzcd}
1.1 \arrow[r, Rightarrow] & 2.1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & 2.2 \arrow[r, Rightarrow] & 4.2 \arrow[r, Rightarrow] & 5.1(=1.1) \\
 & & 4.1 \arrow[ur, Rightarrow] & & \\
 & 3.1 \arrow[d, Rightarrow] & & & \\
 & 3.2 & & &
\end{tikzcd}$$

Thus, the statements 1.1, 2.1, 2.2, 4.1 and 4.2 are equivalent.

References

[1] Bentley, H. L., Hušek, M. and E. Lowen-Colebunders: 2000, 'A Stone-Weierstraß type theorem for an unstructured set - with applications', *Applied Categorical Structures* **8**, 425-434.

[2] Čech, E.: 1966, *Topological Spaces*, Revised edition by M. Frolik and M. Katětov, Interscience, London.

[3] Efremovič, V. A.: 1952, 'The geometry of proximity', Mat. Sbornik **31** (**73**), 189-200.

[4] Gillman, L. and M. Jerison: 1960, *Rings of Continuous Functions*, van Nostrand, Princeton.

[5] Herrlich, H.: 1986, *Einführung in die Topologie*, Heldermann, Berlin.

[6] Preuß, G.: 2002, *Foundations of Topology*, Kluwer, Dordrecht.

[7] Stone, M. H.: 1937, 'Applications of the theory of Boolean rings to general topology', *Trans. Amer. Math. Soc.* 41, 373-481.

[8] Weierstraß, K.: 1885, 'Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen reeller Argumente', S. B. Deutsche Akad. Wiss. Berlin KL. Math. Phys. Tech., 633-639, 789-805.