UNCONDITIONALLY CONVERGENT SERIES OF OPERATORS AND NARROW OPERATORS ON L_1

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ABSTRACT. We introduce a class of operators on L_1 that is stable under taking sums of pointwise unconditionally convergent series, contains all compact operators and does not contain isomorphic embeddings. It follows that any operator from L_1 into a space with an unconditional basis belongs to this class.

1. INTRODUCTION

A famous theorem due to A. Pełczyński [7] states that $L_1[0,1]$ cannot be embedded in a space with an unconditional basis. A somewhat stronger version is also true [4]: If an operator $J\colon L_1[0,1] \to X$ is bounded from below, then it cannot be represented as a pointwise unconditionally convergent series of compact operators. This last theorem in fact also holds for embedding operators $J\colon E \to X$ if E has the Daugavet property; see [5].

We wish to rephrase the theorem using the following definition.

Definition 1.1. Let $\mathscr{U}$ be a linear subspace of $\mathscr{L}(E,X)$, the space of bounded linear operators from E into X. By $\operatorname{unc}(\mathscr{U})$ we denote the set of all operators which can be represented by pointwise unconditionally convergent series of operators from $\mathscr{U}$.

In terms of this definition the above theorem says that an isomorphic embedding operator $J\colon L_1[0,1] \to X$ does not belong to $\operatorname{unc}(\mathscr{K}(L_1[0,1],X))$, where $\mathscr{K}(E,X)$ stands for the space of compact operators from E into X.

Clearly, one can iterate the operation "unc" and consider the classes

$$\operatorname{unc}(\operatorname{unc}(\mathscr{K}(L_1[0,1],X))), \quad \operatorname{unc}(\operatorname{unc}(\operatorname{unc}(\mathscr{K}(L_1[0,1],X)))),$$

etc. Thus the question arises whether one can obtain an isomorphic embedding operator through such a chain of iterations; indeed it is not clear at the outset whether possibly $\operatorname{unc}(\operatorname{unc}(\mathscr{K}(E,X))) = \operatorname{unc}(\mathscr{K}(E,X))$.

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A natural approach to generalise Pełczyński's theorem in this direction is to find a large class of operators $T\colon L_1[0,1] \to X$ which is stable under taking sums of pointwise unconditionally convergent series, contains all compact operators and does not contain isomorphic embeddings.

It was shown by R. Shvidkoy in his Ph.D. Thesis [11] and independently in [3] that in the case $X = L_1[0,1]$, the PP-narrow operators on $L_1[0,1]$ form such a class. Here is the definition.

Let (Ω, Σ, μ) be a fixed nonatomic probability space and $L_p = L_p(\Omega, \Sigma, \mu)$. By Σ^+ we denote the collection of all measurable subsets of Ω having nonzero measure.

Definition 1.2. Let $A \in \Sigma^+$.

(a) A function $f \in L_p$ is said to be a *sign supported on* A if $f = \chi_{B_1} - \chi_{B_2}$, where B_1 and B_2 form a partition of A into two measurable subsets of equal measure.

(b) An operator $T \in \mathscr{L}(L_p, X)$ is said to be *PP-narrow* if for every set $A \in \Sigma^+$ and every $\varepsilon > 0$ there is a sign f supported on A with $\|Tf\| \le \varepsilon$.

The concept of a PP-narrow operator was introduced by Plichko and Popov in [8] under the name *narrow operator*. We use the term "PP-narrow" in order to distinguish such operators from a related concept of a narrow operator from [6], where, incidentally, PP-narrow operators were called L_1-narrow. It should be noted that PP-narrow operators appear implicitly in Rosenthal's papers on sign embeddings (e.g., [10]), where an operator on L_1 is called sign preserving if it is not PP-narrow.

Obviously, no embedding operator is PP-narrow. On the other hand it is clear that a compact operator T is PP-narrow. Indeed, let (r_n) be a Rademacher sequence supported on a set $A \in \Sigma^+$; i.e., the r_n are stochastically independent with respect to the probability space $(A, \Sigma|_A, \mu/\mu(A))$ and $\mu(\{r_n = 1\}) = \mu(\{r_n = -1\}) = \mu(A)/2$. Then $r_n \to 0$ weakly and hence $Tr_n \to 0$ in norm. The same argument shows that weakly compact operators on L_1 are PP-narrow, since L_1 has the Dunford-Pettis property.

The aim of this paper is to find a class of operators with the above properties that works for general X rather than just for $X = L_1[0,1]$. For this purpose we shall introduce the class of hereditarily PP-narrow (for short HPP-narrow) operators in Section 2. We show that they form a linear space of operators (which is false for PP-narrow operators, at least for $p > 1$), and in Section 3 we derive a factorisation scheme for unconditional sums of such operators. This enables us to give an example of a Banach space X for which $\mathrm{unc}(\mathrm{unc}(\mathscr{K}(X,X))) \neq \mathrm{unc}(\mathscr{K}(X,X))$ (Theorem 3.3). In Section 4 we specialise to the case $p = 1$ and obtain that a pointwise unconditionally convergent series of HPP-narrow operators on L_1 is HPP-narrow (Theorem 4.3). As a result, it follows that no embedding operator is in any of the spaces $\mathrm{unc}(\dots(\mathrm{unc}(\mathscr{K}(L_1, X))))$. A further consequence is that every operator from L_1 into a space with an unconditional basis is HPP-narrow and in particular PP-narrow; this implies that L_1 does not even sign-embed into a space with an unconditional basis. These last results are due to Rosenthal in his unpublished paper [9] (not only is this paper unpublished, as a matter of fact it has never been written, as Rosenthal has pointed out to us).

In this paper we deal with real Banach spaces.

2. Haar-like systems and hereditarily PP-narrow operators

We start by introducing some notions that will be used throughout the paper.

Denote

$$\mathscr{A}_0 = \{\emptyset\}, \quad \mathscr{A}_n = \{-1, 1\}^n, \quad \mathscr{A}_\infty = \bigcup_{n=0}^{\infty} \mathscr{A}_n.$$

The elements of $\mathscr{A}_n$ are n-tuples of the form $(\alpha_1, \dots, \alpha_n)$ with $\alpha_k = \pm 1$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathscr{A}_n$ and $\alpha_{n+1} \in \{-1, 1\}$ denote by α, α_{n+1} the $(n+1)$-tuple $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathscr{A}_{n+1}$; also, put $\emptyset, \alpha_1 = (\alpha_1)$. The elements of $\mathscr{A}_\infty$ can be written as a sequence in the following *natural order*:

$$\emptyset,\ -1,\ 1,\ (-1,-1),\ (-1,1),\ (1,-1),\ (1,1),\ (-1,-1,-1),\ \dots .$$

Definition 2.1. Let $A \in \Sigma^+$.

(a) A collection $\{A_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ of subsets of A is said to be a *tree of subsets on A* if $A_\emptyset = A$ and if for every $\alpha \in \mathscr{A}_\infty$ the subsets $A_{\alpha,1}$ and $A_{\alpha,-1}$ form a partition of A_α into two measurable subsets of equal measure.

(b) The collection of functions $\{h_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ defined by $h_\alpha = \chi_{A_{\alpha,1}} - \chi_{A_{\alpha,-1}}$ is said to be a *Haar-like system on A* (corresponding to the tree of subsets A_α, $\alpha \in \mathscr{A}_\infty$).

It is easy to see that after deleting the constant function the classical Haar system is an example of a Haar-like system. Moreover, every Haar-like system is equivalent to this example. In particular we note:

Remark 2.2. (a) Let $\{h_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ be a Haar-like system on A corresponding to a tree of subsets A_α, and let $1 \le p < \infty$. Denote by Σ_1 the σ-algebra on A generated by the subsets A_α. Then the system $\{h_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ in its natural order forms a monotone Schauder basis for the subspace $L_p^0(A, \Sigma_1, \mu)$ of $L_p(A, \Sigma_1, \mu)$ consisting of all $f \in L_p(A, \Sigma_1, \mu)$ with $\int_A f\, d\mu = 0$. Note that, for $\alpha \in \mathscr{A}_n$, $\|h_\alpha\| = \bigl(2^{-n}\mu(A)\bigr)^{1/p}$ for every Haar-like system on A.

(b) Therefore, if $\varepsilon > 0$ and $\{\varepsilon_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ is a family of positive numbers such that $\sum_\alpha \varepsilon_\alpha / \|h_\alpha\| \le \varepsilon/2$ and if $\{x_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ is a family of vectors in a Banach space X such that $\|x_\alpha\| \le \varepsilon_\alpha$, then the mapping $h_\alpha \mapsto x_\alpha$ extends to a bounded linear operator from $L_p^0(A, \Sigma_1, \mu)$ to X of norm $\le \varepsilon$.

Lemma 2.3. *Let $1 \le p < \infty$ and let $T\colon L_p \to X$ be a PP-narrow operator.*

(a) *For every $A \in \Sigma^+$ and every family of numbers $\varepsilon_\alpha > 0$ there is a Haar-like system $\{h_\alpha\colon \alpha \in \mathscr{A}_\infty\}$ on A such that $\|Th_\alpha\| \le \varepsilon_\alpha$ for $\alpha \in \mathscr{A}_\infty$.*

(b) *For every $\varepsilon > 0$ and every $A \in \Sigma^+$ there is a σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and the restriction of T to $L_p^0(A, \Sigma_\varepsilon, \mu)$ has norm $\le \varepsilon$.*

Proof. To construct a tree of subsets and the corresponding Haar-like system for (a) we repeatedly apply the definition of a PP-narrow operator. Namely, let $h_\emptyset$ be a sign supported on A with $\|Th_\emptyset\| \le \varepsilon_\emptyset$. Put, using the notation $\{h = x\} = \{\omega\colon h(\omega) = x\}$,

$$A_{-1} = \{h_\emptyset = -1\}, \quad A_1 = \{h_\emptyset = 1\}.$$

Let h_{-1} and h_1 be signs supported on A_{-1} and A_1 respectively with $\|Th_{\pm 1}\| \le \varepsilon_{\pm 1}$; put

$$A_{-1,-1} = \{h_{-1} = -1\}, \quad A_{-1,1} = \{h_{-1} = 1\},$$
$$A_{1,-1} = \{h_1 = -1\}, \quad A_{1,1} = \{h_1 = 1\}$$

and continue in the above fashion. This yields part (a).

Part (b) follows from (a) and Remark 2.2(b). □

For $1 < p < \infty$ the class of PP-narrow operators on L_p is not stable under taking sums (see [8], p. 59); this is why we have to consider a smaller class of operators that we introduce next. Incidentally, the stability of PP-narrow operators on L_1 under sums is still an open problem.

Definition 2.4. An operator $T\colon L_p \to X$ is said to be *hereditarily PP-narrow* (*HPP-narrow* for short) if for every $A \in \Sigma^+$ and every nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$ on A the restriction of T to $L_p(A, \Sigma_1, \mu)$ is PP-narrow.

Since every compact operator on L_p is PP-narrow and compactness is inherited by restrictions, compact operators on L_p are HPP-narrow. On the other hand, the operator

$$T\colon L_p([0,1]^2) \to L_p[0,1], \quad (Tf)(s) = \int_0^1 f(s,t)\,dt$$

shows that a PP-narrow operator need not be HPP-narrow.

We now show that the set of HPP-narrow operators forms a subspace of $\mathscr{L}(L_p, X)$.

Proposition 2.5. *Let $1 \le p < \infty$ and let $U, V\colon L_p \to X$.*

(a) *If U is PP-narrow and V is HPP-narrow, then $U+V$ is PP-narrow.*
(b) *If U and V are both HPP-narrow, then $U+V$ is HPP-narrow as well.*

Proof. (a) Let $A \in \Sigma^+$ and $\varepsilon > 0$. By Lemma 2.3(b) there is a σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and the restriction of U to $L_p^0(A, \Sigma_\varepsilon, \mu)$ has norm $\le \varepsilon$. Since V is HPP-narrow, there is a Σ_ε-measurable sign f supported on A for which $\|Vf\| \le \varepsilon$. Then $\|(U+V)f\| \le \varepsilon\mu(A)^{1/p} + \varepsilon \le 2\varepsilon$.

(b) follows from (a). □

3. Unconditionally convergent series of HPP-narrow operators

In this section we are going to give an example of a Banach space X for which
$$\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathscr{K}(X,X))) \setminus \mathrm{unc}(\mathscr{K}(X,X)).$$

We begin with a factorisation lemma for unconditional sums of HPP-narrow operators.

Lemma 3.1. *Let $1 \le p < \infty$, X be a Banach space, $T_n\colon L_p \to X$ be HPP-narrow operators with $\sum_{n=1}^\infty T_n$ converging pointwise unconditionally to an operator T and let $M = \sup_\pm \| \sum_{n=1}^\infty \pm T_n\|$. Given $0 < \varepsilon < 1/2$, there exist a Banach space Y and a factorisation*

$$\begin{tikzcd} L_p \arrow[rr, "T"] \arrow[dr, "\tilde{T}"'] & & X \\ & Y \arrow[ur, "W"'] & \end{tikzcd}$$

with $\|\tilde{T}\| \le M$, $\|W\| \le 1$, and there are a nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$, a Haar-like system $\{h_\alpha\}$ forming a basis for $L_p^0(\Omega, \Sigma_1, \mu)$ and operators U, V: $L_p^0(\Omega, \Sigma_1, \mu) \to Y$ with $U + V = \tilde{T}$ on $L_p^0(\Omega, \Sigma_1, \mu)$ such that U maps $\{h_\alpha\}$ to a 1-unconditional basic sequence and $\|V\| \le \varepsilon$.

Proof. Define Y as the space of all sequences $y = (y_1, y_2, \dots)$, $y_n \in X$, such that $\sum_{n=1}^\infty y_n$ converges unconditionally in X. Equip Y with the natural norm
$$\|y\| = \sup_\pm \left\| \sum_{n=1}^\infty \pm y_n \right\|.$$
Put $\tilde{T}f = (T_1 f, T_2 f, \dots)$ and $Wy = \sum_{n=1}^\infty y_n$. Then Y, $\tilde{T}$ and W satisfy the desired factorisation scheme.

Our main task is now to define for this $\tilde{T}$ a Haar-like system $\{h_\alpha\}$ and operators U, V as claimed in the lemma. To do this one uses a standard blocking technique and the stability of HPP-narrow operators under summation (Proposition 2.5). Namely, for every $1 \le n < m \le \infty$ define a projection operator $P_{n,m}\colon Y \to Y$ as follows:
$$P_{n,m}(y_1, y_2, \dots) = (0, 0, \dots, 0, y_n, y_{n+1}, \dots, y_{m-1}, 0, 0, \dots).$$
Let (ε_α) be positive numbers. Select an arbitrary sign $h_\emptyset$ supported on Ω and find $n_\emptyset \in \mathbb{N}$ for which
$$\|P_{n_\emptyset,\infty} \tilde{T} h_\emptyset\| \le \varepsilon_\emptyset.$$
Put
$$U h_\emptyset = P_{1,n_\emptyset} \tilde{T} h_\emptyset, \quad V h_\emptyset = P_{n_\emptyset,\infty} \tilde{T} h_\emptyset.$$

The sign $h_\emptyset$ generates a partition of Ω, i.e.,

$$A_{-1} = \{h_\emptyset = -1\}, \quad A_1 = \{h_\emptyset = 1\}.$$

Since the operator $P_{1,n_\emptyset}\tilde{T}$ is PP-narrow by Proposition 2.5, there is a sign h_{-1} supported on A_{-1} for which

$$\|P_{1,n_\emptyset}\tilde{T}h_{-1}\| \le \frac{1}{2}\varepsilon_{-1}.$$

Find $n_{-1} > n_\emptyset$ such that

$$\|P_{n_{-1},\infty}\tilde{T}h_{-1}\| \le \frac{1}{2}\varepsilon_{-1}.$$

Put

$$Uh_{-1} = P_{n_\emptyset,n_{-1}}\tilde{T}h_{-1}, \quad Vh_{-1} = (P_{1,n_\emptyset} + P_{n_{-1},\infty})\tilde{T}h_{-1}.$$

Continuing in this fashion we obtain a Haar-like system $\{h_\alpha\}$ and operators $U, V\colon \overline{\mathrm{lin}}\{h_\alpha\} \to Y$ such that $U + V = \tilde{T}$ on $\overline{\mathrm{lin}}\{h_\alpha\}$, U maps $\{h_\alpha\}$ to disjoint elements of the sequence space Y and hence to a 1-unconditional basic sequence and V maps $\{h_\alpha\}$ to elements whose norms are controlled by the numbers ε_α; therefore $\|V\| \le \varepsilon$ by Remark 2.2(b) if $\varepsilon_\alpha \to 0$ sufficiently fast. □

Lemma 3.2. *Under the conditions of Lemma 3.1 assume in addition that the operator T is bounded from below by a constant c; i.e.,*

$$\|Tf\| \ge c\|f\| \qquad \forall f \in L_p.$$

Then

$$M = \sup_{\pm}\Bigl\|\sum_{n=1}^{\infty} \pm T_n\Bigr\| \ge \beta_p c,$$

where β_p is the unconditional constant of the Haar system in L_p.

Proof. Let $0 < \varepsilon < 1/2$. Under the above conditions the operator U from Lemma 3.1 maps a Haar-like system $\{h_\alpha\}$ to a 1-unconditional basic sequence. This implies that if U is considered as acting from $\overline{\mathrm{lin}}\{h_\alpha\}$ into $\overline{\mathrm{lin}}\{Uh_\alpha\}$, then $\|U\|\|U^{-1}\| \ge \beta_p$. On the other hand

$$\|U\| \le \|\tilde{T}\| + \|V\| \le M + \varepsilon$$

and

$$\|Uf\| \ge \|\tilde{T}f\| - \varepsilon\|f\| \ge \|Tf\| - \varepsilon\|f\| \ge (c - \varepsilon)\|f\|$$

for all $f \in \overline{\mathrm{lin}}\{h_\alpha\}$, so $\|U^{-1}\| \le (c-\varepsilon)^{-1}$. Hence we have $(M+\varepsilon)(c-\varepsilon)^{-1} \ge \beta_p$, which yields the desired inequality since $\varepsilon > 0$ was arbitrary. □

It is known that $\beta_p \to \infty$ if $p \to 1$ or $p \to \infty$; in fact, Burkholder [2] has shown that

$$\beta_p = \max\Bigl\{p - 1, \frac{1}{p-1}\Bigr\}.$$

Theorem 3.3. *There exists a Banach space X for which*

$$\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathscr{K}(X,X))) \setminus \mathrm{unc}(\mathscr{K}(X,X)).$$

Proof. Consider the space $X = L_{p_1} \oplus_2 L_{p_2} \oplus_2 \dots$ where $1 < p_n < \infty$ and $p_n \to 1$.

Suppose that $\mathrm{Id} = \sum_{n=1}^{\infty} T_n$ pointwise unconditionally with compact operators T_n. The restrictions of T_n to L_{p_j} are also compact and hence HPP-narrow, so by the previous lemma

$$\sup_{\pm} \left\| \sum_{n=1}^{\infty} \pm T_n \right\| \geq \sup_{\pm} \left\| \sum_{n=1}^{\infty} \pm T_n|_{L_{p_j}} \right\| \geq \beta_{p_j} \to \infty.$$

So the assumption of pointwise unconditional convergence of $\sum_{n=1}^{\infty} T_n$ leads to a contradiction, and hence Id does not belong to $\mathrm{unc}(\mathscr{K}(X,X))$.

On the other hand all the natural projections $P_j \colon X \to L_{p_j}$ belong to $\mathrm{unc}(\mathscr{K}(X,X))$ since each L_{p_j} has an unconditional basis. Taking into account the unconditional representation $\mathrm{Id} = \sum_{n=1}^{\infty} P_n$ we obtain that $\mathrm{Id} \in \mathrm{unc}(\mathrm{unc}(\mathscr{K}(X,X)))$. □

4. HPP-narrow operators on L_1

In this section we prove the main result of the paper, namely that the sum of a pointwise unconditionally convergent series of HPP-narrow operators on L_1 is again an HPP-narrow operator.

The following lemma implies that the operator U from Lemma 3.1 factors through c_0.

Lemma 4.1. *Let $\{h_\alpha\}$ be a Haar-like system in L_1, $U \colon L_1 \to X$ be an operator which maps $\{h_\alpha\}$ into an unconditional basic sequence. Then there is a constant C such that for every element of the form $f = \sum_\alpha a_\alpha h_\alpha$ one has*

$$\|Uf\| \leq C \sup_\alpha |a_\alpha|. \tag{4.1}$$

Proof. Without loss of generality we can assume that $\|U\| = 1$, $\|h_\emptyset\| = 1$ and that the unconditional constant of $\{Uh_\alpha\}$ also equals 1 (one can achieve all these goals by an equivalent renorming of X and by multiplication of μ by a constant).

Let us first remark that for every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathscr{A}_n$

$$\|\alpha_1 h_\emptyset + 2\alpha_2 h_{\alpha_1} + 4\alpha_3 h_{\alpha_1,\alpha_2} + \dots + 2^{n-1}\alpha_n h_{\alpha_1,\dots,\alpha_{n-1}}\| \leq 2;$$

indeed, it is easy to check by induction over n that this sum equals

$$2^n \chi_{A_{\alpha_1,\dots,\alpha_n}} - \chi_{A_\emptyset}.$$

Hence

$$\|\alpha_1 Uh_\emptyset + 2\alpha_2 Uh_{\alpha_1} + \dots + 2^{n-1}\alpha_n Uh_{\alpha_1,\dots,\alpha_{n-1}}\| \leq 2,$$

and, since $\{Uh_\alpha\}$ is a 1-unconditional basic sequence,

$$\|Uh_\emptyset + 2Uh_{\alpha_1} + \dots + 2^{n-1} Uh_{\alpha_1,\dots,\alpha_{n-1}}\| \leq 2.$$

Passing from $n-1$ to n in the last inequality and averaging over $\alpha \in \mathscr{A}_n$ we obtain that

$$2 \geq \left\| \frac{1}{2^n} \sum_{\alpha \in \mathscr{A}_n} (Uh_\emptyset + 2Uh_{\alpha_1} + \cdots + 2^{n-1} Uh_{\alpha_1,\dots,\alpha_n}) \right\| = \left\| \sum_{k=0}^{n} \sum_{\alpha \in \mathscr{A}_k} Uh_\alpha \right\|.$$

Again by 1-unconditionality of $\{Uh_\alpha\}$ the last inequality implies that for all $a_\alpha \in [-1, 1]$

$$\left\| \sum_{k=0}^{n} \sum_{\alpha \in \mathscr{A}_k} a_\alpha Uh_\alpha \right\| \leq 2,$$

which gives (4.1) with $C = 2$. □

An inspection of the proof shows that

$$\|Uf\| \leq 2\|U\|\beta^2 \sup_\alpha |a_\alpha|$$

where β denotes the unconditional constant of the basic sequence (Uh_α).

Lemma 4.2. *For every Haar-like system $\{h_\alpha\}$ in L_1 supported on A and every $\delta > 0$ there is a sign*

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathscr{A}_k} a_\alpha h_\alpha \tag{4.2}$$

supported on A with $\sup_\alpha |a_\alpha| \leq \delta$.

Proof. Fix an $m \in \mathbb{N}$ such that $1/m \leq \delta$ and define

$$f_k = \sum_{\alpha \in \mathscr{A}_k} a_\alpha h_\alpha$$

as follows: $f_0 = \frac{1}{m} h_\emptyset$, and for every $\alpha \in \mathscr{A}_n$ put $a_\alpha = 1/m$ if $|\sum_{k=0}^{n-1} f_k| < 1$ on $\operatorname{supp} h_\alpha$ and $a_\alpha = 0$ if $|\sum_{k=0}^{n-1} f_k| = 1$ on $\operatorname{supp} h_\alpha$. Under this construction all the partial sums of the series $\sum_{k=0}^{\infty} f_k$ are bounded by 1 in modulus. Since $\{f_k\}_{k=0}^{\infty}$ is an orthogonal system, the series $\sum_{k=0}^{\infty} f_k$ converges in L_2 (and hence in L_1) to a function f supported on A that can be represented as in (4.2) with $\sup_\alpha |a_\alpha| \leq \delta$. We shall prove that f is a sign.

Obviously $\int_A f\, d\mu = 0$. Consider $B = \{t \in A\colon |f(t)| \neq 1\}$. By our construction we have for each $n \in \mathbb{N}$

$$B \subset \{t \in A\colon f_n(t) \neq 0\} = \left\{t \in A\colon |f_n(t)| = \frac{1}{m}\right\},$$

so $\mu(B) \leq m\|f_n\|$, and since $\|f_n\| \to 0$, we conclude that $\mu(B) = 0$. Therefore f is a sign. □

The previous lemma can also be proved by means of abstract martingale theory. For simplicity of notation let us work with the classical Haar system $h_1, h_2, \dots$ on $[0, 1]$. Let $\xi_n = \sum_{k=1}^{n} h_k$ and $T = \inf\{n\colon |\xi_n| \leq m\}$. Then (ξ_n) is a martingale, T is a stopping time and $(\xi'_n) = (\xi_{n \wedge T})$ is a uniformly bounded martingale. Hence (ξ'_n) converges almost surely and in L_1 to a limit ξ that takes only the values $\pm m$ on $\{T < \infty\}$. But since (ξ_n) fails to converge pointwise, the event $\{T = \infty\}$ has probability 0. This shows that $\xi = \pm m$ almost surely and $\mathbb{E}\xi = 0$. Hence $f = \xi/m$ is the sign we are looking for.

We are now ready for the main result of this paper. An analogous theorem for operators on $C(K)$-spaces was proved in [1].

Theorem 4.3. *Let $T_n \colon L_1 \to X$ be HPP-narrow operators, and suppose that $\sum_{n=1}^{\infty} T_n$ converges pointwise unconditionally to some operator T. Then T is HPP-narrow.*

Proof. Let $A \in \Sigma^+$, and let $\tilde{\Sigma}$ be a nonatomic sub-σ-algebra of $\Sigma|_A$. We have to show that for every $\varepsilon > 0$ there is a sign $f \in L_1(A, \tilde{\Sigma}, \mu)$ supported on A with $\|Tf\| \le \varepsilon$.

Applying Lemma 3.1 to the restrictions of T_n and T to $L_1(A, \tilde{\Sigma}, \mu)$ we get a Haar-like system $\{h_\alpha\}$ forming a basis for some $L_1^0(A, \Sigma_1, \mu)$ and we obtain operators $U, V \colon L_1^0(A, \Sigma_1, \mu) \to Y$, $W \colon Y \to X$ such that $\|W\| \le 1$, $T = W(U + V)$ on $L_1^0(A, \Sigma_1, \mu)$, $\|V\| \le \varepsilon/2$ and U maps $\{h_\alpha\}$ to a 1-unconditional basic sequence. Let C be the constant from (4.1). Taking a sign

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathscr{A}_k} a_\alpha h_\alpha$$

supported on A with $\sup_\alpha |a_\alpha| \le \varepsilon/(2C)$ (Lemma 4.2) we obtain from (4.1) that $\|Uf\| \le \varepsilon/2$. Therefore $\|Tf\| \le \|Uf\| + \|Vf\| \le \varepsilon$. ☐

Corollary 4.4. *For any Banach space X, no embedding operator is contained in* $\mathrm{unc}(\dots(\mathrm{unc}(\mathscr{K}(L_1, X))))$.

Proof. Compact operators are HPP-narrow. ☐

The next corollary is due to Rosenthal [9].

Corollary 4.5. *Every operator T from L_1 into a Banach space X with an unconditional basis is HPP-narrow; in particular it is PP-narrow. Consequently, L_1 does not even sign-embed into a space with an unconditional basis.*

Proof. If P_n, $n = 1, 2, \dots$, are the partial sum projections associated to an unconditional basis of X, then $T = \sum_{n=1}^{\infty} (P_n - P_{n-1})T$ is a pointwise unconditionally convergent series of rank-1 operators. ☐

5. Questions

(1) Can one describe $\mathrm{unc}(\mathscr{K}(L_1, X))$ for general X? What about $X = L_1$?

(2) Describe the smallest class of operators $\mathscr{M} \subset \mathscr{L}(L_1, X)$ that contains the compact operators and is stable under pointwise unconditional sums. In particular, is $\mathrm{unc}(\mathscr{K}(L_1, L_1)) = \mathrm{unc}(\mathrm{unc}(\mathscr{K}(L_1, L_1)))$? Note that X does not embed into a space with an unconditional basis if $\mathscr{M} \neq \mathscr{L}(L_1, X)$.

(3) Can one develop a similar theory for operators on the James space or other spaces that do not embed into spaces with unconditional bases?

(4) Is there a space X with the Daugavet property such that $\mathrm{Id} \in \mathrm{unc}(\dots(\mathrm{unc}(\mathscr{K}(X,X))))$?

(5) Suppose E is a Banach space with the Daugavet property on which the set of narrow operators from E to X is a linear space. (This is not always the case; e.g., it is not so for $E = X = C([0,1],\ell_1)$ [1].) If $T = \sum T_n$ is a pointwise unconditionally convergent series of narrow operators from E into X, must T also be narrow? It is known that under these conditions $\|\mathrm{Id} + T\| \geq 1$ [5]. The answer is positive for $E = C([0,1],\ell_p)$ if $1 < p < \infty$ [1].

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