Green currents on Kähler manifolds

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ABSTRACT. We present an approach to the Green operator and to Green currents for homogeneous Kähler manifolds. To obtain Green currents for a cycle Y in a Kähler manifold X we apply the heat kernel to the Dirac distribution δ_Y. This yields a unique Green current g_Y. We discuss how this current can be effectively computed for Grassmannians.

1. Introduction

The heat kernel of a compact Riemannian manifold X can be seen as a homotopy between the diagonal on $X \times X$, a geometric object, and a harmonic form an object of analysis. Therefore, it is not astonishing that in Arakelov theory, where an interplay of arithmetic, geometry and analysis of Arakelov manifolds is studied, the heat kernel can be applied.
We show that for an analytic cycle Y on a Kähler manifold X the heat flux of the associated Dirac current yields a Green current for Y which is uniquely defined. Moreover, we show how this canonical Green current can be computed for homogeneous Kähler manifolds. Finally, these computations are carried through in the case of the projective one space.
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2. The heat kernel on compact manifolds

2.1. Preliminaries.

2.1.1. *Forms and currents.* Let X be a closed compact Riemannian manifold of dimension n. The metric g defines a volume form Ω on X. We denote the space of smooth k forms on X by $A^k(X,\mathbb{R})$. By $D^k(X,\mathbb{R})$, we denote the space of functionals on $A^{n-k}(X,\mathbb{R})$ which are continuous for the Schwartz topology. These functionals are called currents on X. There exits a natural embedding of $A^k(X,\mathbb{R})$ into $D^k(X,\mathbb{R})$ which assigns a smooth k form α the current $[\alpha]$ given by $[\alpha](\beta) := \int_X \alpha \wedge \beta$. The currents of type $[\alpha]$ are called smooth currents and are well understood. Dirac currents are the paradigm for currents which are not smooth.

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The Dirac current δ_Y of a submanifold $Y \xrightarrow{\iota} X$ is defined by $\delta_Y(\beta) = \int_Y \iota^*\beta$. We see that $\delta_Y \in D^k(X, \mathbb{R})$ where k is the codimension of Y in X.
It follows from Stokes' theorem that for a k form α the equation

$$[d\alpha](\beta) = -(-1)^k[\alpha](d\beta)$$

holds. This motivates the following definition of a differential $d : D^k(X, \mathbb{R}) \to D^{k+1}(X, \mathbb{R})$ and shows that it is compatible with $d : D^k(X, \mathbb{R}) \to D^{k+1}(X, \mathbb{R})$ and the embedding $A^k(X, \mathbb{R}) \to D^k(X, \mathbb{R})$. For $T \in D^k(X, \mathbb{R})$, we define $dT \in D^{k+1}(X, \mathbb{R})$ by

$$dT(\beta) := -(-1)^k T(d\beta)\,.$$

If $f : X \to Y$ is a proper morphism, then we obtain a push forward morphism $f_* : D^k(X, \mathbb{R}) \to D^{k-(\dim(X)-\dim(Y))}(Y, \mathbb{R})$ defined by $f_*T(\beta) = T(f^*\beta)$.
Let us give two examples for this morphism: If X is a closed submanifold of Y and 1_X the constant function on X, then $f_*[1_X] = \delta_X$.
In the second example, we assume that $f : X \to Y$ is a submersion. If $[\alpha]$ is the current associated to the smooth form $\alpha \in A^k(X, \mathbb{R})$, then $f_*[\alpha]$ is associated to a smooth form too. This form is called the integration of α along the fibers of f and denoted by $f_*\alpha$.

2.2. A formal approach to the heat kernel.

2.2.1. *The Laplace operator.* The metric g of the Riemannian manifold X defines an inner product on the cotangent bundle and its symmetric powers. Using the volume form, we obtain an inner product on the spaces $A^k(X, \mathbb{R})$, which we denote by $\langle \alpha, \beta \rangle$. The Hodge $*$ operator is defined by

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta\,.$$

We have $* : A^k(X, \mathbb{R}) \to A^{n-k}(X, \mathbb{R})$ and $**\alpha = (-1)^{nk+k}\alpha$ for $\alpha \in A^k(X, \mathbb{R})$. This and Stokes' theorem yield that the operator

$$d^* = (-1)^{nk} * d* : A^k(X, \mathbb{R}) \to A^{k-1}(X, \mathbb{R})$$

is the adjoint to d, i.e. we have

$$\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle\,.$$

The Laplace operator Δ is the square of $d + d^*$. We have

$$\Delta = dd^* + d^*d = (d + d^*)(d + d^*) : A^k(X, \mathbb{R}) \to A^k(X, \mathbb{R})\,.$$

This operator is positive semi definite. The kernel is the space of harmonic forms $\mathcal{H}^k(X, \mathbb{R})$.

2.2.2. *The spectrum of the Laplace operator on $A^k(X, \mathbb{R})$.* The spectrum of this operator is discrete and eigenvectors to different eigenvalues are orthogonal with respect to the scalar product $\langle , \rangle$. Let $\mathrm{Spec}(\Delta) = \{\lambda_0, \lambda_1, \ldots, \}$ with $\lambda_i \le \lambda_{i+1}$. Let $\mathcal{B} = \{\alpha_0, \alpha_1, \ldots\}$ be a corresponding orthonormal system of eigenvectors, i.e.

$$\langle \alpha_i, \alpha_j \rangle = \delta_{ij} \quad \text{and} \quad \Delta\alpha_i = \lambda_i\alpha_i\,.$$

The eigenvectors span a dense subspace of $A^k(X, \mathbb{R})$, i.e. for any $\beta \in A^k(X, \mathbb{R})$, we have

$$\lim_{m\to\infty} ||\beta - \sum_{k=1}^{m} \langle \beta, \alpha_k \rangle \alpha_k|| = 0\,.$$

2.2.3. *The heat equation and the heat flux of a k form.* Let β be a form in $A^k(X,\mathbb{R})$. A family $(\beta_t)_{t\in\mathbb{R}^+}$ of smooth forms which is C^1 with respect to t and satisfies

$$\begin{array}{rcll} \lim_{t\to 0}\beta_t &=& \beta & \text{(initial condition), and} \\ \frac{\partial}{\partial t}\beta_t &=& -\Delta\beta_t & \text{(heat equation)} \end{array}$$

is called the heat flux of β. The heat flux of an element α_i of $\mathcal{B}$ is given by $(\exp(-\lambda_i\cdot t)\cdot\alpha_i)_{t\in\mathbb{R}^+}$. Next we give the universal solution for this equation which is called the *heat kernel for k forms.* To do so, we consider the following maps between manifolds:

$$X \xleftarrow{\mathrm{pr}_1} X\times X \xrightarrow{\mathrm{pr}_2} X\,.$$

On $X\times X$, we define the heat kernel for k forms to be the family $(p_t^k)_{t\in\mathbb{R}^+}$ of n forms given by

$$p_t^k := (-1)^{n+k}\sum_{i\geq 0}\exp(-\lambda_i\cdot t)\cdot\mathrm{pr}_1^*\alpha_i\wedge\mathrm{pr}_2^* *\alpha_i\,.$$

The associated operator $P_t^k : A^k(X,\mathbb{R})\to A^k(X,\mathbb{R})$, which assigns β the form $\mathrm{pr}_{1*}(p_t^k\wedge\mathrm{pr}_2^*\beta)$ solves the heat equation. At least formally it is easy to show that $P_t^k(\sum_{i\geq 0}a_i\cdot\alpha_i)=\sum_{i\geq 0}\exp(-\lambda_i\cdot t)\cdot a_i\cdot\alpha_i$ is the heat flux.

2.2.4. *The heat kernel.* The sum $p_t=\sum_{k=0}^n p_t^k$ of all k form heat kernels yields the heat kernel. The associated operator P_t fulfills $P_t=\sum_{k=0}^n P_t^k$. This kernel is called the heat kernel on X. The next theorem summarizes the main properties of the heat kernel $p_t\in A^n(X\times X\times\mathbb{R}^+,\mathbb{R})$ and the associated operator $P_t : A^\bullet(X,\mathbb{R})\to A^\bullet(X,\mathbb{R})$ defined by $P_t(\beta)=\mathrm{pr}_{1*}(p_t\wedge\mathrm{pr}_2^*\beta)$. The formal access provided above gives evidence for most of the statements except the local description and the smoothness of p_t. By Δ_X, we denote the diagonal in $X\times X$ and by τ the interchanging homomorphism on $X\times X$, i.e. $\tau(x_1,x_2)=(x_2,x_1)$.

THEOREM 2.1 (Properties of the heat kernel on a compact manifold). *Let X be a compact manifold of dimension n. Then the forms p_t and p_t^k and the associated operators P_t and P_t^k fulfill the following properties.*

(1) *The heat kernel p_t is a smooth n form on $X\times X$ for all $t>0$.*
(1') *The heat kernel p_t^k is a smooth n form on $X\times X$ for all $k=0,\dots,n$ and $t>0$.*
(2) *The operator $P_t : A^\bullet(X)\to A^\bullet(X)$ "solves" the heat equation for all $\alpha\in A^\bullet(X)$, i.e. $\alpha_t=P_t(\alpha)$ is the heat flux of α.*
(2') *The operator $P_t^k : A^k(X)\to A^k(X)$ "solves" the heat equation for all $\alpha\in A^k(X)$, i.e. $\alpha_t=P_t^k(\alpha)$ is the heat flux of α.*
(3) *In $D^n(X\times X,\mathbb{R})$, we have: $\lim_{t\to 0}[p_t]=\delta_{\Delta_X}$.*
(4) *The heat form p_t fulfills $\tau^*p_t=(-1)^n p_t$.*
(4') *The heat forms p_t^k and p_t^{n-k} are related by the equality $\tau^*p_t^k=(-1)^n p_t^{n-k}$.*
(5) *For a smooth form $\alpha\in A^\bullet(X)$. The limit $\alpha_\infty=\lim_{t\to\infty}P_t(\alpha)$ is the harmonic projection of α, i.e. $\Delta\alpha_\infty=0$, and all integrals in the following formal computation, are well defined:*

$$\alpha_\infty-\alpha=\int_0^\infty\frac{\partial}{\partial t}\alpha_t dt=\int_0^\infty -\Delta\alpha_t dt=-\Delta\int_0^\infty(\alpha_t-\alpha_\infty)dt\,.$$

(6) *Let $x\in X$ be a point with local coordinates $x_1,\dots,x_n$ such that the tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_{i=1..n}$ form an orthonormal basis at x. If $x_1',x_2',\dots,x_n'$, and*

$x_1'', x_2'', \ldots x_n''$ are the corresponding coordinates at the point $(x,x) \in X \times X$, then on a neighborhood of (x,x) the heat kernel p_t is given by

$$p_t(x', x'') = \frac{\exp\left(\frac{-||x'-x''||^2}{4t}\right)}{(4\pi t)^n} d(x_1' - x_1'') \wedge d(x_1' - x_1'') \wedge \ldots \wedge d(x_n' - x_n'') + \rho\,,$$

with ρ a form with bounded coefficients. If $x \neq y$, then p_t is given by forms with bounded coefficients on a neighborhood of (x,y).

2.3. The heat flux of Dirac currents. We want to extend the notion of the heat flux to currents. The following result tells us that we can proceed in a similar manner to that in the case of the differentiation of currents in 2.1.1. Indeed, lemma 2.2 tells us that up to a sign, the current associated to the heat flux of a smooth form can be computed by applying the current to the heat flux of forms.

LEMMA 2.2. *If $\alpha \in A^k(X,\mathbb{R})$ and $\beta \in A^{n-k}$ are two smooth forms, then we have*

$$[P_t(\alpha)](\beta) = (-1)^n[\alpha](P_t(\beta))\,.$$

PROOF. The proof uses the definition of the operator P_t, property (4') of theorem 2.1, and interchanges the projections of $X \times X$.

$$\begin{aligned}
[P_t(\alpha)](\beta) &= \int_X P_t(\alpha) \wedge \beta = \int_X \mathrm{pr}_{1*}(p_t^k \wedge \mathrm{pr}_2^*\alpha) \wedge \beta \\
&= \int_{X\times X} p_t^k \wedge \mathrm{pr}_2^*\alpha \wedge \mathrm{pr}_1^*\beta = (-1)^n \int_{X\times X} \tau^* p_t^{n-k} \wedge \mathrm{pr}_2^*\alpha \wedge \mathrm{pr}_1^*\beta \\
&= (-1)^{n+k(n-k)} \int_{X\times X} \tau^* p_t^{n-k} \wedge \mathrm{pr}_1^*\beta \wedge \mathrm{pr}_2^*\alpha \\
&= (-1)^{n+k(n-k)} \int_X P_t(\beta) \wedge \alpha \\
&= (-1)^n \int_X \alpha \wedge P_t(\beta) = (-1)^n[\alpha](P_t(\beta))\,.
\end{aligned}$$

2.3.1. *Definition of the heat flux of currents.* For a current $T \in D^k(X,\mathbb{R})$, we define its heat flux $(P_t(T))_{t\in\mathbb{R}^+}$ by $P_t(T)(\beta) := (-1)^n T(P_t(\beta))$.

2.3.2. *Smoothness of the heat flux of Dirac currents.* Analogous to the proof of 2.2, we can show that the heat flux $P_t(\delta_Y)$ of a closed submanifold $Y \to X$ of codimension k can be represented by smooth forms, i.e. there exists a family $(\mu_{Y,t})_{t\in\mathbb{R}^+}$ with the following properties

$$\mu_{Y,t} \in A^k(X,\mathbb{R}) \qquad \lim_{t\to 0}[\mu_{Y,t}] = \delta_Y\,.$$

This also holds for closed subsets Y of X, provided they have a desingularization $\tilde{Y} \to Y$ by a smooth compact fold. See 2.2 in [**7**] for more details.

2.3.3. *The harmonic projection and the Green operator.* The limit of P_t when $t \to \infty$ is the harmonic projection operator. We write

$$\mu_Y := \lim_{t\to\infty} P_t(\delta_Y) = \lim_{t\to\infty} \mu_{Y,t}$$

for the harmonic projection of δ_Y. This way we can consider the heat kernel as a homotopy between the identity and the harmonic projection. Having part (5) of theorem 2.1 in mind, we can write

$$\mu_{Y,t} - \mu_Y = \Delta \int_t^\infty (\mu_{Y,s} - \mu_Y) ds$$

for $t > 0$. It is shown in [**7**] that we can pass to the limit $t = 0$ when we allow currents. We obtain for the current $G(Y) = \int_0^\infty (\mu_{Y,t} - \mu_Y) dt$ the equality

$$\delta_Y - [\mu_Y] = \Delta G(Y) .$$

The Hodge theorem states that every k form α can be written as $\alpha = \alpha_{\mathrm{har}} + \Delta(\alpha_{\perp\mathrm{har}})$ with $\alpha_{\mathrm{har}} \in \mathcal{H}^k(X, \mathbb{R})$ and $\alpha_{\perp\mathrm{har}}$ orthogonal to $\mathcal{H}^k(X, \mathbb{R})$. The above equality is the equivalent for currents. Therefore, $G(Y)$ is the Green operator of δ_Y.

2.3.4. *The heat flux of closed currents.* If $Y \to X$ is a closed submanifold, then we have $d(\delta_Y) = 0$ by Stokes' theorem. The harmonic projection μ_Y is closed too. Thus, we obtain $d\Delta(G(Y)) = 0$. Since d and Δ commute, we eventually get $\Delta(dG(Y)) = 0$. Thus, $dG(Y)$ is harmonic and exact which, by the Hodge decomposition theorem, yields $dG(Y) = 0$. Writing down the equation $\delta_Y - [\mu_Y] = \Delta G(Y)$, we see that

$$\delta_Y - [\mu_Y] = (dd^* + d^*d) G(Y) = d(d^* G(Y)) .$$

Thus, we not only see that δ_Y and μ_Y define the same cohomology class in the de Rham cohomology, but we obtain that their difference is the differential of $d^* G(Y)$.

2.3.5. *The standard example: A point on the circle.* Let $X = S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ with the standard volume form dx and the corresponding metric. The eigenvectors for the Laplacian on functions are given by

$$\Delta(1_X) = 0 \qquad \Delta(\sin(n \cdot x)) = n^2 \sin(n \cdot x) \qquad \Delta(\cos(n \cdot x)) = n^2 \cos(n \cdot x) .$$

Applying the Hodge $*$ operator, we obtain the eigenvectors for the Laplacian on one forms. Applying our formulas, we can write down the heat kernel on $X \times X$ to be

$$p_t = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{n \geq 1} e^{-n^2 t} \left(\sin(n \cdot x_1) \sin(n \cdot x_2) + \cos(n \cdot x_1) \cos(n \cdot x_2) \right) \right) (dx_1 - dx_2) .$$

We consider the point $Y = 0 \in X$. We obtain the heat flux of the Dirac current δ_Y by restricting the above heat kernel to $X \times \{0\}$ and get

$$P_t(\delta_Y) = \left[\frac{1}{\pi} \left(\frac{1}{2} + \sum_{n \geq 1} e^{-n^2 t} \cos(n \cdot x) \right) dx \right] .$$

If we write $P_t(\delta_Y) = [f_t(x) dx]$, then f_t is a smooth function on X. We have plotted the graphs of f_t for $t \in \{\frac{1}{10}; \frac{1}{2}; 2\}$ in the next figure.

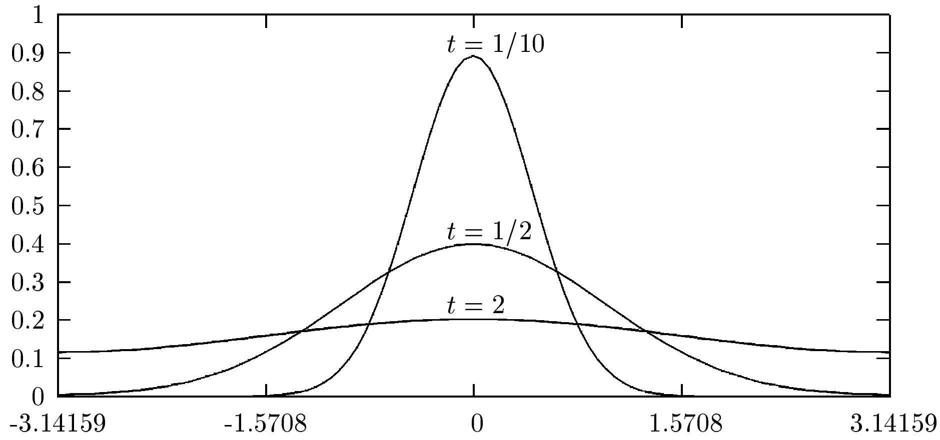

Figure 1: The functions $f_{\frac{1}{10}}$, $f_{\frac{1}{2}}$, and f_2.

We see that the harmonic projection μ_Y of δ_Y is the form $\frac{dx}{2\pi}$. The Green operator $G(Y)$ is given by

$$G(Y) = \frac{1}{\pi}\left(\sum_{n\geq 1} \frac{\cos(n\cdot x)}{n^2}\right) dx\,.$$

Next is a plot of the continuous function g on the circle with the property $G(Y) = *g$, i.e.

$$g(x) = \frac{1}{\pi}\left(\sum_{n\geq 1} \frac{\cos(n\cdot x)}{n^2}\right)\,.$$

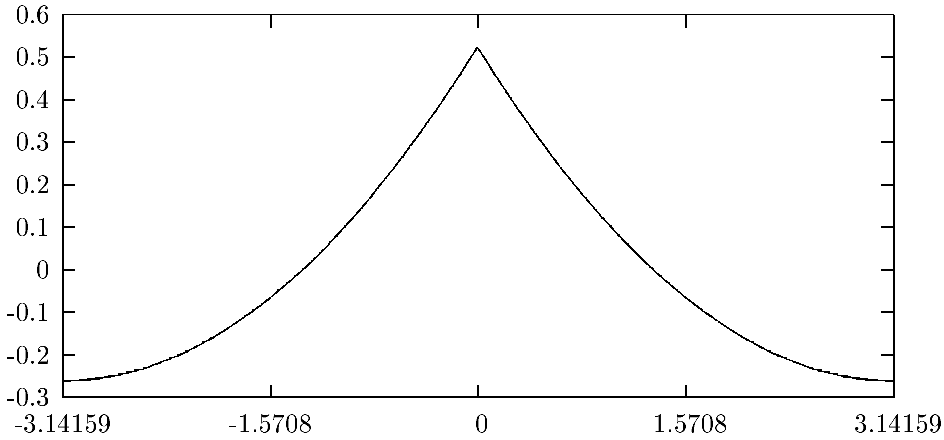

Figure 2: The graph of g.

The differential dh of the almost function $h = d^*G(Y)$ is the difference of $\delta_Y - [\mu_Y]$. We compute that

$$h = \frac{1}{\pi}\left(\sum_{n\geq 1} \frac{\sin(n\cdot x)}{n}\right)\,.$$

We end this example with the graph of h.

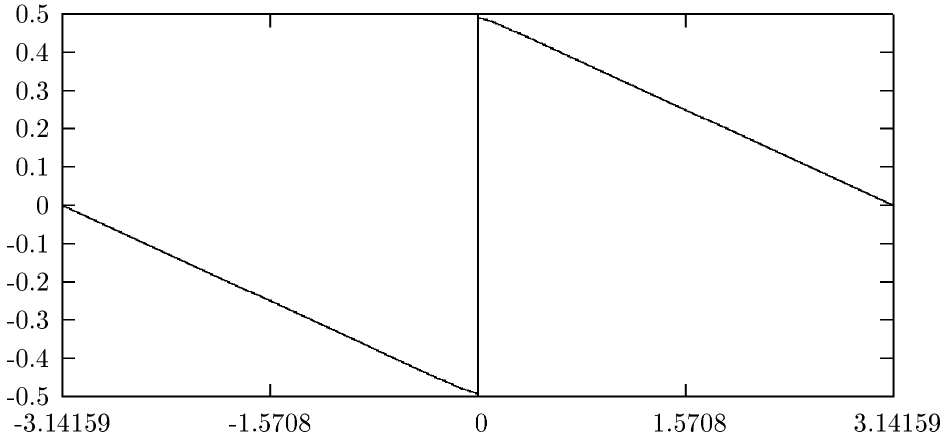

Figure 3: The distribution h.

2.4. Canonical Green currents on Kähler manifolds.

2.4.1. *Complex cycles on Kähler manifolds.* Here we consider the case when X is a Kähler manifold and $Y \subset X$ is locally the zero set of holomorphic equations. We denote the Kähler form by ω and the adjoint operator of $(\alpha \mapsto \alpha \wedge \omega)$ by Λ. We recall that the operator d^c is defined by $d^c = \frac{1}{4\pi i}(\partial - \bar\partial)$ and satisfies $dd^c = \frac{1}{2\pi i}\bar\partial\partial$. The Hodge decomposition and commutator relations yield

$$[\mu_Y] - \delta_Y = dd^c(4\pi\Lambda G(Y)) .$$

Therefore, the form $g_Y = 4\pi\Lambda G(Y)$ is called the canonical Green current of Y. This Green current, its properties, and its relation to intersection theory on X are the subject of the author's article [**7**] where more details can be found. Let us mention that the harmonic projection of g_Y is zero.

2.4.2. *Example: Divisors on Kähler manifolds.* In the case where Y is a divisor, the equation $[\mu_Y] - \delta_Y = dd^c g_Y$ determines g_Y up to a constant. However, the property that the harmonic projection of g_Y is zero means $\int_X g_Y \omega^{\wedge n} = 0$, which allows us to compute this constant. In order to find g_Y, we proceed therefore as follows: Take the line bundle $L := \mathcal{O}(Y)$. Equip L with a metric h such that the first Chern form (see [**5**] or IV.2 in [**11**]) of the Hermitian line bundle (L, h) is harmonic. This metric h is unique up to a constant scalar. Let $s \in \Gamma(L)$ be the global section which defines the divisor Y. We set $\tilde g_Y = -\log ||s||_h^2$. This current satisfies $[\mu_Y] - \delta_Y = dd^c \tilde g_Y$. To obtain a current with trivial harmonic projection, we just set $g_Y = \tilde g_Y - \frac{\int_X \tilde g_Y \omega^{\wedge n}}{\int_X \omega^{\wedge n}}$.

On the $\mathbb{P}^1$ with the Fubini-Study metric, for example, we get the canonical Green current

$$g_Y = \log(1 + (z\bar z)^{-1}) - 1$$

for the point Y defined by $z = 0$. See 6.1 in [**7**] for a further discussion of this example.

2.4.3. *Lagrangians on Kähler manifolds.* Let n be the complex dimension of the Kähler manifold X and $L \subset X$ be a Lagrangian submanifold, i.e. L is of real dimension n and the restriction of the Kähler form ω to L is zero. The last condition would be expressed in terms of currents by $\delta_L(\omega \wedge \alpha) = 0$ for all $\alpha \in A^{n-2}(X)$. Since the heat flux commutes with ω, we see that the harmonic projection $\mu_L = P_t(\delta_L)$ of δ_L is a harmonic n form which is in the kernel of the map $\mathcal{H}^n(X) \to \mathcal{H}^{n+2}(X)$ which assigns a harmonic n form α the harmonic form $\omega \wedge \alpha$. These forms are called the primitive n forms and represent the primitive cohomology classes. (See page 122 in the book [**6**] by Griffiths and Harris for the definition of primitive cohomology classes.) Thus, Lagrangians on Kähler manifolds represent primitive cohomology classes.

2.4.4. *The heat kernel in intersection theory.* Let X be a compact manifold and Y and Z be two cycles in X. Note that we supposed Y and be Z to be oriented cycles in order to define the distribution δ_Y and δ_Z, respectively. As seen in 2.3.4, we have that

$$\delta_Y - [\mu_Y] = d(d^* G(Y))\,.$$

Thus, the distribution δ_Y and the harmonic form μ_Y (considered as a distribution) differ by an exact distribution. Hence, they can be identified in cohomology. Let us compare both objects. On the one hand, we have the singular distribution δ_Y. We have, however, a concrete geometric object, namely Y, which gives this distribution a well defined geometric meaning. On the other hand, we have a smooth distribution $[\mu_Y]$ which can easily be twisted with other distributions. It is easy to define wedge products of smooth forms, but they lack the nice geometric meaning, because they are equally distributed over X. If the two cycles Y and Z intersect transversally, then $Y \cap Z$ is their intersection product. However, if $Y = Z$, then we could take $\mu_Z \wedge \delta_Y$ to be the intersection of both cycles. Here the wedge product of a k form α with a distribution $T \in D^l(X, \mathbb{R})$ is defined by $(\alpha \wedge T)(\beta) = (-1)^{k \cdot l} T(\alpha \wedge \beta)$.
It is the heat kernel which gives a homotopy between both objects. To see this, we consider the $X \times X$ distribution $\delta_{Y \times Z}$. Since the limit of $[p_t]$ for $t \to 0$ is the diagonal, we see that $\lim_{t \to 0} \mathrm{pr}_{1*}(p_t \wedge \delta_{Y \times Z}) = \delta_{Y \cap Z}$ provided Y and Z intersect properly. On the other hand, we have $\lim_{t \to \infty} \mathrm{pr}_{1*}(p_t \wedge \delta_{Y \times Z}) = \delta_Y \wedge \mu_Z$.
This fits into the concept of algebraic intersection theory (see Chapter 8 in [**2**]) where arbitrary intersection products are reduced to intersections with the diagonal.

2.4.5. *Green currents in Arakelov theory.* Let X be an Arakelov variety, i.e. a regular projective scheme which is flat over $\mathrm{Spec}(\mathbb{Z})$ and, furthermore, equipped with a metric on $X(\mathbb{C})$ which is invariant under complex conjugation. The arithmetic Chow group $\widehat{\mathrm{CH}}^p(X)$ is the abelian group generated by all pairs (Z, g_Z) where $Z \subset X$ is a cycle of codimension p and g_Z is a Green current for $Z(\mathbb{C}) \subset X(\mathbb{C})$ modulo the elements of the form $(\mathrm{div}(f), -\log ||f||^2)$ and $(0, \partial u + \bar{\partial} v)$ where f is a rational function on a subvariety of codimension $p-1$, $u \in D^{p-2,p-1}(X(\mathbb{C}))$ and $v \in D^{p-1,p-2}(X(\mathbb{C}))$.
There exists an intersection pairing $\widehat{\mathrm{CH}}^p(X) \times \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)$ which gives $\widehat{\mathrm{CH}}(X) = \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(X)$ the structure of a commutative ring. For two transversally intersecting cycles Y and Z, the intersection product of $[Y, g_Y]$ and $[Z, g_Z]$ can be represented by $(Y \cap Z, g_Y * g_Z)$. For $dd^c g_Y = \omega_Y - \delta_Y$ and $dd^c g_Z = \omega_Z - \delta_Z$ the $*$-product $g_Y * g_Y$ of the Green currents g_Y and g_Z is given by

$$g_Y * g_Z = g_Y \wedge \delta_Z + \omega_Y \wedge g_Z\,.$$

This current is singular along the cycle Z. Thus, the $*$-product is far from being the canonical Green current of $Y \cap Z$. However, the difference of the canonical current $G_{Y\cap Z}$ for $Y \cap Z$ and $g_Y * G_Z$ encodes how Y and Z intersect at the finite places of X.

3. The spectrum of the Laplace operator on $\mathbb{P}^1$

3.1. Eigenspaces for the Laplace operator on homogeneous spaces. We repeat here the results of [**8**]. Let $X = G/K$ be a homogeneous space equipped with a G-invariant metric. We assume that X is compact. Let U be a hermitian vector space with an action of K preserving the metric on U. Hence, we have a hermitian vector bundle $E := G \times_K U$ on X.

$$\begin{array}{ccc} G \times U & \longrightarrow & E = G \times_K U \\ \downarrow & & \downarrow \\ G & \xrightarrow{p} & X = G/K \end{array}$$

We obtain a canonical isomorphism $\mathcal{C}^\infty(G, U) = \mathcal{C}^\infty(G, p^*E)$. On $\mathcal{C}^\infty(G, U)$ we have a G-action given by $g \cdot \varphi(x) = \varphi(g^{-1}x)$. We have in $\mathcal{C}^\infty(G, U)$ the subspace of the K-invariant sections

$$\mathcal{C}^\infty(G, K, U) = \{\varphi \in \mathcal{C}^\infty(G, U) \,|\, \varphi(g \cdot k) = k^{-1}\varphi(g) \text{ for all } k \in K\}\,,$$

which is naturally identified with $\mathcal{C}^\infty(X, E)$. For an eigenvalue λ of Δ, we denote by E_λ its eigenspace. Obviously, E_λ is a G-invariant subspace of $\mathcal{C}^\infty(X, E)$ of finite dimension. Decomposing it into its irreducible components, we obtain a further decomposition of E_λ. If, on the other hand, V is an irreducible representation of G with a G-embedding $\iota : V \to \mathcal{C}^\infty(G, K, U)$, then, since Δ is G-invariant, $\iota(V)$ has to be contained in an eigenspace E_λ, for a certain λ. Thus, we have to determine the set $\mathrm{Hom}_G(V, \mathcal{C}^\infty(G, K, U))$ for all irreducible representations V of G. By Frobenius reciprocity we have

$$\begin{array}{rcl} \mathrm{Hom}_K(V, U) & \xrightarrow{\sim} & \mathrm{Hom}_G(V, \mathcal{C}^\infty(G, K, U)) \\ \psi & \mapsto & \Psi \text{ with } \Psi(v)(g) = \psi(g^{-1}v)\,. \end{array}$$

3.2. The case of the Grassmannian variety.

3.2.1. *The setup for the Grassmannian manifold.* We want to point out how to carry out the program of 3.1 for the Grassmannian variety $X := \mathrm{Grass}(m+n, n)$ of complex dimension n quotients of the vector space $\mathbb{C}^{m+n}$. That is, we have $G = \mathrm{SU}(n+m)$ and $K = S(\mathrm{U}(m) \times \mathrm{U}(n))$. Since the tangent space of $\mathrm{U}(k)$ is the space of all complex $k \times k$ matrices A with $A + \bar{A}^T = 0$ (plus the requirement that A is traceless when considering $\mathrm{SU}(k)$), we conclude that the tangent space $X = G/K$ in the point [1] can be identified with the complex $m \times n$ matrices. With the embedding

$$\begin{array}{rcl} T_{[1]}X = M_{m,n}(\mathbb{C}) & \to & T_1\mathrm{SU}(m+n) \\ A & \mapsto & \begin{pmatrix} 0 & A \\ -\bar{A}^T & 0 \end{pmatrix}, \end{array}$$

we identify $T_{[1]}X$ with a subspace of $T_1\mathrm{SU}(m+n)$. On $T_1\mathrm{SU}(m+n)$, we have a natural metric $||B||^2 = -\frac{\mathrm{tr}(B \cdot B)}{2}$. Using this identification, the action of K on $T_{[1]}X$ is given by

$$K \times T_{[1]}X \to T_{[1]}X \qquad ((g_1, g_2), A) \mapsto g_1 \cdot A \cdot g_2^{-1}\,.$$

Thus, the metric is K-invariant and defines a metric on the tangent bundle of X, the Fubini-Study metric.

3.2.2. *Determining the (p,q)-Eigenforms of the Laplace operator.* We set $U = \Lambda^p(T^{01}_{[1]}X) \otimes \Lambda^q(T^{10}_{[1]}X)$. We have to determine for all irreducible G representations V the K homomorphisms from V to U. The Lie algebras of G and K have the same Cartan subalgebra. Thus, we can reduce the determination of these homomorphisms to weight considerations of the representations. Indeed, only the representation theoretical background presented in Chapter 15 of [**3**] is needed to work out these homomorphisms. These computations can effectively be done for small values of m and n. In particular, for $n = 1$ things work well. This is illustrated in the next subsection.

3.3. The Green current of a point in $\mathbb{P}^1_{\mathbb{C}}$ again.

3.3.1. *Eigenfunctions of Δ in $\mathcal{C}^\infty(\mathbb{P}^1, \mathbb{C})$.* We apply the above to $G = \mathrm{SU}(2)$, $K = S(\mathrm{U}(1) \times \mathrm{U}(1))$, $X = \mathbb{P}^1$, and $U = \mathbb{C}$ with the trivial action of K. We consider the space $W = \mathbb{C}\langle x\rangle \oplus \mathbb{C}\langle y\rangle$ with the natural G-action. It implies representations of G on $V_n := \mathrm{Sym}^n(W)$ for all integers $n \geq 0$. The representations $\{V_n\}_{n\geq 0}$ are a complete set of inequivalent irreducible representations of G.
The decomposition of V_n into irreducible K-spaces is given by

$$V_n = \mathbb{C}\langle x^n\rangle \oplus \mathbb{C}\langle x^{n-1}y\rangle \oplus \ldots \oplus \mathbb{C}\langle y^n\rangle$$

where $\mathrm{U}(1) \cong K$ acts via $(\xi, x^k y^{n-k}) \mapsto \xi^{2k-n} x^k y^{n-k}$ on $\mathbb{C}\langle x^k y^{n-k}\rangle$. We obtain that any K-homomorphism from V_n to U must send $x^k y^{n-k}$ to zero unless $n = 2k$. Thus, we conclude that $\mathrm{Hom}_K(V_n, \mathbb{C})$ is zero for n odd, and of dimension one for n even. Hence, there exists a unique G-equivariant embedding $\Phi_{2n} : V_{2n} \to \mathcal{C}^\infty(G, K, U)$ up to scalars. The above remarks imply that $\Phi_{2n}(V_{2n})$ is contained in an eigenspace E_λ for a certain λ. Thus, a Hilbert space basis for $\mathcal{C}^\infty(\mathbb{P}^1, \mathbb{C})$, consisting of eigenfunctions of the Laplace operator on $\mathbb{P}^1$, is given by the set $\{f_{n,k} := \Phi_{2n}(x^k y^{2n-k})\}_{n\geq 0,\, k=0,\ldots,2n}$.

3.3.2. Next we compute the eigenfunctions on the open subset $\mathbb{C} \cong U_0 \subset \mathbb{P}^1$ with the complex coordinate z. Over U_0 we have a section σ of p given by

$$\sigma(z) = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}.$$

The standard action of $\sigma(z)^{-1}$ on W is given by

$$\sigma(z)^{-1}(x) = \frac{1}{\sqrt{1+|z|^2}}(x - zy) \text{ and } \sigma(z)^{-1}(y) = \frac{1}{\sqrt{1+|z|^2}}(\bar{z}x + y).$$

Eventually, we can give the local form of $f_{n,k}$ in U_0

$$\begin{aligned} f_{n,k}(z) &= \frac{x^n y^n - \text{part of } \left((\sigma(z)^{-1}(x))^k(\sigma(z)^{-1}(x))^{2n-k}\right)}{(1+|z|^2)^n} \\ &= \frac{1}{(n!)^2(1+|z|^2)^n}\frac{\partial^n}{\partial x^n}\frac{\partial^n}{\partial y^n}\left((\sigma(z)^{-1}(x))^k(\sigma(z)^{-1}(x))^{2n-k}\right) \\ &= \begin{cases} \frac{\bar{z}^{n-k}}{(1+|z|^2)^n}\sum_{l=0}^{k}(-1)^{k-l}\begin{pmatrix} k \\ l \end{pmatrix}\begin{pmatrix} 2n-k \\ n-l \end{pmatrix}|z|^{2k-2l} & \text{, for } k \leq n \\ \frac{z^{k-n}}{(1+|z|^2)^n}\sum_{l=0}^{2n-k}(-1)^{k+n+l}\begin{pmatrix} k \\ n-l \end{pmatrix}\begin{pmatrix} 2n-k \\ l \end{pmatrix}|z|^{2l} & \text{, for } k \geq n. \end{cases} \end{aligned}$$

Finally, we give the eigenfunctions for the first eigenvalues.

$$\begin{array}{lll} f_{0,0}(z) \equiv 1 & f_{1,0}(z) = \frac{2\bar{z}}{1+|z|^2} & f_{1,1}(z) = \frac{1-|z|^2}{1+|z|^2} \\ f_{1,2}(z) = \frac{-2z}{1+|z|^2} & f_{2,0}(z) = \frac{6\bar{z}^2}{(1+|z|^2)^2} & f_{2,1}(z) = \frac{3\bar{z}(1-|z|^2)}{(1+|z|^2)^2} \\ f_{2,2}(z) = \frac{1-4|z|^2+|z|^4}{(1+|z|^2)^2} & f_{2,3}(z) = \frac{-3z(1-|z|^2)}{(1+|z|^2)^2} & f_{2,4}(z) = \frac{6z^2}{(1+|z|^2)^2}\,. \end{array}$$

3.3.3. Recalling that the Laplace operator $\Delta = dd^* + d^*d$ for functions on U_0 has the form $\Delta = -4\pi(1+z\bar{z})^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$, we easily compute that $\Delta f_{n,0} = 4\pi n(n+1)f_{n,0}$. Since $\Phi_{2n}(V_{2n})$ is contained in an eigenspace, we conclude

$$\Delta f_{n,k} = 4\pi n(n+1)f_{n,k} \quad \text{for } k = 0,\ldots 2n\,.$$

Knowing all eigenvalues of Δ with their multiplicities, we can write down the zeta function $\zeta_{\mathbb{P}^1_{FS}}$ of $\mathbb{P}^1$ with the Fubini-Study metric

$$\zeta_{\mathbb{P}^1_{FS}}(s) = \sum_{n\geq 1}\frac{2n+1}{(4\pi n(n+1))^s}\,.$$

3.3.4. The orthogonality of eigenspaces of Δ implies that the scalar product

$$(f_{n,k}, f_{n',k'}) = \int_{\mathbb{P}^1} f_{n,k}\overline{f_{n',k'}}\omega$$

is zero unless $n = n'$. From our formula for the $f_{n,k}$, it follows immediately that the scalar product $(f_{n,k}, f_{n,k'})$ is zero for $k \neq k'$. Hence, the $f_{n,k}$ form an orthogonal basis. By straightforward calculations we obtain

$$||f_{n,n}||^2 = \frac{1}{2n+1}\,.$$

3.3.5. Now we are able to derive a formula for the canonical harmonic green current of the Dirac distribution δ_0. We set

$$\delta_0 = \sum_{n\geq 0, k=0\ldots 2n} a_{n,k}f_{n,k}\omega\,.$$

Applying this distribution to the function $f_{m,l}$, we obtain

$$f_{m,l}(0) = \int_{\mathbb{P}^1}\sum_{n\geq 0, k=0\ldots 2n} a_{n,k}f_{n,k}f_{m,l}\omega\,.$$

Using the equality $f_{m,l} = (-1)^{m+l}\overline{f_{m,2m-l}}$, and the pairwise orthogonality of the $f_{n,k}$, this yields $f_{m,l}(0) = a_{m,2m-l}\int_{\mathbb{P}^1} f_{m,2m-l}f_{m,l}\omega$. Since $f_{n,k}(0) = \delta_{n,k}$, we obtain

$$a_{m,l} = (2m+1)\delta_{m,l}\,.$$

Thus, we get for the Dirac distribution δ_0

$$\delta_0 = \sum_{n\geq 0}(2n+1)f_{n,n}\omega\,,$$

which allows us to write down the heat flow $\mu_{0,t}$ of δ_0

$$\mu_{0,t} = \sum_{n\geq 0}(2n+1)\exp\left(-4\pi tn(n+1)\right)f_{n,n}\omega\,.$$

We deduce, that the Green operator $G(0)$ of δ_0 is given by

$$G(0) = \sum_{n>0}\frac{(2n+1)f_{n,n}}{4\pi n(n+1)}\omega\,,$$

and that the canonical harmonic Green current has the form

$$g_0 = \sum_{n>0}\frac{(2n+1)f_{n,n}}{n(n+1)}\,.$$

The uniqueness of the canonical harmonic Green current and 2.4.2 thus implies

$$\log\left(1+\frac{1}{z\bar{z}}\right) - 1 = \sum_{n>0}\sum_{l=0}^{n}\frac{(2n+1)(-1)^l\begin{pmatrix} n \\ l \end{pmatrix}^2||z||^{2l}}{n(n+1)(1+||z||^2)^n}\,.$$

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