On multi-dimensional random walk models approximating symmetric space-fractional diffusion processes

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Abstract

In the paper the multi-dimentional analog of the Gillis-Weiss random walk model is studied. The convergence of this random walk to a space-fractional diffusion process governed by a symmetric operator defined as a hypersingular integral or the inverse of the Riesz potential in the sense of distributions is proved.

1 Introduction

In this paper we will study multidimensional random walk models approximating the Cauchy problem for fractional diffusion equations with symmetric spatial operator of fractional order. We will follow the method profitably used by Gorenflo and Mainardi for the study of scaling limits of discrete random walks, which was implemented in one-dimensional case in the series of their works (see [2] and references therein), where a number of discrete random walk models approximating fractional diffusion processes with symmetric and non-symmetric Lévy-Feller fractional differential operators in the governing equation were studied. Extending this to the multi-dimensional case is not trivial and requires to introduce some new elements. We will comment these later.

Multi-dimensional random walk is often used in modeling various processes in different areas [1], [8]. See, for instance, [5] for applications to economics and [6] to finance, [12] for modeling of copepod behaviour of animals in zoology, [7] for modeling of river flows. In the recent book [8] by M.Meerschaert and Scheffler the wide range of problems in multi-dimensional stochastic processes is highlighted, with emphasis on operator stable probability distributions, and there are also given comments on the historical development and indications of their different applications.

R.Gorenflo and F.Mainardi [2] have constructed several discrete random walk models by choosing suitable transition probabilities, which approximate stable Lévy-Feller processes for the full range of scaling order $\alpha \in (0, 2]$. Our treatment can be considered as a multi-dimentional generalization of the one-dimensional Gillis-Weiss model [3] presented in [2] by Gorenflo and Mainardi. The necessary and sufficient conditions for a random vector to belong to the domain of attraction of nondegenerate nonnormal stable law was found in [13] (see also [8]).

Our paper is organized as follows. In Section 2 we give auxiliary materials and introduce terminology that will be used in the paper. In Section 3 we mainly recall some properties of pseudo-differential operators considered in our previous paper [10] referring to it for details and lay out some elementary properties of symbols. These properties will be essentially used later in the study of the diffusion limits of random walks. In the next section we formulate in terms of random walk the problem we are going to study. In Section 5 we formulate the main result obtained in this paper.

2 Preliminaries

Let $\mathbf{R^N}$ and $\mathbf{Z^N}$ be the N-dimensional euclidean space with coordinates $\mathbf{x} = (x_1, ..., x_N)$ and the N-dimensional integer-valued lattice with nodes $j = (j_1, ..., j_N)$, respectively. Denote by $x_j = (hj_1, ..., hj_N), j \in \mathbf{Z}^N$, the nodes of the uniform lattice $\mathbf{Z}_h^N$ defined as $(h\mathbf{Z})^N$ with a positive number h, the mesh width.

Suppose that a particle is located at the origin $x_0 = \mathbf{0} = (0, ..., 0)$ at the initial time $t = 0$ and at time instances $t_1 = \tau, t_2 = 2\tau, ..., t_n = n\tau, ...$ jumps moving through nodes of the lattice $\mathbf{Z}_h^N$. Denote by p_j the probability of jumping from a point $x_k \in \mathbf{Z}_h^N$ to a point $x_{j+k} \in \mathbf{Z}_h^N$, where j and k are in $\mathbf{Z}^N$. By this we automatically assume that the particle jumps are isotropic for all directions. The numbers $p_j, j \in \mathbf{Z}^N$, are called transition probabilities. They satisfy the following conditions of non-negativity and normalization

(a) $p_j \geq 0, j \in \mathbf{Z}^N$; (b) $\sum_{j \in \mathbf{Z}^N} p_j = 1$.

We denote by p_j the value of a discrete function $p : \mathbf{Z^N} \to R^1$ at a point $j \in \mathbf{Z^N}$. For given two discrete functions, p and q we define a convolution $p * q$ by the rule

$$(p * q)_j = \sum_{k \in Z^N} p_k q_{j-k}, j \in \mathbf{Z}^N.$$

Also for a given discrete function p we define its Fourier transform (Fourier series) $\hat{p}(\xi) = F[p](\xi), \xi \in \mathbf{R}^N$, by the formula

$$\hat{p}(\xi) = \sum_{k \in \mathbf{Z}^N} p_k e^{-ik\xi},$$

and we call $\hat{p}(-\xi)$ the characteristic function for p.

In general we denote by $u(t, x)$ the probability density of a random walk, namely for given time $t > 0$ the probability of sojourn of the particle at the point $x \in \mathbf{R}^N$ at time $t, t > 0$. For the discrete random walk introduced above we use the symbol $y_j(t_n)$ emphasizing discreteness of the process, writing again $u(t, x)$ in the limit case. Heuristically we consider $y_j(t_n)$ as an approximation of $h^N u(t_n, x_j) \approx \int_{C_j} u(t_n, x) dx$, the total probability of sojourn inside a cubical cell C_j with the center x_j and side lenth h with $u(t, x)$ as density of a process in $\mathbf{R^N}$.

The lemma below is well known. For completeness we provide a brief proof.

Lemma 1 *For the density probabilities $y_j(t_n)$ the following statements hold true:*

(i) $y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n), j \in \mathbf{Z}^N$;

(ii) $y_j(t_n) = (\underbrace{p * ... * p}_{n\,times})_j$.

Proof. The probability of jumping of a particle from the point x_{j-k} to the point x_j at the time t_n is $p_k y_{j-k}(t_n)$. Summing probabilities of all possible jumps we get

$$y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n)$$

or making use of the convolution notation,

$$y_j(t_{n+1}) = (p * y(t_n))_j.$$

Using this action for $n, n-1, ..., 2$ we obtain (ii).

In our further considerations we will need some properties of cubature formulas. Let f be a continuous function integrable over $\mathbf{R}^N$ function. Then the following rectangular cubature formula

$$\int_{I\!R^N} f(x)dx = h^N \sum_{j \in \mathbf{Z}^N} f(x_j) + o(1). \tag{1}$$

is valid [9].

3 Pseudodifferential operators and symbols

In this section we consider some properties of pseudo-differential operators $A(D), D = (D_1, .., D_N), D_j = \frac{\partial}{i\partial x_j}, j = 1, ..N$, with a continuous symbol $A(\xi)$ not depending on x defined in $\mathbf{R}^N$. For a test function $\varphi(x)$ taken from the classical space $S(\mathbf{R}^N)$, the Fourier transform

$$\hat{\varphi}(\xi) = F[\varphi](\xi) = \int_{\mathbf{R}^N} \varphi(x) e^{-ix\xi} dx$$

is well defined and belongs again to $S(\mathbf{R}^N)$. Let $S'(\mathbf{R}^N)$ be the space of tempered distributions, i.e. the dual space to $S(\mathbf{R}^N)$. The Fourier transform for distributions $f \in S'(\mathbf{R}^N)$ is usually defined by the extension formula $< \hat{f}(\xi), \varphi(\xi) >=< f(x), \hat{\varphi}(x) >$.

Assume G to be an open domain in $\mathbf{R}^N$. Let a function f be in $L_p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, whose Fourier transform (taken in the sense of distributions) $\hat{f}(\xi)$ has a compact support in G. The set of all such functions endowed with the convergence in the following sense is denoted by $\Psi_{G,p}(\mathbf{R}^n)$: a sequence of functions $f_m \in \Psi_{G,p}(\mathbf{R}^n)$ is said to converge to an element $f_0 \in \Psi_{G,p}(\mathbf{R}^n)$ iff: (i) there exists a compact set $K \subset G$ such that $\operatorname{supp} \hat{f}_m \subset K$ for all $m = 1, 2, ...$; (ii) $\|f_m - f_0\|_p = (\int_{\mathbf{R}^n} |f_m - f_0|^p dx)^{\frac{1}{p}} \to 0$ for $m \to \infty$. In the case $p = 2$ we write simply $\Psi_G(\mathbf{R}^n)$ instead of $\Psi_{G,2}(I\!R^n)$ and also if $G = \mathbf{R}^N$ we omit it in the index writing $\Psi_p(\mathbf{R}^n)$. Note that according to the Paley-Wiener theorem functions in $\Psi_{G,p}(\mathbf{R}^n)$ are entire functions of finite exponential type (see [14], [15]).

Let $q = p/(p-1)$. Denote by $\Psi'_{-G,q}(\mathbf{R}^n)$ the space of all linear bounded functionals defined on the space $\Psi_{G,p}(\mathbf{R}^n)$ endowed with the weak (dual with respect to $\Psi_{G,p}(\mathbf{R}^n)$) topology. Namely, we say that a sequence of functionals $g_m \in \Psi'_{-G,q}(\mathbf{R}^n)$ converges to an element $g_0 \in \Psi'_{-G,q}(\mathbf{R}^n)$ in the weak sense if for all $f \in \Psi_{G,p}(\mathbf{R}^n)$ the sequence of numbers $< g_m, f >$ converges to $< g_0, f >$ as $m \to \infty$. By $< g, f >$ we mean the value of $g \in \Psi'_{-G,q}(\mathbf{R}^n)$ on an element $f \in \Psi_{G,p}(\mathbf{R}^n)$.

Let $A(\xi)$ be a continuous function defined in $G \subset \mathbf{R}^N$. For a function $\varphi \in \Psi_{G,p}(\mathbf{R}^n)$ a pseudo-differential operator $A(D)$ with the symbol $A(\xi)$ is defined by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^N}\int_G A(\xi)F\varphi(\xi)e^{ix\xi}d\xi, \tag{2}$$

with the integral taken over G. Note that in general this may not have sense even for infinitely differentiable functions with finite support (see [10]).

We define the operator $A(-D)$ acting in the space $\Psi'_{-G,q}(\mathbf{R}^N)$ by the extention formula

$$< A(-D)f, \varphi > = < f, A(D)\varphi >,\ f \in \Psi'_{-G,q}(\mathbf{R}^n),\ \varphi \in \Psi_{G,p}(\mathbf{R}^n). \tag{3}$$

We recall (see [10]) that the pseudo-differential operators $A(D)$ and $A(-D)$ act as

$$A(D) : \Psi_{G,p}(\mathbf{R}^N) \to \Psi_{G,p}(\mathbf{R}^N), A(-D) : \Psi'_{-G,q}(\mathbf{R}^N) \to \Psi'_{-G,q}(\mathbf{R}^N)$$

and are continuous under some conditions on their symbols.

Lemma 2. *For the symbol $A(\xi)$ of an operator $A(D)$ the following relations are true:*

(i) $A(\xi)1(x) = e^{-ix\xi}(A(D)e^{ix\xi})$; *(ii)* $A(\xi) = (A(D)e^{ix\xi})|_{x=0}$.

Note. *In relation (i) $A(\xi)1(x)$ means the direct product of two functions. Moreover the equality in (i) should be understood in the sense of the space $\Psi'_{-G,q}(\mathbf{R}^N)$.*

P r o o f. Taking into account $e^{ix\xi} \in \Psi_q'(\mathbf{R}^N)$ and the duality rules we have

$$< e^{-ix\xi}(A(-D)e^{ix\xi}), \varphi(x) >=< A(-D)e^{ix\xi}, \varphi(x)e^{ix\xi} >=$$
$$< e^{ix\xi}, A(D)(\varphi(x)e^{ix\xi}) >=< \delta(\eta-\xi), A(\eta)F[\varphi(x)e^{ix\xi}](\eta) >$$
$$=< \delta(\eta-\xi), A(\eta)\hat{\varphi}(\eta-\xi) >= A(\xi)\hat{\varphi}(0) =$$
$$A(\xi) < \hat{\delta}, \hat{\varphi} >=< A(\xi)1(x), \varphi(x) >,$$

hence the assertion (i).

In order to prove relation (ii) we have to show that $G(x,\xi) = A(D)e^{ix\xi}$ is continuous in x, and so one can calculate its value for $x = 0$. Moreover this value is equal to $A(\xi)$. Indeed, let $k_m(x)$ be a sequence of smooth functions converging to $\delta(x)$ in the weak sense. Then $\mu_m = \hat{k}_m$ converges to $1(x)$ as $m \to \infty$. Consequently, in $\Psi_q'(\mathbf{R}^N)$ we have

$$A(D)(e^{ix\xi}\mu_m(x)) = \frac{1}{(2\pi)^N}\int A(\eta)F(\mu_m(x)e^{ix\xi})(\eta)e^{-ix\eta}d\eta =$$

$$\frac{1}{(2\pi)^N}\int A(\eta)k_m(\eta-\xi)e^{-ix\eta}d\eta = \frac{1}{(2\pi)^N} < A(\eta)e^{ix\eta}, k_m(\eta-\xi) >$$

Consequently (the limit is taken in the weak sense)

$$A(D)e^{ix\xi} = \lim_{m\to\infty} A(D)(\mu_m(x)e^{ix\xi}) =$$
$$\frac{1}{(2\pi)^N} < A(\eta)e^{ix\eta}, \delta(\eta-\xi) >= A(\xi)e^{ix\xi}.$$

The latter expression is continuous as the product of two continuous functions. Hence

$$G(x,\xi)_{x=0} = (A(D)e^{ix\xi})|_{x=0} = A(\xi)e^{-ix\xi}|_{x=0} = A(\xi).$$

Q.e.d.

In our further random walk constructions $G = \mathbf{R}^N$, so that symbols of pseudo-differential operators are continuous functions in the whole space. The symbol of the Laplace operator $A(D) = \Delta$ as easily seen is $-|\xi|^2$. Both relations in Lemma 2 for symbols can be easily verified for this symbol. The pseudo-differential operator $A(D) = D_0^\alpha$ with the symbol $-|\xi|^\alpha$ can be represented with the help of a hypersingular integral (see, e.g. [4])

$$D_0^\alpha f(x) = -\frac{1}{d(\alpha,l)}\int_{\mathbf{R}^N}\frac{\Delta_y^l f(x)}{|y|^{N+\alpha}}dy \tag{4}$$

where $0 < \alpha < l$, l is a positive integer, Δ_y^l is the finite difference of the order l in the y direction, either centered or non-centered, and $d(\alpha, l)$ is a constant defined in dependence on what type of difference, centered or non-centered, is taken (see for details [4]). Note that in this paper we consider only centered case of finite difference Δ_y^l in definition of D_0^α.

Let l be a given positive integer. Denote by τ_y a shift operator with spatial vector-step y

$$(\tau_y f)(x) = f(x-y), x, y \in \mathbf{R}^N.$$

Using this operator we determine the symmetric difference operator of order l

$$(\Delta_y^l f)(x) = (\tau_{-\frac{y}{2}} - \tau_{\frac{y}{2}})^l f(x) = \sum_{k=0}^{l} (-1)^k \left(\begin{array}{c} l \\ k \end{array} \right) f(x + (\frac{l}{2} - k)y).$$

Let a constant $d(\alpha, l)$ be defined as (see [4])

$$d(\alpha, l) = \frac{\pi^{1+N/2} A_l(\alpha)}{2^\alpha \Gamma(1+\frac{\alpha}{2}) \Gamma(\frac{N+\alpha}{2}) \sin(\alpha\pi/2)}, \tag{5}$$

with $A_l(\alpha)$ determined by the formula

$$A_l(\alpha) = 2 \sum_{k=0}^{[l/2]} (-1)^{k-1} \left(\begin{array}{c} l \\ k \end{array} \right) (\frac{l}{2} - k)^\alpha. \tag{6}$$

Moreover $d(\alpha, l) \neq 0$ for all $\alpha > 0$ and for even l, but $d(\alpha, l)$ is identically zero for odd orders l. We accept that the hypersingular operator D_0^α in (4) is defined with so introduced Δ_y^l and $d(\alpha, l)$.

In the construction of random walk models leading to the diffusion equations governed by pseudo-differential operators in space the sign of the normalizing constant $d(\alpha, l)$ is important. Let $l = 2$. Then after slightly rearranging we have

$$D_0^\alpha f(x) = b(\alpha) \int_{\mathbf{R}^N} \frac{f(x-y) - 2f(x) + f(x+y)}{|y|^{N+\alpha}} dy, \tag{7}$$

where

$$b(\alpha) = \frac{\alpha \Gamma(\frac{\alpha}{2}) \Gamma(\frac{N+\alpha}{2}) sin \frac{\alpha\pi}{2}}{2^{2-\alpha} \pi^{1+N/2}}. \tag{8}$$

It is seen from (8) that the value $\alpha = 2$ is singular.

We note also that D_0^α can be considered as a fractional power of the Laplace operator, namely $D_0^\alpha = -(-\Delta)^{\alpha/2}$. From Lemma 2 it follows that

$$D_0^\alpha e^{ix\xi}|_{x=0} = b(\alpha) \int_{\mathbf{R}^N} \frac{\Delta_y^2 e^{ix\xi}}{|y|^{N+\alpha}} dy|_{x=0} = -|\xi|^\alpha, 0 < \alpha < 2.$$

Taking into account the cubature formula (1) for the integral in the right hand side of (4) we have (with $f_j = f(hj)$ and $|k|$ as Euclidean norm of $k = (k_1, ..., k_N) \in \mathbf{Z}^N$)

$$\int_{\mathbf{R}^N} \frac{\Delta_y^2 f(x_j)}{|y|^{N+\alpha}} dy = h^\alpha \sum_{k \in \mathbf{Z}^N} \frac{\Delta_k^2 f_j}{|k|^{N+\alpha}} + o(1).$$

4 Fractional differential equations and random walk

Consider the fractional order diffusion equation

$$\frac{\partial}{\partial t} u(t,x) = D_0^\alpha u(t,x),\ t > 0,\ x \in \mathbf{R}^n, \tag{9}$$

where $D_0^\alpha, 0 < \alpha < 2$, is the pseudo-differential operator defined in the previous section which has the symbol $-|\xi|^\alpha$. In accordance with $\lim_{\alpha \to 2} |\xi|^\alpha = |\xi|^2$ we will accept $D_0^2 = \Delta$, where Δ is the Laplace operator. In the general case we have formally $D_0^\alpha = -(-\Delta)^{\alpha/2}$. A weak solution, namely a distribution $G_\alpha(t,x)$, which satisfies (9) and the condition

$$G_\alpha(0,x) = \delta(x), x \in \mathbf{R}^N, \tag{10}$$

with the Dirac function $\delta(x)$, in the sense of distributions, is called a fundamental solution of the Cauchy problem (9), (10).

It is clear that in the case $\alpha = 2$ we have the classical heat conduction equation

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x),\ t > 0,\ x \in \mathbf{R}^n,$$

whose fundamental solution is the Gauss probability density

$$G_2(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4\pi t}}.$$

In the case $\alpha = 1$ the corresponding fundamental solution is given by the Cauchy-Poisson probability density (see [11])

$$G_1(t,x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \frac{1}{(|x|^2 + t^2)^{(n+1)/2}}.$$

It is well known that for the functions $G_q(t,x), q = 1, 2$, the relations [11]

$$\hat{G}_2(t,\xi) = e^{-|\xi|^2} \text{ and } \hat{G}_1(t,\xi) = e^{-t|\xi|}$$

hold.

For other values of α, $0 < \alpha < 2$, applying the Fourier transform and its inverse the fundamental solution to the Cauchy problem (9), (10) can be represented in the form

$$G_\alpha(t,x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{-t|\xi|^\alpha} e^{ix\xi} d\xi. \tag{11}$$

The question we want to explore is that for the existence of a random walk which approximates the diffusion process governed by the equation (9). In the random walk terminology we are using this means that the fundamental solution is a probability density of the diffusion limit of some discrete random walk.

Bearing this in mind let us describe the discrete random walk model approximating the fractional diffusion process governed by the equation (9) in terms of probability theory. Let $\mathbf{X}$ be an N-dimensional random vector [8] which takes values in $\mathbf{Z}^N$. Let the random vectors $\mathbf{X}_1, \mathbf{X}_2, ...$ also be N-dimensional independent identically distributed random vectors, all having their probability distribution common with $\mathbf{X}$. We introduce a spatial grid $\{x_j = jh, j \in \mathbf{Z}^N\}$, with $h > 0$ and temporal grid $\{t_n = n\tau, n = 0, 1, 2, ...\}$ with a step $\tau > 0$. Consider the sequence of random vectors

$$\mathbf{S}_n = h\mathbf{X}_1 + h\mathbf{X}_2 + ... + h\mathbf{X}_n, n = 1, 2, ...$$

taking $\mathbf{S}_0 = \mathbf{0}$ for convenience. We interpret $\mathbf{X}_1, \mathbf{X}_2, ...,$ as the jumps of a particle sitting in $x = x_0 = \mathbf{0}$ at the starting time $t = t_0 = 0$ and making a jump $\mathbf{X}_n$ from $\mathbf{S}_{n-1}$ to $\mathbf{S}_n$ at the time instance $t = t_n$. Then the position $\mathbf{S}(t)$ of the particle at time t is

$$\sum_{1 \le k \le t/\tau} \mathbf{X}_k.$$

Recall that the probability of sojourn of the particle in x_j at the time t_n was denoted by $y_j(t_n)$. Taking into account the recursion $\mathbf{S}_{n+1} = \mathbf{S}_n + h\mathbf{X}_n$ we have

$$y_j(t_{n+1}) = \sum_{k\in\mathbf{Z}^N} p_k y_{j-k}(t_n), j \in \mathbf{Z}^N, n = 0, 1, ...$$

The convergence of the sequence $\mathbf{S}_n$ when $n \to \infty$ means convergence of the discrete probability law $(y_j(t_n))_{j\in\mathbf{Z}^N}$ properly rescaled as explained in the next Section, to the probability law with a density $u(t, x)$ in the sense of distributions (in law). This is equivalent to the locally uniform convergence of the corresponding characteristic functions (see for details [8]). This will be used in the next Section to prove the convergence of the constructed random walks to the fundamental solution of the governing diffusion equation (limit process).

5 Main result

Consider the Cauchy problem

$$\frac{\partial}{\partial t}u(t, x) = D_0^\alpha u(t, x),\ t > 0,\ x \in \mathbf{R}^N, \tag{12}$$

with the initial condition

$$u(0, x) = \delta(x), x \in \mathbf{R}^N, \tag{13}$$

where $0 < \alpha < 2$, D_0^α is the pseudo-differential operator defined above with the symbol $-|\xi|^\alpha$; $D_0^2 = \Delta$.

Assume that the hypersingular integral in the right hand side of (12) is defined by the centered finite difference of second order ($l = 2$), i.e by the formula (7). Using the cubature formula for discrete approximation of the right hand side of (12), namely

$$D_0^\alpha u(t, x_j) \approx b(\alpha) \sum_{k\in\mathbf{Z}^N} \frac{u_{j+k}(t) - 2u_j(t) + u_{j-k}(t)}{|k|^{N+\alpha}h^\alpha}$$

where $b(\alpha)$ is norming constant, defined in (8). Replacing $\frac{\partial u}{\partial t}$ by the first order difference ratio

$$\frac{\partial u}{\partial t} \approx \frac{u_j(t_{n+1}) - u_j(t_n)}{\tau}$$

with the time step $\tau = t_{n+1} - t_n$ we have the relation

$$y_j(t_{n+1}) = \sum_{k \in \mathbf{Z}^N} p_k y_{j-k}(t_n)$$

which shows that the transition probabilities have the form

$$p_k = \begin{cases} 1 - \mu b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} |m|^{-N-\alpha}, & \text{if } k = 0; \\ \\ \mu b(\alpha) |k|^{-N-\alpha}, & \text{if } k \neq 0, \end{cases}$$

where the scaling parameter is $\mu = \frac{2\tau}{h^\alpha}$.

We will require that the transition probabilities satisfy the properties:

(i) $\sum_{k \in \mathbf{Z}^N} p_k = 1$; (ii)$p_k \geq 0, k \in \mathbf{Z}^N$.

Now we formulate the main result of this paper. In the formulation we will use the notations introduced in the previous section.

Theorem. Let the transition probabilities $p_k = P(\mathbf{X} = x_k), k \in \mathbf{Z}^N$, *of the random vector* $\mathbf{X}$ *be given as follows:*

a) if $0 < \alpha < 2$, *then*

$$p_k = \begin{cases} 1 - \mu b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} \frac{1}{|m|^{N+\alpha}}, & \textit{if } k = 0; \\ \\ \mu b(\alpha) |k|^{-(N+\alpha)}, & \textit{if } k \neq 0, \end{cases}$$

with μ *satisfying the condition*

$$0 < \mu \leq \frac{1}{b(\alpha) \sum_{m \in \mathbf{Z}^N \setminus \{0\}} |m|^{-N-\alpha}}.$$

and the space and time steps h *and* τ *being connected by the scaling relation* $\tau = \tau(h) = \mu h^\alpha / 2$.

b) if $\alpha = 2$, *then*

$$p_k = \begin{cases} \frac{1}{2N}, & \textit{if } |k| = 1; \\ \\ 0, & \textit{if } |k| = 0, \end{cases}$$

with $\tau = \frac{h^2}{2N}$.

Then the sequence of random vectors $\mathbf{S}_n = h\mathbf{X}_1 + ... + h\mathbf{X}_n$, *converges as* $n \to \infty$ *in the sense of distributions to the random vector whose probability* *density is the fundamental solution of the Cauchy problem (12), (13), i.e. $G(t,x)$ defined in (11).*

P r o o f. We have to show that the sequence of random vectors $\mathbf{S}_n$ tends to the random vector with pdf

$$G(t,x)=\frac{1}{(2\pi)^N}\int_{\mathbf{R}^N}e^{-t|\xi|^\alpha}e^{ix\xi}d\xi.$$

It is obvious that the Fourier transform of $G(t,x)$ with respect to the variable x is the function $\hat{G}(t,\xi)=e^{-t|\xi|^\alpha}$. Let $\hat{p}(-\xi)$ be the characteristic function corresponding to the discrete function p_k, $k\in\mathbf{Z}^N$, that is

$$\hat{p}(-\xi)=\sum_{k\in\mathbf{Z}^N}p_ke^{ik\xi}.$$

As a consequence of Lemma 1 and the well known fact that convolution goes over in multiplication by the Fourier transform, the characteristic function of $y_j(t_n)$ can be represented in the form

$$\hat{y}_j(t_n,-\xi)=\hat{p}^n(-\xi).$$

Taking this into account it suffices to show that

$$\hat{p}^n(-h\xi)\to e^{-t|\xi|^\alpha},n\to\infty. \tag{14}$$

The latter is equivalent to

$$\lim_{h\to 0}\frac{\ln\hat{p}(-h\xi)}{\tau(h)}=-|\xi|^\alpha.$$

where $\tau(h)=\frac{t}{n}=\mu h^\alpha/2$. But for us it is more convenient to use the process (14). ¿From the continuity of e^s it is readily seen that if a sequence s_n converges to s for $n\to\infty$, then

$$\lim(1+\frac{s_n}{n})^n=e^s. \tag{15}$$

We have

$$\hat{p}^n(-h\xi)=(1-\frac{\mu b(\alpha)}{2}\sum_{k\in\mathbf{Z}^N\setminus\{0\}}\frac{2}{|k|^{N+\alpha}}(1-e^{ik\xi h}))^n=$$

$$(1+\frac{tb(\alpha)\sum_{k\in\mathbf{Z}^N\setminus\{0\}}\frac{\Delta^2e^{ik\xi h}}{|kh|^{N+\alpha}}h^N}{n})^n.$$

It follows from (1) and Lemma 2 (ii) that

$$b(\alpha)\sum_{k\in Z^N}\frac{\Delta^2 e^{ik\xi h}}{|kh|^{N+\alpha}}h^N$$

tends to $(D_0^\alpha e^{ix\xi})_{|x=0} = -|\xi|^\alpha$ as $h \to 0$ (or, the same, $n \to \infty$) for all $\alpha \in (0,2)$. Hence in accordance with (15) we have

$$\hat{u}(t_n, -\xi) = \hat{p}^n(-h\xi) \to e^{-t|\xi|^\alpha}, n \to \infty.$$

The case $\alpha = 2$ with the transition probabilities given in b) of Theorem can be treated as an exercise. Q.e.d.

Comments. 1. The obtained result relates to the fully symmetric random walk. In one-dimensional case the symmetric random walk model (called Gillis-Weiss model) was studied by R.Gorenflo and F.Mainardi in [2]. They used in the definition of Feller's potential the centered second order difference. Note that in [2] $\alpha = 2$ is singular (as well as in Theorem). The method used for proving the above theorem is actually applicable for operators with symmetric spatial part more general then D_0^α. Namely, if the symbol $A(\xi)$ of an operator $A(D)$ satisfies the condition $C_1|\xi|^\alpha \le -A(\xi) \le C_2|\xi|^\alpha$, where C_1, C_2 are some positive constants, and $0 < \alpha \le 2$, then the analogous result holds true.

2. As it was mentioned above the one-dimensional case was studied in the paper [2]. There convergence was proved by using the asymptotic behaviour of an integral obtained by summing a formal Laurent series of a complex variable z. Instead of this (which is no longer feasable in multi-dimensional case) we used the properties of symbols and cubature formulas. To compare with the one-dimensional case we note that the coefficient $b(1)$ of the inverse Riesz potential takes the value $1/\pi$, which coincides with $b(1) = 1/\pi$ used in [2].

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