NON-SYMMETRIC CONVENIENT TOPOLOGY AND ITS RELATIONS TO CONVENIENT TOPOLOGY

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Dedicated to Guillaume Brümmer on the occasion of his 70th birthday

Abstract. Preuniform convergence spaces are introduced as a generalization of semiuniform convergence spaces with the advantage that the construct **PUConv** of preuniform convergence spaces is not only a strong topological universe, i.e. it fulfills nice convenient properties, such as the construct **SUConv** of semiuniform convergence spaces, but it allows one to study even non-symmetric topological concepts as well as quasiuniform concepts. Furthermore, a completion for preuniform convergence spaces is investigated from which the usual Hausdorff completion of a separated uniform space as well as the T_0-quasiuniform bicompletion of a T_0-quasiuniform space in the sense of P. Fletcher and W. F. Lindgren can be derived.

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0. Introduction

Convenient Topology consists in the study of strong topological universes in which symmetric convergence structures, such as symmetric topological structures and various generalizations, and uniform convergence structures, such as uniform structures and various generalizations, are available, whereas in *Non-Symmetric Convenient Topology* strong topological universes are studied, in which non-symmetric convergence structures, such as topological structures and various generalizations, and quasiuniform convergence structures, such as quasiuniform structures and various generalizations, are at hand. Furthermore, in both cases, such a strong topological universe should be easily described by means of suitable axioms and should not be too big. Thus, the construct **SUConv** of semiuniform convergence spaces is a good candidate for this purpose in Convenient Topology (cf. [6]). Omitting the 'symmetry' axiom in the definition of a semiuniform convergence space, we obtain a preuniform convergence space, and the construct **PUConv** of preconvergence spaces is mainly studied in Non-Symmetric Convenient Topology.

It is well-known that the construct **QUnif** of quasiuniform spaces contains the construct **Unif** of uniform spaces in the sense of A. Weil [7] as a bireflective and bicoreflective subconstruct, and the construct **Top** of topological spaces as a bicoreflectively embedded subconstruct. This situation has an analogue in Non-Symmetric Convenient Topology: **SUConv** is a bireflective and bicoreflective subconstruct of **PUConv**, and the construct **GConv** of generalized convergence spaces can be bicoreflectively embedded in **PUConv**. Consequently, each preuniform convergence space has an underlying generalized convergence space, namely its bicoreflective **GConv**-modification. Further, **SUConv** contains the construct $\mathbf{KConv}_S$ of symmetric Kent convergence spaces as a bicoreflectively embedded subconstruct. Therefore, each preuniform convergence space has an underlying symmetric Kent convergence space, namely its bicoreflective $\mathbf{KConv}_S$-modification. Hence, there are two possibilities of defining filter convergence in a preuniform convergence space $\mathbf{X}$:

1^0 convergence in the underlying symmetric Kent convergence space, now called convergence in $\mathbf{X}$, and

2^0 convergence in the underlying generalized convergence space, now called preconvergence in $\mathbf{X}$.

It is easily checked that each convergent filter is preconvergent, but not vice versa.

If a filter on a preuniform convergence space is called a Cauchy filter iff it is a Cauchy filter on its bicoreflective **SUConv**-modification (or equivalently: on its bireflective **SUConv**-modification), then completeness (i.e. convergence of Cauchy filters) and completions can be studied in **PUConv**, and it turns out that the simple completion, known from Convenient Topology, exists in **PUConv**. Since **Unif** and **QUnif** are bireflective in **PUConv**, every preuniform convergence space has an underlying uniform space and an underlying quasiuniform space, namely its bireflective **Unif**-modification and its bireflective **QUnif**-modification respectively.

It turns out that for a T_0-quasiuniform space (resp. separated uniform space)

the underlying quasiuniform space (resp. uniform space) of its simple completion is the T_0-quasiuniform bicompletion in the sense of P. Fletcher and W. F. Lindgren [2] (resp. the usual Hausdorff completion).
By the way, the natural function space structure in **GConv**, i.e. the structure of continuous convergence, can be derived from the natural function space structure in **PUConv** by bicoreflective modification.
The terminology of this paper corresponds to [6].

1. Preliminaries

1.1 Definitions. 1) A *preuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such the following are satisfied:
UC_1) The filter generated by $\{(x,x)\}$, i.e. $\dot{x} \times \dot{x}$, belongs to $\mathcal{J}_X$ for each $x \in X$.
UC_2) $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$.
If $(X, \mathcal{J}_X)$ is a preuniform convergence space, then the elements of $\mathcal{J}_X$ are called *quasiuniform filters.*
2) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between preuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X$.
3) The construct of preuniform convergence spaces (and uniformly continuous maps) is denoted by **PUConv**.

1.2 Proposition. PUConv *is a strong topological universe.*

Proof. cf. [5; 3.10]

1.3 Remarks. 1) If X is a set and $F(X)$ denotes the set of all filters on X, then further axioms for subsets $\mathcal{J}_X$ of $F(X \times X)$ (besides UC_1) and UC_2)) can be considered:
UC_3) $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1} : F \in \mathcal{F}\} \in \mathcal{J}_X$ where $F^{-1} = \{(x,y) : (y,x) \in F\}$,
UC_4) $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \cap \mathcal{G} \in \mathcal{J}_X$,
UC_5) $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{G} \in \mathcal{J}_X$ imply $\mathcal{F} \circ \mathcal{G} \in \mathcal{J}_X$ (whenever $\mathcal{F} \circ \mathcal{G}$ exists, i.e. $F \circ G = \{(x,y) : \exists\ z \in X \text{ with } (x,z) \in G \text{ and } (z,y) \in F\} \neq \phi$ for every $F \in \mathcal{F}$ and every $G \in \mathcal{G}$), where $\mathcal{F} \circ \mathcal{G}$ is the filter generated by the filter base $\{F \circ G : F \in \mathcal{F}, G \in \mathcal{G}\}$.

A preuniform convergence is called
a) a *semiuniform convergence space* provided that UC_3) is fulfilled,
b) a *semiuniform limit space* provided that UC_3) and UC_4) are fulfilled,
c) a *uniform limit space* provided that UC_3), UC_4) and UC_5) are fulfilled,
d) a *preuniform limit space* provided that UC_4) is fulfilled, and
e) a *quasiuniform limit space* provided that UC_4) and UC_5) are fulfilled.
The corresponding full subconstructs of **PUConv** are denoted by **SUConv**, **SULim**, **ULim**, **PULim**, and **QULim** respectively.
Furthermore, a preuniform convergence space $(X, \mathcal{J}_X)$ is called *quasiuniform* (resp. *uniform*) provided that there is a quasiuniformity (resp. uniformity) $\mathcal{V}$ on X such that $\mathcal{J}_X = [\mathcal{V}] := \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{V}\}$. The corresponding full subconstruct of **PUConv** is denoted by **QUnif** (resp. **Unif**). Obviously, **QUnif** (resp. **Unif**) is concretely isomorphic to the usual construct of quasiuniform spaces (and uniformly continuous maps) in the sense of [2] (resp. the usual construct of uniform spaces [and uniformly continuous maps] in the sense of A. Weil [7], where no separation axiom is assumed).
2) In contrast to the situation for **PUConv** the subconstructs **QUnif** and **Unif** are neither cartesian closed nor extensional (cf. [1] or [6; 3.1.9.2 and 3.2.7.2], where the same example for **Unif** and **QUnif** can be used). Though in **Unif** products of quotients are quotients (cf. [3]) the corresponding question for **QUnif** is open.
3) It is well-known that **ULim** is cartesian closed (cf. [4]), but its 'non-symmetric' analogue **QULim** is not cartesian closed (cf. [1; 1.5.3]). None of both constructs is extensional (cf. [1]).

1.4 Proposition. **SUConv** *is a bireflective and bicoreflective* (*full and isomorphism-closed*) *subconstruct of* **PUConv**. *In particular, for each* $(X, \mathcal{J}_X) \in |$**PUConv**$|$,
$1_X : (X, \mathcal{J}_X^c) \to (X, \mathcal{J}_X)$ *with* $\mathcal{J}_X^c = \{\mathcal{F} \in \mathcal{J}_X : \mathcal{F}^{-1} \in \mathcal{J}_X\}$
is the bicoreflection of $(X, \mathcal{J}_X)$ *w.r.t* **SUConv**, *and*
$1_X : (X, \mathcal{J}_X) \to (X, \mathcal{J}_X^r)$ *with* $\mathcal{J}_X^r = \mathcal{J}_X \cup \{\mathcal{F} \in F(X \times X) : \mathcal{F}^{-1} \in \mathcal{J}_X\}$
is the bireflection of $(X, \mathcal{J}_X)$ *w.r.t* **SUConv**.

1.5 Definition. Let $(X, \mathcal{J}_X)$ be a preuniform convergence space. Then $(X, \mathcal{J}_X^c)$ (resp. $(X, \mathcal{J}_X^r)$) is called its *bicoreflective* (*resp. bireflective*) **SUConv**-*modification.*

1.6 Remarks. 1) a) Since the bicoreflective **SUConv**-modification of a pre-

uniform limit space, a quasiuniform limit space or a quasiuniform space is a semiuniform limit space, a uniform limit space or a uniform space respectively, we obtain the following:
α) **SULim** is bicoreflective in **PULim**,
β) **ULim** is bicoreflective in **QULim**, and
γ) **Unif** is bicoreflective in **QUnif** (cf. [2; p.2] for this construction of a uniform space from a quasiuniform space).
b) For each $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$, let $(\mathcal{J}_X)_L = \{\mathcal{F} \in F(X \times X) : \exists\, \mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathcal{J}_X$ with $\bigcap_{i=1}^{n} \mathcal{F}_i \in \mathcal{F}\}$, and $(\mathcal{J}_X)_Q = \{\mathcal{F} \in F(X \times X) : \exists\, \mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathcal{J}_X$ with $\mathcal{F}_1 \circ \ldots \circ \mathcal{F}_n \subset \mathcal{F}\}$. Then
α) $1_X : (X, \mathcal{J}_X) \to (X, (\mathcal{J}_X^r)_L)$ is the bireflection of $(X, \mathcal{J}_X) \in |\mathbf{PULim}|$ w.r.t. **SULim**, and
β) $1_X : (X, \mathcal{J}_X) \to (X, ((\mathcal{J}_X^r)_Q)_L)$ is the bireflection of $(X, \mathcal{J}_X) \in |\mathbf{QULim}|$ w.r.t. $|\mathbf{ULim}|$.
Furthermore,
γ) $1_X : (X, [\mathcal{V}]) \to (X, [\mathcal{U}])$ is the bireflection of $(X, [\mathcal{V}]) \in \mathbf{QUnif}$ w.r.t **Unif**, provided that $\mathcal{U} = \{V \in \mathcal{V}$: there is a sequence $(V_n)_{n \in \mathbb{N}}$ with $V_n \in \mathcal{V}$ and $V_n^{-1} \in \mathcal{V}$ for each $n \in \mathbb{N}$, and $V_1 = V$, such that $V_{n+1}^2 \subset V_n$ for each $n \in \mathbb{N}\}$.
2) Each of the constructs in the following list

$$\mathbf{PUConv} \supset \mathbf{PULim} \supset \mathbf{QULim} \supset \mathbf{QUnif}$$

is a bireflective (full and isomorphism-closed) subconstruct of the preceding ones (similar to [6; 2.3.2.3]).
3) It has been proved by A. Behling [1] that **PULim** is a topological universe. Since it is easily checked that quotients in **PULim** are formed as in **PUConv**, and since **PULim** is bireflective in **PUConv** and thus, products in **PULim** are formed as in **PUConv**, we obtain that in **PULim** quotients are productive because **PUConv** has this property. Hence, **PULim** *is a strong topological universe.*

2. Preconvergence spaces

2.1 Definitions. 1) Let $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$. A filter $\mathcal{F}$ on X *preconverges* to $x \in X$, denoted by $(\mathcal{F}, x) \in q_{\mathcal{J}_X}$ or $\mathcal{F} \xrightarrow{q_{\mathcal{J}_X}} x$, iff $\dot{x} \times \mathcal{F} \in \mathcal{J}_X$.
2) A preuniform convergence space $(X, \mathcal{J}_X)$ is called a *preconvergence space* provided that $\mathcal{J}_X = \mathcal{J}_{q_{\mathcal{J}_X}}$, where $\mathcal{J}_{q_{\mathcal{J}_X}} = \{\mathcal{G} \in F(X \times X)$: there is some $(\mathcal{F}, x) \in q_{\mathcal{J}_X}$ with $\mathcal{G} \supset \dot{x} \times \mathcal{F}\}$, in other words: a preconvergence space is a

preuniform convergence space which is 'generated' by its preconvergent filters.
3) a) A *generalized convergence space* is a pair (X, q), where X is a set and $q \subset F(X) \times X$ such that the following are satisfied:
C_1) $(\dot{x}, x) \in q$ for each $x \in X$
C_2) $(\mathcal{F}, x) \in q$ whenever $(\mathcal{G}, x) \in q$ and $\mathcal{G} \subset \mathcal{F}$.
b) A map $f : (X, q) \to (X', q')$ between generalized convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.

2.2 Corollary. *If $(X, \mathcal{J}_X)$ is a preuniform convergence space, then $(X, q_{\mathcal{J}_X})$ is a generalized convergence space.*

2.3 Proposition. *For each generalized convergence space (X, q) there is finest preuniform convergence space structure $\mathcal{J}_q$ on X which induces q, i.e. $q_{\mathcal{J}_q} = q$, where $q_{\mathcal{J}_q} = \{(\mathcal{F}, x) \in F(X) \times X) : \dot{x} \times \mathcal{F} \in \mathcal{J}_q\}$.*

Proof. Let $\mathcal{J}_q = \{\mathcal{G} \in F(X \times X)$: there is some $(\mathcal{F}, x) \in q$ with $\mathcal{G} \supset \dot{x} \times \mathcal{F}\}$. Obviously, $(X, \mathcal{J}_q)$ is a preuniform convergence space. Furthermore, $q_{\mathcal{J}_q} = q$:
a) $(\mathcal{F}, x) \in q$ implies $(\mathcal{F}, x) \in q_{\mathcal{J}_q}$ since $\dot{x} \times \mathcal{F} \supset \dot{x} \times \mathcal{F}$.
b) If $(\mathcal{F}, x) \in q_{\mathcal{J}_q}$ there is some $(\mathcal{G}, y) \in q$ such that $\dot{x} \times \mathcal{F} \supset \dot{y} \times \mathcal{G}$ which implies $x = y$ and $\mathcal{F} \supset \mathcal{G}$. Thus, $(\mathcal{F}, x) \in q$.
If $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$ such that $q_{\mathcal{J}_X} = q$, then $\mathcal{J}_q \subset \mathcal{J}_X$: Let $\mathcal{G} \in \mathcal{J}_q$. Hence, there is some $(\mathcal{F}, x) \in q = q_{\mathcal{J}_X}$, i.e. $\dot{x} \times \mathcal{F} \in \mathcal{J}_X$, such that $\mathcal{G} \supset \dot{x} \times \mathcal{F}$. Consequently, $\mathcal{G} \in \mathcal{J}_X$.

2.4 Proposition. *The construct* **PConv** *of preconvergence spaces* (*and uniformly continuous maps*) is bicoreflective in **PUConv** *and concretely isomorphic to the construct* **GConv** *of generalized convergence spaces* (*and continuous maps*).

Proof. A) $1_X : (X, \mathcal{J}_{q_{\mathcal{J}_X}}) \to (X, \mathcal{J}_X)$ is the desired bicoreflection of $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$ w.r.t **PConv**:
1. $(X, \mathcal{J}_{q_{\mathcal{J}_X}})$ is a preconvergence space since, by 2.3, $q_{\mathcal{J}_X} = q_{\mathcal{J}_{q_{\mathcal{J}_X}}}$, and consequently, $\mathcal{J}_{q_{\mathcal{J}_X}} = \mathcal{J}_{q_{\mathcal{J}_{q_{\mathcal{J}_X}}}}$.
2. $1_X : (X, \mathcal{J}_{q_{\mathcal{J}_X}}) \to (X, \mathcal{J}_X)$ is uniformly continuous, i.e. $\mathcal{J}_{q_{\mathcal{J}_X}} \subset \mathcal{J}_X$, which is easily checked.
3. If $(Y, \mathcal{J}_Y) \in |\mathbf{PConv}|$ and $f : (Y, \mathcal{J}_Y) \to (X, \mathcal{J}_X)$ is uniformly continuous,

then $f:(Y,\mathcal{J}_Y)\to(X,\mathcal{J}_{q_{\mathcal{J}_X}})$ is uniformly continuous, too:
Let $\mathcal{G}\in\mathcal{J}_Y=\mathcal{J}_{q_{\mathcal{J}_Y}}$. Thus, $\mathcal{G}\supset\dot{y}\times\mathcal{F}$ with $\dot{y}\times\mathcal{F}\in\mathcal{J}_Y$, which implies $f\times f(\dot{y}\times\mathcal{F})=\dot{f(y)}\times f(\mathcal{F})\subset f\times f(\mathcal{G})$, and since, by assumption, $f\times f(\dot{y}\times\mathcal{F})\in\mathcal{J}_X$, it follows that $f\times f(\mathcal{G})\in\mathcal{J}_{q_{\mathcal{J}_X}}$.
B) The following are valid:
a) $q_{\mathcal{J}_q}=q$ for each generalized convergence space structure q (cf. 2.3.).
b) $\mathcal{J}_{q_{\mathcal{J}_X}}=\mathcal{J}_X$ for each preconvergence space structure $\mathcal{J}_X$ (cf. 2.1.2)).
c) If $f:(X,q)\to(X',q')$ is a continuous map between generalized convergence spaces, then $f:(X,\mathcal{J}_q)\to(X',\mathcal{J}_{q'})$ is uniformly continuous: Let $\mathcal{G}\in\mathcal{J}_q$, i.e. there is some $(\mathcal{F},x)\in q$ with $\mathcal{G}\supset\dot{x}\times\mathcal{F}$. By assumption, $(f(\mathcal{F}),f(x))\in q'$. Furthermore, $f\times f(\mathcal{G})\supset\dot{f(x)}\times f(\mathcal{F})$, i.e. $f\times f(\mathcal{G})\in\mathcal{J}_{q'}$.
d) If $f:(X,\mathcal{J}_X)\to(Y,\mathcal{J}_Y)$ is a uniformly continuous map between preuniform convergence spaces, then $f:(X,q_{\mathcal{J}_X})\to(Y,q_{\mathcal{J}_Y})$ is continuous: Let $(\mathcal{F},x)\in q_{\mathcal{J}_X}$, i.e. $\dot{x}\times\mathcal{F}\in\mathcal{J}_X$. By assumption, $f\times f(\dot{x}\times\mathcal{F})=\dot{f(x)}\times f(\mathcal{F})\in\mathcal{J}_Y$, i.e. $(f(\mathcal{F}),f(x))\in q_{\mathcal{J}_Y}$.
It follows from a)-d) that **PConv** is concretely isomorphic to **GConv**.

2.5 Corollary. **PConv** *is closed under formation of quotients and coproducts* (*both formed in* **PUConv**), *and contains all discrete* **PUConv**-*objects.*

2.6 Remark. By 2.4 we need not distinguish between preconvergence spaces and generalized convergence spaces.

2.7 Definition. Let $(X,\mathcal{J}_X)\in|\mathbf{PUConv}|$. Then $(X,q_{\mathcal{J}_X})$ is called its *underlying generalized convergence space.*

2.8 Corollary. *If* $(f_i:(X,\mathcal{J}_X)\to(X_i,\mathcal{J}_{X_i}))_{i\in I}$ *is an initial source in* **PUConv**, *then* $(f_i:(X,q_{\mathcal{J}_X})\to(X_i,q_{\mathcal{J}_{X_i}}))_{i\in I}$ *is an initial source in* **GConv**.

2.9 Proposition. *The underlying generalized convergence space of a quasiuniform space* (*regarded as a preuniform convergence space*) *is a topological space.*

Proof. Let $(X,[\mathcal{V}])\in|\mathbf{PUConv}|$ be quasiuniform. Then
1. $(X,q_{[\mathcal{V}]})$ is a pretopological space:
Let $x\in X,\mathcal{M}=\{\mathcal{F}\in F(X):\mathcal{F}\xrightarrow{q_{[\mathcal{V}]}}x\}$, and $\mathcal{V}(x)=\bigcap\mathcal{M}$. Then $\mathcal{V}(x)\xrightarrow{q_{[\mathcal{V}]}}x$ since $\dot{x}\times\mathcal{V}(x)\supset\mathcal{V}$ (namely, for each $\mathcal{F}\in\mathcal{M},\dot{x}\times\mathcal{F}\supset\mathcal{V}$,

i.e. if $V \in \mathcal{V}$ there is some $F_{\mathcal{F}} \in \mathcal{F}$ such that $\{x\} \times F_{\mathcal{F}} \subset V$, and consequently $W = \bigcup\{F_{\mathcal{F}} : \mathcal{F} \in \mathcal{M}\} \in \mathcal{V}(x)$ such that $\{x\} \times W \subset V$).
2. For each $x \in X, \mathcal{V}(x) = \{V(x) : V \in \mathcal{V}\}$, since the following are equivalent:
(1)$\mathcal{F} \xrightarrow{q_{[\mathcal{V}]}} x$,
(2)$\mathcal{F} \supset \{V(x) : V \in \mathcal{V}\}$.
[(1) $\Rightarrow$ (2). Let $V \in \mathcal{V}$. By assumption, there is some $F \in \mathcal{F}$ such that $\{x\} \times F \subset V$. Hence, $F \subset V(x)$ which implies $V(x) \in \mathcal{F}$.
(2) $\Rightarrow$ (1). Let $V \in \mathcal{V}$. Then $V(x) \in \mathcal{F}$, and $V \supset \{x\} \times V(x)$, i.e. $V \in \dot{x} \times \mathcal{F}$.]
3. It is well-known that there is a unique topology on X such that for each $x \in X, \{V(x) : V \in \mathcal{V}\}$ is the neighborhood filter of x w.r.t. this topology (cf. e.g. [2]).
1.-3. imply that $(X, q_{[\mathcal{V}]})$ is a topological space.

2.10 Remarks. 1) 2.3 is similar to the well-known fact that for each topology there is a finest quasiuniformity inducing this topology.
2) 2.4 is similar to the fact that the construct **FQUnif** of fine quasiuniform spaces is bicoreflective in **QUnif** and concretely isomorphic to the construct **Top** of topological spaces.
3) There is another possibility to introduce a generalized convergence structure in a preuniform convergence space $(X, \mathcal{J}_X)$:
$\mathcal{F} \in F(X)$ *converges to* x, denoted by $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$, iff $(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) \in \mathcal{J}_X$. Since $(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) = (\mathcal{F} \times \mathcal{F}) \cap (\dot{x} \times \mathcal{F}) \cap (\mathcal{F} \times \dot{x}) \cap (\dot{x} \times \dot{x}) \subset \dot{x} \times \mathcal{F}$, we obtain, that *each convergent filter in a preuniform convergence space preconverges. On the other hand, a preconvergent filter in a preuniform convergence space need not be convergent* as the following *counterexample* shows:
Let $\mathbb{R}_t$ be the usual topological space of real numbers, and let $(\mathbb{R}, \mathcal{J}_{q_{\mathbb{R}}})$ be its corresponding preconvergence space, i.e. $\mathcal{F} \in \mathcal{J}_{q_{\mathbb{R}}}$ iff there is some $x \in \mathbb{R}$ such that $\mathcal{F} \supset \dot{x} \times \mathcal{U}(x)$, where $\mathcal{U}(x)$ denotes the neighborhood filter of x in $\mathbb{R}_t$. Then the preconvergent filters in $(\mathbb{R}, \mathcal{J}_{q_{\mathbb{R}}})$ are exactly the convergent filters in $\mathbb{R}_t$ (cf. 2.3). In particular, the elementary filter $\mathcal{F}_e$ of $(\frac{1}{n})_{n \in \mathbb{N}}$ is a preconvergent filter in $(\mathbb{R}, \mathcal{J}_{q_{\mathbb{R}}})$, but it is not convergent in $(\mathbb{R}, \mathcal{J}_{q_{\mathbb{R}}})$ since $\mathcal{F}_e \neq \dot{\alpha}$ for each $\alpha \in \mathbb{R}$. Note: *A filter $\mathcal{F}$ on a preconvergence space $(X, \mathcal{J}_X)$ converges to $x \in X$ iff $\mathcal{F} = \dot{x}$.*
4) a) *There is no difference between preconvergence and convergence in uniform limit spaces.* (If $(X, \mathcal{J}_X) \in |\mathbf{ULim}|$ and $\dot{x} \times \mathcal{F} \in \mathcal{J}_X$, then $(\dot{x} \times \mathcal{F})^{-1} = \mathcal{F} \times \dot{x} \in \mathcal{J}_X$ and $\mathcal{F} \times \mathcal{F} = (\dot{x} \times \mathcal{F}) \circ (\mathcal{F} \times \dot{x}) \in \mathcal{J}_X$ which imply $(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) = (\mathcal{F} \times \mathcal{F}) \cap (\dot{x} \times \mathcal{F}) \cap (\mathcal{F} \times \dot{x}) \cap (\dot{x} \times \dot{x}) \in \mathcal{J}_X$).

b) *There is no difference between preconvergence and convergence in* **Fil**-*determined preuniform convergence spaces*, where a preuniform convergence space $(X, \mathcal{J}_X)$ is called **Fil**-determined provided that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there is some $\mathcal{G} \in F(X)$ with $\mathcal{G} \times \mathcal{G} \in \mathcal{J}_X$, and $\mathcal{F} \supset \mathcal{G} \times \mathcal{G}\}$. This is easily checked. Obviously, every **Fil**-determined preuniform convergence space is a semiuniform convergence space.
5) In order to prove 2.9 it is essential that preconvergence is not defined via $\mathcal{F} \times \dot{x}$ (cf. "(1) $\Rightarrow$ (2)" in parts 2 of the proof of 2.9) which would also lead to a generalized convergence structure.

2.11 Proposition. **PConv** *is closed under formation of subspaces and finite products* (*both formed in* **PUConv**).

Proof. 1) Let $(Y, \mathcal{J}_Y) \in |\mathbf{PConv}|$ and $(X, \mathcal{J}_X)$ a subspace of $(Y, \mathcal{J}_Y)$ in **PUConv**, where $X \subset Y$. It suffices to prove that $\mathcal{J}_X \subset \mathcal{J}_{q_{\mathcal{J}_X}}$ since the inverse inclusion is always valid. Let $\mathcal{H} \in \mathcal{J}_X$. Then $(i \times i)(\mathcal{H}) \in \mathcal{J}_Y = \mathcal{J}_{q_{\mathcal{J}_Y}}$ (where $i : X \to Y$ denotes the inclusion map), i.e. $i \times i(\mathcal{H}) \supset \dot{y} \times \mathcal{G}$ for some $(\mathcal{G}, y) \in q_{\mathcal{J}_Y}$. This implies $\mathcal{H} \supset (i \times i)^{-1}(\dot{y} \times \mathcal{G}) = i^{-1}(\dot{y}) \times i^{-1}(\mathcal{G}) = \dot{y} \times i^{-1}(\mathcal{G})$, where $\dot{y} \times i^{-1}(\mathcal{G}) \in \mathcal{J}_X$ because $(i \times i)(\dot{y} \times i^{-1}(\mathcal{G})) \supset \dot{y} \times \mathcal{G} \in \mathcal{J}_Y$, i.e. $\mathcal{H} \in \mathcal{J}_{q_{\mathcal{J}_X}}$.
2) Let $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ be a finite family of preconvergence spaces and let $(X, \mathcal{J}_X)$ be their product in **PUConv**. In order to prove $\mathcal{J}_X \subset \mathcal{J}_{q_{\mathcal{J}_X}}$, let $\mathcal{H} \in \mathcal{J}_X$. Hence, $p_i \times p_i(\mathcal{H}) \in \mathcal{J}_{X_i}$ for each $i \in I$ where $p_i : X \to X_i$ denotes the i-th projection. By assumption, for each $i \in I, p_i \times p_i(\mathcal{H}) \supset \dot{x}_i \times \mathcal{F}_i$ with $\dot{x}_i \times \mathcal{F}_i \in \mathcal{J}_{X_i}$. If $j : \prod\limits_{i \in I} X_i \times X_i \to \prod\limits_{i \in I} X_i \times \prod\limits_{i \in I} X_i$ denotes the canonical isomorphism, then $j(\prod\limits_{i \in I} \dot{x}_i \times \mathcal{F}_i) = (\dot{x_i})_{i \in I} \times \prod\limits_{i \in I} \mathcal{F}_i \subset j(\prod\limits_{i \in I}(p_i \times p_i)(\mathcal{H})) \subset \mathcal{H}$ such that $(\dot{x_i})_{i \in I} \times \prod\limits_{i \in I} \mathcal{F}_i \in \mathcal{J}_X$ (note: $p_i \times p_i((\dot{x_i})_{i \in I} \times \prod\limits_{i \in I} \mathcal{F}_i) = \dot{x}_i \times \mathcal{F}_i \in \mathcal{J}_{X_i}$ for each $i \in I$), i.e. $\mathcal{H} \in \mathcal{J}_{q_{\mathcal{J}_X}}$.

2.12 Remarks. **PConv** *is not countably productive in* **PUConv** as the following *example* shows:
Let $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ be a family of preconvergence spaces such that $|I| = \aleph_0$ and $|X_i| \geq 2$ for all $i \in I$. For each $i \in I$, choose some $\mathcal{G}_i \in \mathcal{J}_{X_i}$. If the product $(X, \mathcal{J}_X)$ of $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ in **PUConv** were a preconvergence space, there would exist $x \in X$ and $\mathcal{F} \in F(X)$ such that
(*) $\quad \dot{x} \times \mathcal{F} \subset j(\prod\limits_{i \in I} \mathcal{G}_i) \qquad$ (j as under 2.11)

and $\dot{x} \times \mathcal{F} \in \mathcal{J}_X$ (note: $(p_i \times p_i)(j(\prod_{i\in I} \mathcal{G}_i) = \mathcal{G}_i \in \mathcal{J}_{X_i}$ for each $i \in I$, i.e. $j(\prod_{i\in I} \mathcal{G}_i) \in \mathcal{J}_X$). Let $F \in \mathcal{F}$. By $(*)$, $\{x\} \times F \supset j(\prod_{i\in I} M_i)$ with $M_i \in \mathcal{G}_i$ for each $i \in I$, and $M_i = X_i \times X_i$ for all but finitely many $i \in I$. Thus, for all but finitely many $i \in I, p_i \times p_i[\{x\} \times F] = \{x_i\} \times p_i[F] \supset p_i \times p_i[j[\prod_{i\in I} M_i]] = M_i = X_i \times X_i$. Consequently, $X_i \subset \{x_i\}$ for all but finitely many $i \in I$ in contrast to the assumption $|X_i| \geq 2$ for all $i \in I$.

2.13 Corollary. PConv *is a strong topological universe.*

Proof. Since **PUConv** is a topological universe, and **PConv** is a bicoreflective subconstruct, it follows from 2.11 that **PConv** is a topological universe, too (cf. [6; 3.1.7 and 3.2.5]). Furthermore, in **GConv** products of quotients are quotients, and by 2.4, **GConv** $\cong$ **PConv**. Thus, in **PConv** quotients are productive.

2.14 Remarks. The *structure of continuous convergence* in **GConv** *can be derived from the natural function space structure in* **PUConv**, called *the uniformly continuous* **PUConv**-*structure*, which is defined similarly to the uniformly continuous **SUConv**-structure (cf. [6; 3.1.9.③]):
Let (X, q) and (X', q') be generalized convergence spaces, and let $\mathbf{X} = (X, \mathcal{J}_q)$ and $\mathbf{X}' = (X', \mathcal{J}_{q'})$ be the corresponding preconvergence spaces. Then the underlying generalized convergence space of the natural function space $\mathbf{X}'^{\mathbf{X}}$ in **PUConv** is the natural function space in **GConv**, whose function space structure is the structure of continuous convergence (cf. [6; 3.1.9.⑥ a)] for the definition).

3. Completions

3.1 Definition. Let $(X, \mathcal{J}_X)$ be a preuniform convergence space.
1) A filter $\mathcal{F}$ on X is called a *Cauchy filter* (or more exactly a $\mathcal{J}_X$-Cauchy filter) provided that $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$.
2) $(X, \mathcal{J}_X)$ is called *complete* provided that each $\mathcal{J}_X$-Cauchy filter converges.

3.2 Remarks. 1) If $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$, then $\mathcal{F} \in F(X)$ is a Cauchy filter iff $\mathcal{F}$ is a Cauchy filter in $(X, \mathcal{J}_X^c)$ (or equivalently, a Cauchy filter in $(X, \mathcal{J}_X^r)$).

2) Let $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$, and $\mathcal{F} \in F(X)$. Then $\mathcal{F}$ converges to $x \in X$ iff $\mathcal{F}$ converges to x in $(X, \mathcal{J}_X^c)$ (or equivalently, $\mathcal{F}$ converges to x in $(X, \mathcal{J}_X^r)$).
3) It follows from 1) and 2) that $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$ is complete iff $(X, \mathcal{J}_X^c)$ is complete (or equivalently, $(X, \mathcal{J}_X^r)$ is complete).
4) Obviously, preconvergence spaces and convergence spaces are complete, where a convergence space is a preuniform convergence space $(X, \mathcal{J}_X)$ such that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \text{there is some } (\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}} \text{ with } \mathcal{F} \supset \mathcal{G} \times \mathcal{G}\}$.

3.3 Definition. Let $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$, and let $(Y, \mathcal{J}_Y)$ be a complete preuniform convergence space containing $(X, \mathcal{J}_X)$ as a dense subspace (i.e. $\overline{X} = \{y \in Y\colon \text{there is some } \mathcal{G} \in F(Y) \text{ converging to } y \text{ such that } A \in \mathcal{G}\} = Y$). If $i : X \to Y$ denotes the inclusion, then $i : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ is called a *completion* of $(X, \mathcal{J}_X)$. Occasionally $(Y, \mathcal{J}_Y)$ is already called a completion of $(X, \mathcal{J}_X)$.

3.4 Remark. If (X, γ) is a filter space (i.e. $\gamma \subset F(X)$ such that 1^0 $\dot{x} \in \gamma$ for each $x \in X$, and 2^0 $\mathcal{F} \in \gamma$ whenever $\mathcal{F}$ contains some $\mathcal{G} \in \gamma$), then an equivalence relation $\sim$ on γ is defined by

$\mathcal{F} \sim \mathcal{G} \Leftrightarrow$ There are finitely many $\mathcal{F}_0, \ldots, \mathcal{F}_n \in \gamma$ with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_n = \mathcal{G}$ such that $\sup\{\mathcal{F}_{i-1}, \mathcal{F}_i\}$ exists for each $i \in \{1, \ldots, n\}$.

The equivalence class of $\mathcal{F} \in \gamma$ w.r.t. $\sim$ is denoted by $[\mathcal{F}]$.
If $(X, \mathcal{J}_X) \in |\mathbf{PUConv}|$, then $(X, \gamma_{\mathcal{J}_X})$ is a filter space, where $\gamma_{\mathcal{J}_X}$ is the set of all $\mathcal{J}_X$-Cauchy filters.

3.5 Theorem. *Let $(X, \mathcal{J}_X)$ be a preuniform convergence space. Put $Y = X \cup \{[\mathcal{F}] : \mathcal{F} \in \gamma_{\mathcal{J}_X} \text{ does not converge}\}$, and let $i : X \to Y$ be the inclusion map. Define a* **PUConv**-*structure $\mathcal{J}_Y$ on Y by*

$$\mathcal{F} \in \mathcal{J}_Y \text{ iff there is some } \mathcal{G} \in \mathcal{B}_Y \text{ such that } \mathcal{F} \supset \mathcal{G},$$

where $\mathcal{B}_Y = \{i \times i(\mathcal{G}) : \mathcal{G} \in \mathcal{J}_X\} \cup \{i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) \times (i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) : \mathcal{F} \in \gamma_{\mathcal{J}_X} \text{ does not converge}\}$. Then $i : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ (or shortly: $(Y, \mathcal{J}_Y)$) is a completion of $(X, \mathcal{J}_X)$, *called the* **simple completion**. *The set of all $\mathcal{J}_Y$-Cauchy filters is generated by*

$$\{i(\mathcal{F}) : \mathcal{F} \in \gamma_{\mathcal{J}_X} \text{ converges}\} \cup \{i(\mathcal{F}) \cap [\dot{\mathcal{F}}] : \mathcal{F} \in \gamma_{\mathcal{J}_X} \text{ does not converge}\}.$$

Proof. Similar to [6; 4.4.5].

3.6 Remarks. 1) The simple completion preserves (and reflects) the property ‘semiuniform convergence space’.

2) If $\mathcal{R}$: **PUConv** $\rightarrow$ **Unif** denotes the bireflector, then $\mathcal{R}((X,\mathcal{J}_X)) = (X,[\mathcal{U}])$ is called *the underlying uniform space* of $(X,\mathcal{J}_X) \in |\mathbf{PUConv}|$, where $\mathcal{U}$ is the finest uniformity which is contained in each $\mathcal{F} \in \mathcal{J}_X$, i.e. $\mathcal{U} = \{V \in \mathcal{V}$: there is sequence $(V_n)_{n\in\mathbb{N})}$ with $V_n \in \mathcal{V}$ and $V_n^{-1} \in \mathcal{V}$ for each $n \in \mathbb{N}$, and $V_1 = V$, such that $V_{n+1}^2 \subset V_n$ for each $n \in \mathbb{N}\}$, where $\mathcal{V} = \bigcap\limits_{\mathcal{F}\in\mathcal{J}_X} \mathcal{F}$.
Now *the Hausdorff completion of a separated uniform space is obtained as the underlying uniform space of its simple completion* (cf. [6; 4.4.21]).

3.7 Definition. A preuniform convergence space $(X,\mathcal{J}_X)$ is called *separated* provided that each filter $\mathcal{F}$ on X converges to at most one point $x \in X$.

3.8 Extension lemma. *Let $(X,\mathcal{J}_X)$ be a preuniform convergence space, $i : (X,\mathcal{J}_X) \rightarrow (Y,\mathcal{J}_Y)$ its simple completion, $(X',\mathcal{J}_{X'})$ a separated preuniform convergence space, and $f : (X,\mathcal{J}_X) \rightarrow (X',\mathcal{J}_{X'})$ a uniformly continuous map. Then there is a unique uniformly continuous map $\overline{f} : (Y,\mathcal{J}_Y) \rightarrow (X',\mathcal{J}_X')$ such that $\overline{f} \circ i = f$.*

Proof. Analogous to [6; 4.4.13].

3.9 Proposition. *If $(X,\mathcal{J}_X)$ is a quasiuniform limit space, then $(X,\gamma_{\mathcal{J}_X})$ is a Cauchy space.*

Proof. Let $\mathcal{F},\mathcal{G} \in \gamma_{\mathcal{J}_X}$ such that $\sup\{\mathcal{F},\mathcal{G}\}$ exists. Then $(\mathcal{F}\times\mathcal{F})\circ(\mathcal{G}\times\mathcal{G}) = \mathcal{G}\times\mathcal{F} \in \mathcal{J}_X$, and $(\mathcal{G}\times\mathcal{G})\circ(\mathcal{F}\times\mathcal{F}) = \mathcal{F}\times\mathcal{G} \in \mathcal{J}_X$.
Consequently $(\mathcal{F}\cap\mathcal{G})\times(\mathcal{F}\cap\mathcal{G}) = \mathcal{F}\times\mathcal{F}\cap\mathcal{F}\times\mathcal{G}\cap\mathcal{G}\times\mathcal{F}\cap\mathcal{G}\times\mathcal{G} \in \mathcal{J}_X$ since $\mathcal{F}\times\mathcal{F}, \mathcal{G}\times\mathcal{G} \in \mathcal{J}_X$ by assumption. Hence, $\mathcal{F}\cap\mathcal{G} \in \gamma_{\mathcal{J}_X}$.

3.10 Corollary. *Let $(X,\mathcal{J}_X)$ be a quasiuniform limit space and $\mathcal{F} \in \gamma_{\mathcal{J}_X}$. Then the following are equivalent*:
(1) *$\mathcal{F}$ does not converge.*
(2) *$[\mathcal{F}] \neq [\dot{x}]$ for each $x \in X$.*

Proof. cf. [6; 4.4.4].

3.11 Proposition. *Let $(X,\mathcal{J}_X)$ be a separated quasiuniform limit space. Then $j : X \rightarrow \{[\dot{x}] : x \in X\}$ defined by $j(x) = [\dot{x}]$ is bijective.*

Proof. Obviously, j is surjective. In order to prove that j is injective, let $[\dot{x}] = [\dot{y}]$. Since $(X, \gamma_{\mathcal{J}_X})$ is a Cauchy space by 3.9, $\dot{x} \cap \dot{y} \in \gamma_{\mathcal{J}_X}$. By the definition of convergence in $(X, \mathcal{J}_X)$, $\dot{x} \cap \dot{y}$ converges to x and y. By assumption, $x = y$.

3.12 Convention. Let $(X, \mathcal{J}_X)$ be a separated quasiuniform limit space, and let $Y' = \{[\mathcal{F}] : \mathcal{F} \in \gamma_{\mathcal{J}_X}\}$. Then $Y' = \{[\dot{x}] : x \in X\} \cup \{[\mathcal{F}] : \mathcal{F} \in \gamma_{\mathcal{J}_X}$ does not convergence$\}$. Since $j : X \to \{[\dot{x}] : x \in X\}$ is bijective, X and $\{[\dot{x}] : x \in X\}$ can be identified, i.e. $Y = Y'$, where $(Y, \mathcal{J}_Y)$ is the simple completion of $(X, \mathcal{J}_X)$.

3.13 Remarks. 1) Let $(X, \mathcal{V})$ be a quasiuniform space, and $(X, \mathcal{U})$ its bicoreflective **Unif**-modification. Then the following are equivalent:
(1) $(X, \mathcal{X}_{\mathcal{V}})$ is a T_0-space, where $\mathcal{X}_{\mathcal{V}} = \{O \subset X$: for each $x \in O$, there is some $V \in \mathcal{V}$ such that $V(x) \subset O\}$.
(2) $(X, \mathcal{X}_{\mathcal{U}})$ is a T_2-space, where $\mathcal{X}_{\mathcal{U}} = \{O \subset X$: for each $x \in O$, there is some $U \in \mathcal{U}$ such that $U(x) \subset O\}$.
(3) $(X, \mathcal{V})$ is separated (in the sense of 3.7).
2) P. Fletcher and W. F. Lindgren [2; 3.33] have constructed (in their terminology) a T_0-bicompletion for each T_0-quasiuniform space, i.e. in our terminology: a *separated quasiuniform completion* $(\tilde{X}, \tilde{\mathcal{V}})$ for each separated quasiuniform space $(X, \mathcal{V})$, in other words: a separated complete quasiuniform space $(\tilde{X}, \tilde{\mathcal{V}})$ containing $(X, \mathcal{V})$ (up to isomorphism) as a dense subspace, as follows:
Let $\tilde{X}$ be the set of all minimal Cauchy filters on $(X, \mathcal{V})$ (or equivalently on $(X, \mathcal{U})$). For each $V \in \mathcal{V}$, let $\tilde{V} = \{(\mathcal{F}, \mathcal{G}) \in \tilde{X} \times \tilde{X}$: there are some $F \in \mathcal{F}$ and some $G \in \mathcal{G}$ such that $F \times G \subset V\}$. Put $\tilde{\mathcal{V}} = (\{\tilde{V} : V \in \mathcal{V}\})$, i.e. $\tilde{\mathcal{V}}$ is generated by $\{\tilde{V} : V \in \mathcal{V}\}$. The embedding $r_X : (X, \mathcal{V}) \to (\tilde{X}, \tilde{\mathcal{V}})$ is defined by $r_X(x) = \mathcal{U}(x)$ (= neighborhood filter of x w.r.t. $\mathcal{X}_{\mathcal{U}}$ [cf. 1)]). *This completion is unique up to isomorphism* ([2; 3.34]).
3) If $\mathcal{R}_q$: **PUConv** $\to$ **QUnif** denotes the bireflector, then $\mathcal{R}_q((X, \mathcal{J}_X)) = (X, [\mathcal{V}])$ is called *the underlying quasiuniform space of* $(X, \mathcal{J}_X) \in |$**PUConv**$|$. In particular, $\mathcal{V}$ is the finest quasiuniformity on X which is contained in each $\mathcal{F} \in \mathcal{J}_X$, i.e. $\mathcal{V} = \{V \in \mathcal{W}$: there is sequence $(V_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{W}$ such that $V_1 = V$, and $V_{n+1}^2 \subset V_n$ for each $n \in \mathbb{N}\}$, where $\mathcal{W} = \bigcap_{\mathcal{F} \in \mathcal{J}_X} \mathcal{F}$.

3.14 Theorem. *Let* $(X, [\mathcal{V}])$ *be a separated quasiuniform space* (*regarded as a preuniform convergence space*). *Then the underlying quasiuniform space* *of its simple completion is the above mentioned separated quasiuniform completion of* $(X, \mathcal{V})$ *(up to isomorphism).*

Proof. Since $(X, [\mathcal{V}])$ is separated, the underlying set of its simple completion may be identified with
$X^* = \{[\mathcal{F}] : \mathcal{F}$ is a $[\mathcal{V}]$-Cauchy filter$\}$, and X with $\{[\dot{x}] : x \in X\}$ (cf. 3.12). Each equivalence class $[\mathcal{F}]$ contains a minimal $[\mathcal{V}]$-Cauchy filter, namely $\mathcal{U}(\mathcal{F}) = (\{U[F] : U \in \mathcal{U}, F \in \mathcal{F}\})$, which is independent of the choice of the representative, where $\mathcal{U}(\dot{x})$ is the neighborhood filter of $x \in X$ w.r.t. $\mathcal{X}_{\mathcal{U}}$.
If $i : X \to X^*$ denotes the inclusion map, and $\mathcal{V}^*$ is the quasiuniformity on X^* which is generated by $\{V^* : V \in \mathcal{V}\}$, where $V^* = \{([\mathcal{F}], [\mathcal{G}]) \in X^* \times X^*$: there are $M_1 \in \mathcal{U}(\mathcal{F})$ and $M_2 \in \mathcal{U}(\mathcal{G})$ such that $M_1 \times M_2 \subset V\} = \{([\mathcal{F}], [\mathcal{G}]) \in X^* \times X^*$: there are $W \in \mathcal{U}, F \in \mathcal{F}$, and $G \in \mathcal{G}$ such that $W[F] \times W[G] \subset V\}$ for each $V \in \mathcal{V}$, then $i : (X, \mathcal{V}) \to (X^*, \mathcal{V}^*)$ is the above mentioned separated quasiuniform completion of $(X, \mathcal{V})$.
Let $i : (X, [\mathcal{V}]) \to (Y, \mathcal{J}_Y)$ be the simple completion of $(X, [\mathcal{V}])$, where $Y = X^*$, and let $\mathcal{W} = \bigcap\limits_{\mathcal{F} \in \mathcal{J}_{X^*}} \mathcal{F}$. Then $\mathcal{W} = (i \times i)(\mathcal{V}) \cap \bigcap \{(i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) \times (i(\mathcal{F}) \cap [\dot{\mathcal{F}}]) :$
$\mathcal{F}$ is a $[\mathcal{V}]$-Cauchy filter$\} = (i \times i)(\mathcal{V}) \cap \bigcap i(\mathcal{U}(\mathcal{F})) \cap [\dot{\mathcal{U}(\mathcal{F})}] \times ((i(\mathcal{U}(\mathcal{F})) \cap [\dot{\mathcal{U}(\mathcal{F})}]) : \mathcal{F}$ is a $[\mathcal{V}]$-Cauchy filter$\}$, and
(1) $\mathcal{V}^* \subset \mathcal{W}$:
By the extension lemma 3.8, there is a unique uniformly continuous map $h : (X^*, \mathcal{J}_{X^*}) \to (X^*, [\mathcal{V}^*])$ such that the diagram

$$\begin{tikzcd}
X, [\mathcal{V}]) \arrow[rr, "i"] \arrow[dr, "i"'] & & (X^*, [\mathcal{V}^*]) \\
& (X^*, \mathcal{J}_{X^*}) \arrow[ur, "h"'] &
\end{tikzcd}$$

commutes, where for each $[\mathcal{V}]$-Cauchy filter $\mathcal{F}, i(\mathcal{F})$ converges to $h([\mathcal{F}])$ in $(X^*, [\mathcal{V}^*])$. On the other hand, $i(\mathcal{F})$ converges to $[\mathcal{F}]$ in $(X^*, \mathcal{V}^*)$, i.e. in $(X^*, [\mathcal{V}^*])$, and since $(X^*, \mathcal{V}^*)$ is separated, $h([\mathcal{F}]) = [\mathcal{F}]$ for each $[\mathcal{V}]$-Cauchy filter $\mathcal{F}$, i.e. $h = 1_{X^*}$. Consequently, $\mathcal{J}_{X^*} \subset [\mathcal{V}^*]$, which implies $\mathcal{V}^* \subset \bigcap \{\mathcal{F} : \mathcal{F} \in \mathcal{J}_{X^*}\} = \mathcal{W}$.
Furthermore,
(2) $\mathcal{V}^* = \mathcal{W} \circ \mathcal{W} \circ \mathcal{W}$:
a) Let $W \in \mathcal{W}$. Then there is some $V \in \mathcal{V}$, and for each $[\mathcal{F}] = [\,\mathcal{U}(\mathcal{F})] \in X^*$,

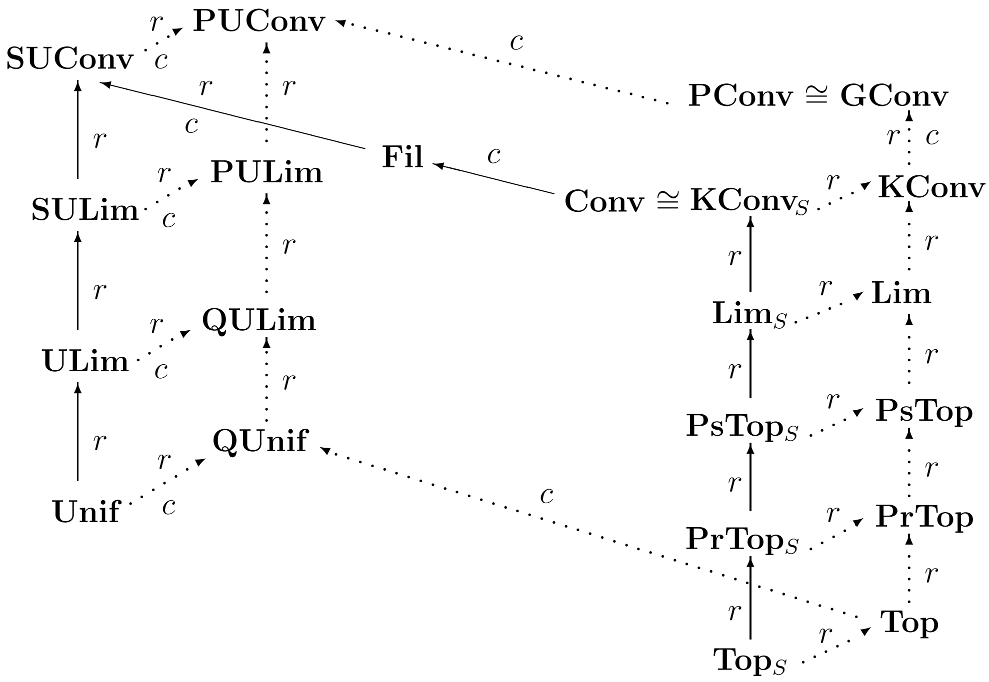
there is a set $C_{[\mathcal{F}]} \in \mathcal{U}(\mathcal{F})$ such that

$$(*)\ V \cup \bigcup_{[\mathcal{F}]\in X^*} (C_{[\mathcal{F}]} \cup \{[\mathcal{F}]\}) \times (C_{[\mathcal{F}]} \cup \{[\mathcal{F}]\}) \subset W.$$

Let $([\mathcal{F}],[\mathcal{G}]) \in V^*$. Then $V \in \mathcal{U}(\mathcal{F}) \times \mathcal{U}(\mathcal{G})$, which implies $V \cap (C_{[\mathcal{F}]} \times C_{[\mathcal{G}]}) \neq \phi$, i.e. there is some $([\dot{x}],[\dot{y}]) \in V \cap (C_{[\mathcal{F}]} \times C_{[\mathcal{G}]})$. Hence, it follows from $(*)$
$([\mathcal{F}],[\dot{x}]) \in (C_{[\mathcal{F}]} \cup \{[\mathcal{F}]\}) \times (C_{[\mathcal{F}]} \cup \{[\mathcal{F}]\}) \subset W$,
$([\dot{x}],[\dot{y}]) \in V \subset W$, and
$([\dot{y}],[\mathcal{G}]) \in (C_{[\mathcal{G}]} \cup \{[\mathcal{G}]\}) \times (C_{[\mathcal{G}]} \cup \{[\mathcal{G}]\}) \subset W$.
Hence, $([\mathcal{F}],[\mathcal{G}]) \in W^3$, i.e. $V^* \subset W^3$. This proves $\mathcal{W}^3 = \mathcal{W} \circ \mathcal{W} \circ \mathcal{W} \subset \mathcal{V}^*$.
b) Since $\mathcal{V}^*$ is a quasiuniformity, $\mathcal{V}^* = \mathcal{V}^* \circ \mathcal{V}^*$. Thus, by (1), $\mathcal{V}^* = \mathcal{V}^* \circ \mathcal{V}^* = \mathcal{V}^* \circ \mathcal{V}^* \circ \mathcal{V}^* \subset \mathcal{W} \circ \mathcal{W} \circ \mathcal{W}$.
Now, let $\mathcal{R}$ be a quasiuniformity on $Y = X^*$ such that $\mathcal{R} \subset \mathcal{W}$. Then $\mathcal{R} = \mathcal{R} \circ \mathcal{R} \circ \mathcal{R} \subset \mathcal{W} \circ \mathcal{W} \circ \mathcal{W} = \mathcal{V}^*$, and since by (1) $\mathcal{V}^* \subset \mathcal{W}, \mathcal{V}^*$ is the finest quasiuniformity which is contained in $\mathcal{W}$, i.e. the finest quasiuniformity which is contained in each $\mathcal{F} \in \mathcal{J}_{X^*}$. Thus, $(X^*,[\mathcal{V}^*])$ is the underlying quasiuniform space of $(X^*,\mathcal{J}_{X^*})$.

4. Diagram of relations between various subconstructs of PUConv

In the following diagram r (resp. c) stands for embedding as a bireflective (resp. bicoreflective) subconstruct. Concerning the constructs not mentioned before, see [6].



(Note: Though $|\mathbf{KConv}_S| \subset |\mathbf{GConv}|$, we have $|\mathbf{Conv}| \cap |\mathbf{PConv}| = \{$discrete **PUConv**-objects$\}$.)

References

[1] Behling, A.: *Einbettung uniformer Räume in topologische Universen*, Ph.D. thesis, Free University, Berlin 1992.

[2] Fletcher, P. and W. F. Lindgren: *Quasi-Uniform Spaces*, Lecture Notes in Pure and Applied Mathematics **77**, Dekker, New York 1982.

[3] Hušek, M. and M. D. Rice: Productivity of coreflective subcategories of uniform spaces, *Gen. Top. Appl.* **9** (1978), 295-306.

[4] Lee, R.: The category of uniform convergence spaces is cartesian closed, *Bulletin Austral. Math. Soc.* **15** (1976), 461-465.

[5] Preuß, G.: Cauchy spaces and generalizations, *Math. Japonica* **38** (1993), 803-812.

[6] Preuß, G.: *Foundations of Topology - An Approach to Convenient Topology*, Kluwer, Dordrecht 2002.

[7] Weil, A.: *Sur les espaces à structures uniformes et sur la topologie générale*, Hermann, Paris 1937.

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